Lecture 6: $E(X)$, $Var(X)$, & $Cov(X, Y)$

Sta230/Mth230

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Random Variables

We have been using them for a while now in a variety of forms but it is good to explicitly define what we mean

Random Variable

A real-valued* function on the sample space Ω

Example: If Ω is the 36 element space resulting from rolling two fair six-sided dies (r and g), then the following are all random variables

$$X(r,g) = r$$
$$Y(r,g) = |r-g|$$
$$Z(r,g) = r+g$$

Random Variables, cont.

Random variables are in essence a fancy way of describing an event.

Previous example:

$$\Omega = \{(r,g) : 1 \le r, g \le 6\}$$
$$Y(r,g) = |r-g|$$

What is the event for $P(Y=1)$ in terms of $\omega \in \Omega$?

Random Variables - Vocabulary

Range of a random variable

- Set of all possible values

Distribution of a random variable

- Specification of $P(X \in A)$ for every set A.
- If X has a countably large range then we can define $f(x) = P(X = x)$ as $P(X \in A) = \sum_{x \in A} f(x)$

Expected Value

The expected value of a random variable is defined as follows

Discrete Random Variable:

$$E[X] = \sum_{\text{all } x} xP(X = x)$$

Continous Random Variable:

$$E[X] = \int_{\text{all } x} xP(X = x)dx$$

This is a natural generalization of what we do when deciding if a casino game is fair.

Properties of Expected Value

- **Constants** - $E(c) = c$ if c is constant
- **Indicators** - $E(I_A) = P(A)$ where I_A is an indicator function
- **Functions** - $E[g(X)] = \begin{cases} \sum_{\text{all } x} g(x)\ P(X = x) & \text{if discrete} \\ \int_x g(x)\ P(X = x)\ dx & \text{if continuous} \end{cases}$
- **Constant Factors** - $E(cX) = cE(x)$
- **Addition** - $E(X + Y) = E(X) + E(Y)$
- **Multiplication** - $E(XY) = E(X)E(Y)$ if X and Y are independent.

Variance

Another common property of random variables we are interested in is the Variance which measures the squared deviation from the mean.

$$Var(X) = E\left[(X - E(X))^2\right] = E(X - \mu)^2$$

One common simplification:

$$\begin{aligned} Var(X) &= E(X - \mu)^2 \\ &= E(X^2 - 2\mu X + \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Standard Deviation:

$$SD(X) = \sqrt{Var(X)}$$

Properties of Variance

What is $Var(aX + b)$ when a and b are constants?

Which gives us:

$$Var(aX) = a^2 Var(X)$$
$$Var(X + c) = Var(X)$$
$$Var(c) = 0$$

Properties of Variance, cont.

What about $Var(X + Y)$?

Covariance

What about when X and Y are not independent?

$$E(XY) \neq E(X)E(Y) \quad \Rightarrow \quad E(XY) - \mu_x\mu_y \neq 0$$

This quantity is known as Covariance, and is roughly speaking a generalization of variance to two variables

$$\begin{aligned} Cov(X, Y) &= E[(X - E(X))(Y - E(Y))] \\ &= E[(X - \mu_x)(Y - \mu_y)] \\ &= E[XY + \mu_x\mu_y - X\mu_y - Y\mu_x]) \\ &= E(XY) - \mu_x\mu_y \end{aligned}$$

Properties of Covariance

- $Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)] = E(XY) - \mu_x\mu_y$
- $Cov(X, Y) = 0$ if X and Y are independent
- $Cov(X, c) = 0$
- $Cov(X, X) = Var(X)$
- $Cov(aX, bY) = ab\ Cov(X, Y)$
- $Cov(X + a, Y + b) = Cov(X, Y)$

Properties of Variance, cont.

A general formula for the variance of the linear combination of two random variables:

From which we can see that

$$Var(X + Y) = Var(X) + Var(Y) + Cov(X, Y)$$
$$Var(X - Y) = Var(X) + Var(Y) - Cov(X, Y)$$

Properties of Variance, cont.

For a completely general formula for the variances of a linear combination of n random variables:

$$
\begin{aligned}
Var\left(\sum_{i=1}^{n} c_i X_i\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(c_i X_i, c_j X_j) \\
&= \sum_{i=1}^{n} c_i^2 \; Var(X_i) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} c_i c_j \; Cov(X_i, X_j)
\end{aligned}
$$

Bernoulli Random Variable

Let $X \sim \text{Bern}(p)$, what is $E(X)$ and $Var(X)$?

Binomial Random Variable

Let $X \sim \text{Binom}(n, p)$, what is $E(X)$ and $Var(X)$?

We can redefine $X = \sum_{i=1}^{n} Y_i$ where $Y_1, \cdots, Y_n \sim \text{Bern}(p)$, and since we are *sampling with replacement* all Y_i and Y_j are independent.

Hypergeometric Random Variable - $E(X)$

Lets consider a simple case where we have an urn with m black marbles and $N - m$ white marbles. Let B_i be an indicator variable for the ith marble being black.

$$
B_i = \begin{cases} 1 & \text{if } i\text{th draw is black} \\ 0 & \text{otherwise} \end{cases}
$$

In the case where $N = 2$ and $m = 1$ what is $P(B_i) = 1$ for all i?

$$
\Omega = \{BW, WB\}
$$

$$
P(B_1) = 1/2, \;\; P(B_2) = 1/2
$$

$$
P(W_1) = 1/2, \;\; P(W_2) = 1/2
$$

Hypergeometric Random Variable - $E(X)$ - cont.

What about when $N = 3$ and $m = 1$?

$$\Omega = \{BW_1W_2, BW_2W_1, W_1BW_2, W_2BW_1, W_1W_2B, W_2W_1B\}$$

$$P(B_1) = 1/3,\ P(B_2) = 1/3,\ P(B_3) = 1/3$$
$$P(W_1) = 2/3,\ P(W_2) = 2/3,\ P(W_3) = 2/3$$

Proposition

$$P(B_i = 1) = m/N \text{ for all } i$$

Hypergeometric Random Variable - $E(X)$ - cont.

Let $X \sim \text{Hypergeo}(N, m, n)$ then $X = B_1 + B_2 + \cdots + B_n$

Hypergeometric Random Variable - $E(X)$ - 2nd way

Let $X \sim \text{Hypergeo}(N, m, n)$, what is $E(X)$?

Hypergeometric Random Variable - $Var(X)$

Let $X \sim \text{Hypergeo}(N, m, n)$, what is $Var(X)$?

Hypergeometric Random Variable - Variance, cont.

Hypergeometric Random Variable - Variance, cont.

Poisson Random Variable - $E(X)$

Let $X \sim \text{Poisson}(\lambda)$, what is $E(X)$?

Poisson Random Variable - $Var(X)$

Let $X \sim \text{Poisson}(\lambda)$, waht is $Var(X)$?

St. Petersburg Lottery

We start with \$1 on the table and a coin.

At each step: Toss the coin; if it shows Heads, take the money. If it shows Tails, I double the money on the table.

Let X be the amount you win, what is $E(X)$?