Lecture 18: Operations and Conditional Distributions

Sta230 / Mth230

Colin Rundel

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Operations

In the context of random variables these are random variables that are defined in terms of a function of two (or more) other random variables.

This is generally the class of problem where we are trying to find something like the following:

- $Z = X + Y$
- $Z = \sqrt{X^2 + Y^2}$
- $Z = X/Y$
- $Z = \min(X, Y)$

Distribution of Sums

If X, Y have a joint density $f(x, y)$, then $X + Y$ has the following density

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f(x, z-x)\; dx \\ &= \int_{-\infty}^{\infty} f(z-y, y)\; dy \end{aligned}$$

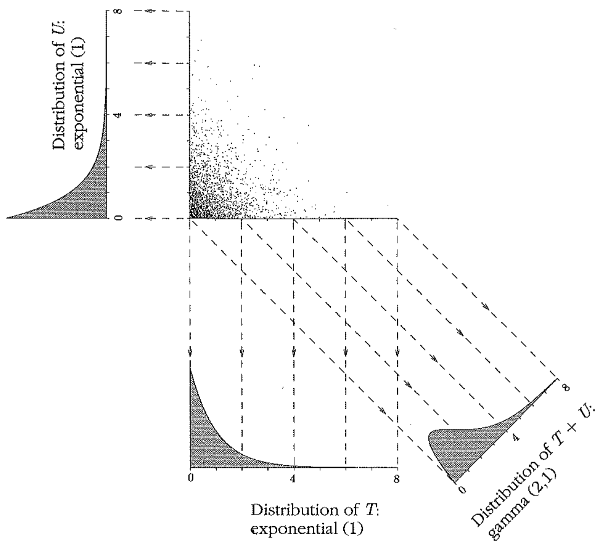
In the case where X and Y are independent, we can use the convolution formula

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\; dx \\ &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\; dy \end{aligned}$$

Example - Sum of Exponentials

Let X, Y be independent exponentially distributed with rate λ, find $f_{X+Y}(z)$ and the distribution of Z.

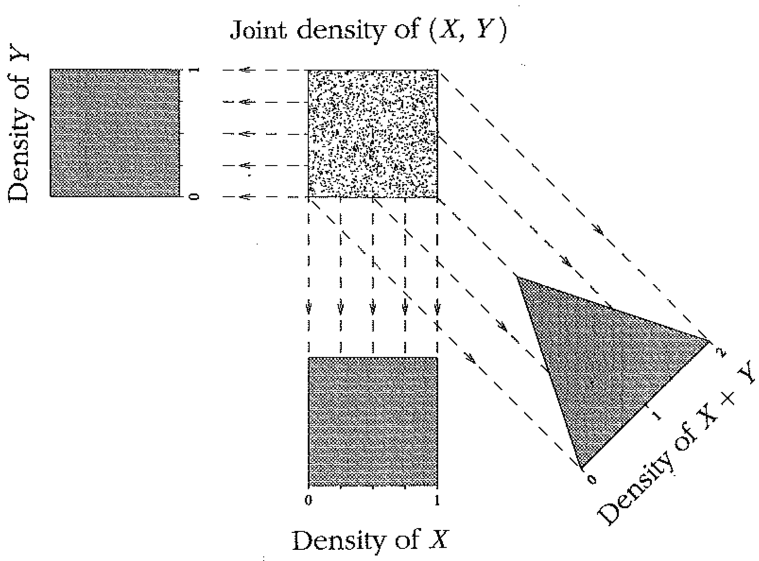
Example - Sum of Exponentials



Example - Sum of Two Uniforms

Let $X, Y \sim \text{Unif}(0,1)$, find $f_{X+Y}(z)$.

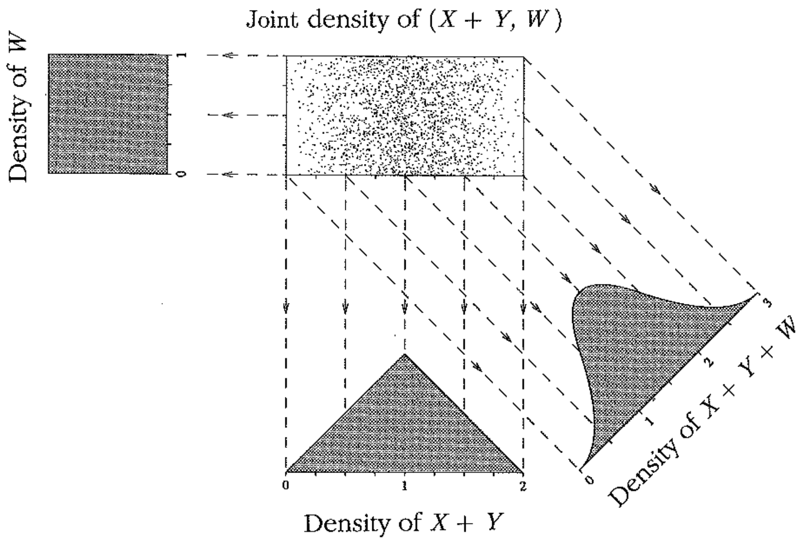
Example - Sum of Two Uniforms



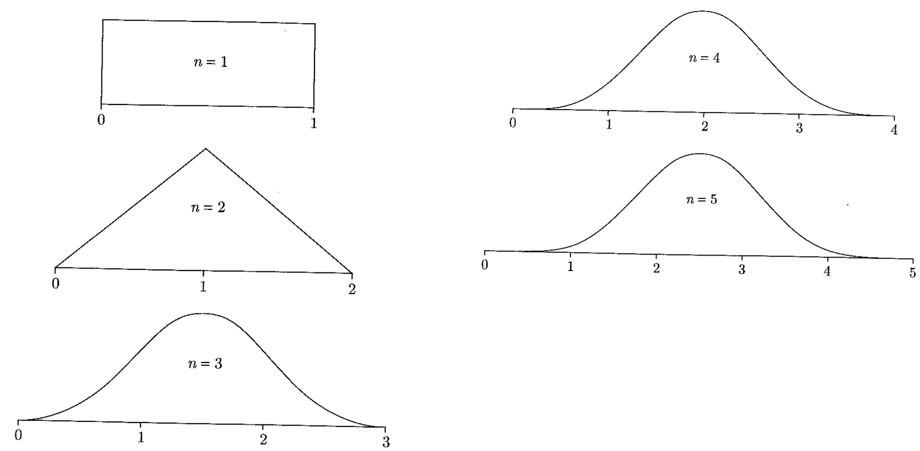
Example - Sum of Three Uniforms

Let $X, Y, W \sim \text{Unif}(0,1)$ where $Z = X + Y$ and $T = X + Y + W$. Find $f_T(t)$.

Example - Sum of Three Uniforms



Example - Sum of Multiple Uniforms



Distribution of Ratios

If X, Y have a joint density $f(x, y)$, then $Z = Y/X$ has the following density

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x| f(x, xz)\; dx$$

In the case where X and Y are independent, we can use the convolution formula

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz)\; dx$$

Example - Ratios of Normals

Let $X, Y \sim \mathcal{N}(0, \sigma^2)$, find $f_{X/Y}(z)$.

Example - Ratios of Uniforms

Let $X, Y \sim \text{Unif}(0,1)$, find $f_{Y/X}(z)$.

Product Distribution

If X, Y have a joint density $f(x,y)$, then $Z = XY$ has the following density

$$\begin{aligned} f_{XY}(z) &= \int_{-\infty}^{\infty} \frac{1}{|x|} f(x, z/x)\, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{|y|} f(z/y, y)\, dy \end{aligned}$$

In the case where X and Y are independent, we can use the convolution formula

$$\begin{aligned} f_{XY}(z) &= \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y(z/x)\, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{|y|} f_X(z/y) f_Y(y)\, dx \end{aligned}$$

Example - Product of Uniforms

Let $X, Y \sim \text{Unif}(0,1)$, find $f_{XY}(z)$.

Where do these distributions come from?

If X, Y have a joint density $f(x,y)$, then $Z = g(X,Y)$ has a density $f_{g(X,Y)}(z)$ which is determined by integrating over all values of X or Y and then considering a change of variables for the other to Z.

Distribution of Sums - $Z = X + Y$:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x) \Bigg/ \left|\frac{dz}{dy}\right| dx = \int_{-\infty}^{\infty} f(x, z-x)\, dx$$

Distribution of Ratios - $Z = Y/X$:

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} f(x, zx) \Bigg/ \left|\frac{dz}{dy}\right| dx = \int_{-\infty}^{\infty} |x| f(x, zx)\, dx$$

Product Distribution - $Z = XY$:

$$f_{XY}(z) = \int_{-\infty}^{\infty} f(x, z/x) \Bigg/ \left|\frac{dz}{dy}\right| dx = \int_{-\infty}^{\infty} f(x, z/x)/|x|\, dx$$

Example - Sum of Poissons

Let X and Y be Poisson random variables with rates λ_1 and λ_2, what is the distribution of $Z = X + Y$.

Example - de Groot Ex 3.9.9

Let X and Y have a continuous joint density

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $Z = X/Y$, find $f_Z(z)$.

If $Z' = XY$, find $f_{Z'}(z)$.