

Lecture 5

Random Effects Models

9/19/2018

Random Effects Models

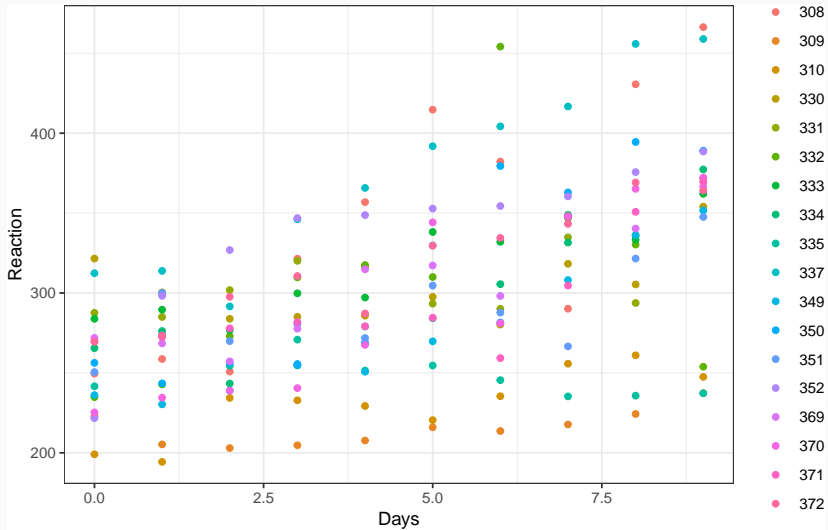
Sleep Study Data

The average reaction time per day for subjects in a sleep deprivation study. On day 0 the subjects had their normal amount of sleep . Starting that night they were restricted to 3 hours of sleep per night. The observations represent the average reaction time on a series of tests given each day to each subject.

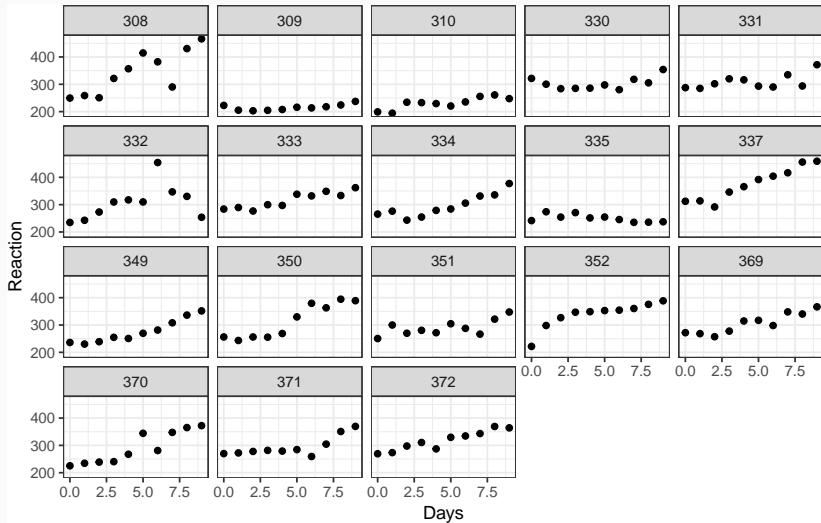
```
sleep = lme4::sleepstudy %>% tbl_df()
```

```
sleep
```

```
## # A tibble: 180 x 3
##   Reaction Days Subject
##   <dbl> <dbl> <fct>
## 1    250.     0 308
## 2    259.     1 308
## 3    251.     2 308
## 4    321.     3 308
## 5    357.     4 308
## 6    415.     5 308
## 7    382.     6 308
## 8    290.     7 308
## 9    431.     8 308
## 10   466.     9 308
## # ... with 170 more rows
```



EDA (small multiples)



Bayesian Linear Model

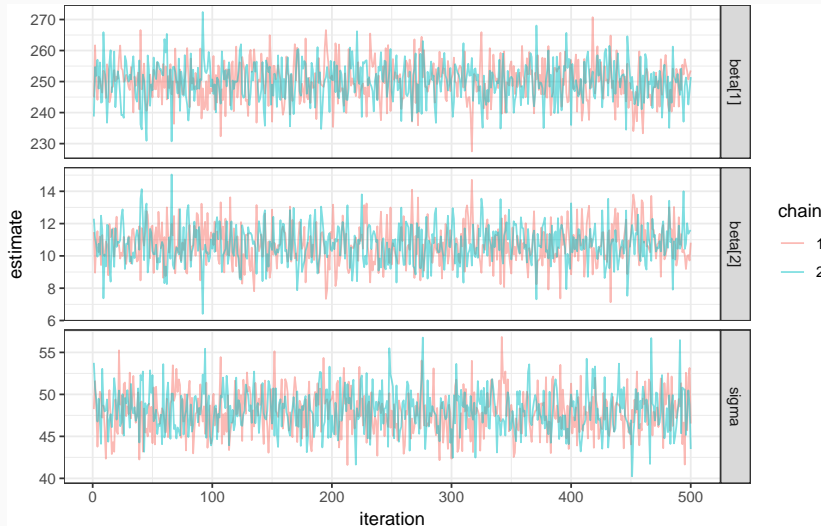
```
sleep_lm = "model{
  # Likelihood
  for(i in 1:length(y)){
    y[i] ~ dnorm(mu[i], tau)
    mu[i] = beta[1] + beta[2]*x[i]

    y_pred[i] ~ dnorm(mu[i],tau)
  }

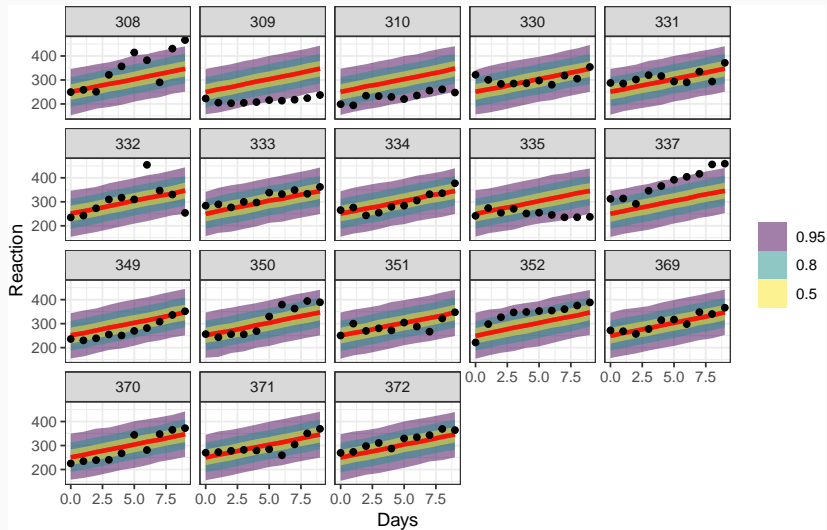
  # Prior for beta
  beta[1] ~ dnorm(0,1/10000)
  beta[2] ~ dnorm(0,1/10000)

  # Prior for sigma / tau
  sigma ~ dunif(0, 100)
  tau = 1/(sigma*sigma)
}"
```

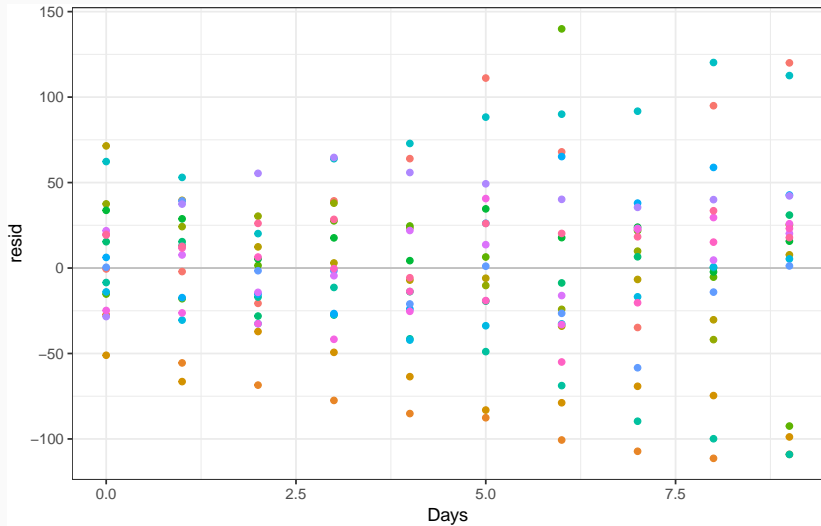
MCMC Diagnostics



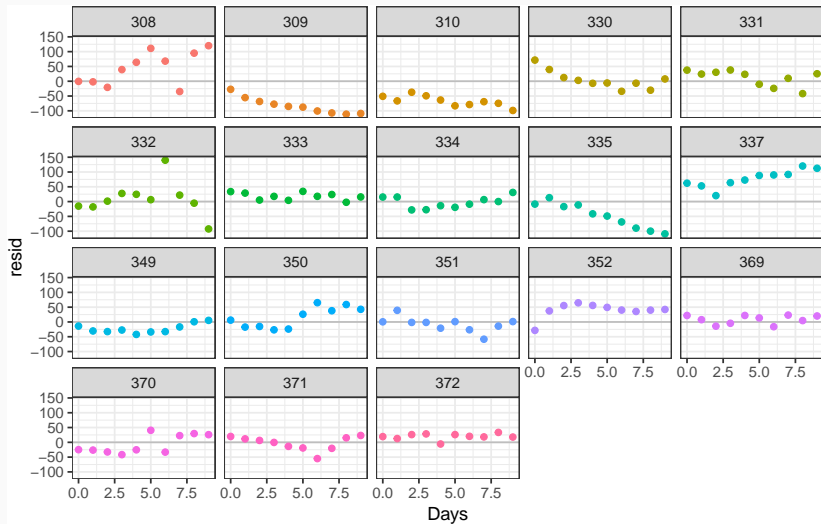
Model fit



Residuals



Residuals by subject



Random Intercept Model

```
sleep = sleep %>%  
  mutate(Subject_index = as.integer(Subject))
```

```
sleep[c(1:2,11:12,21:22,31:32),]
```

```
## # A tibble: 8 x 4
```

```
##   Reaction  Days Subject Subject_index
```

```
##   <dbl> <dbl> <fct>         <int>
```

```
## 1    250.     0 308             1
```

```
## 2    259.     1 308             1
```

```
## 3    223.     0 309             2
```

```
## 4    205.     1 309             2
```

```
## 5    199.     0 310             3
```

```
## 6    194.     1 310             3
```

```
## 7    322.     0 330             4
```

```
## 8    300.     1 330             4
```

Let i represent each observation and $j(i)$ be subject in observation i then

$$y_i = \alpha_{j(i)} + \beta \times \text{Day}_i + \epsilon_i$$

$$\alpha_j \sim \mathcal{N}(\beta_\alpha, \sigma_\alpha^2)$$

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

$$\beta_\alpha \sim \mathcal{N}(0, 10^4)$$

$$\beta \sim \mathcal{N}(0, 10^4)$$

$$\sigma, \sigma_\alpha \sim \text{Unif}(0, 10^2)$$

Random Intercept Model - JAGS

```
sleep_ri = "model{
  for(i in 1:length(Reaction)) {
    Reaction[i] ~ dnorm(mu[i],tau)
    mu[i] = alpha[Subject_index[i]] + beta*Days[i]

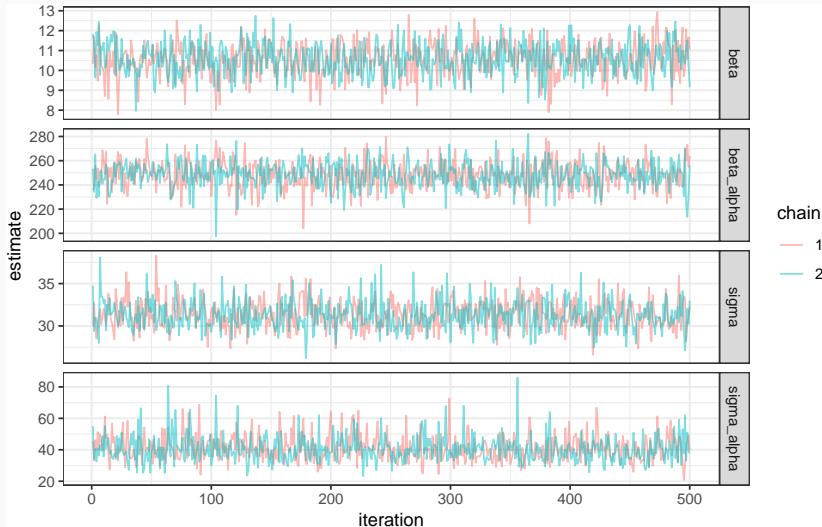
    y_pred[i] ~ dnorm(mu[i],tau)
  }

  for(j in 1:18) {
    alpha[j] ~ dnorm(beta_alpha, tau_alpha)
  }

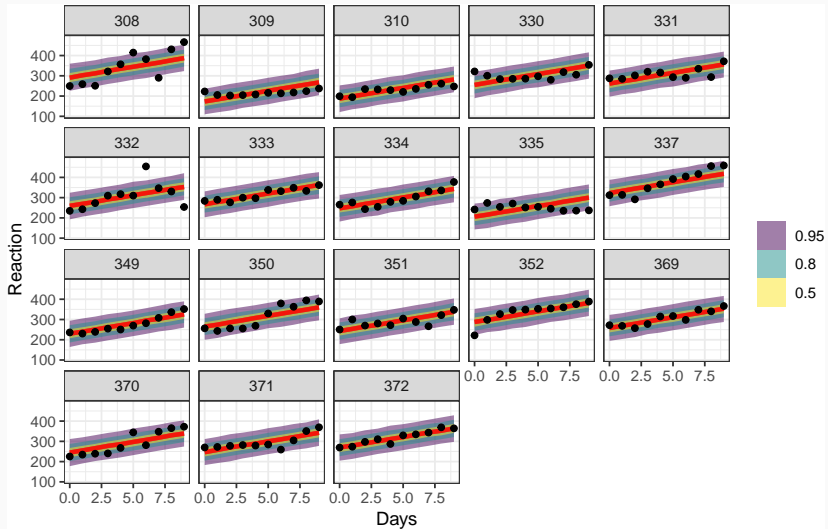
  beta_alpha ~ dnorm(0,1/10000)
  sigma_alpha ~ dunif(0, 100)
  tau_alpha = 1/(sigma_alpha*sigma_alpha)

  beta ~ dnorm(0,1/10000)
  sigma ~ dunif(0, 100)
  tau = 1/(sigma*sigma)
}"
```

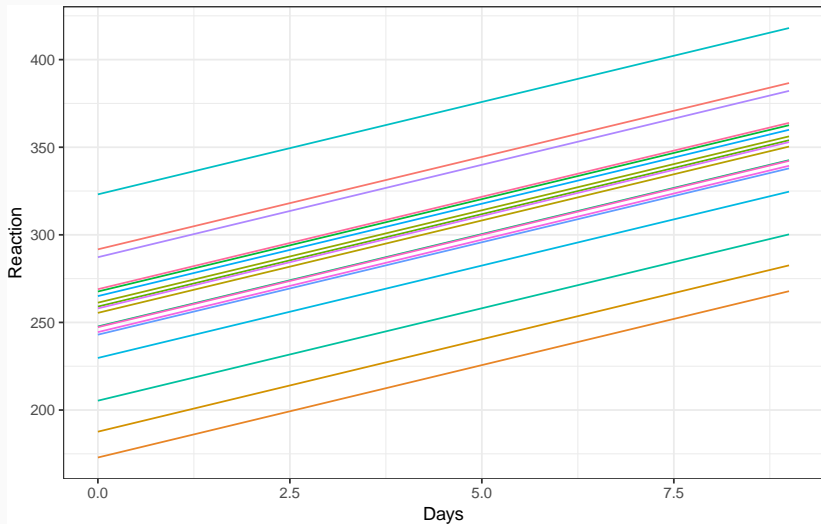
MCMC Diagnostics



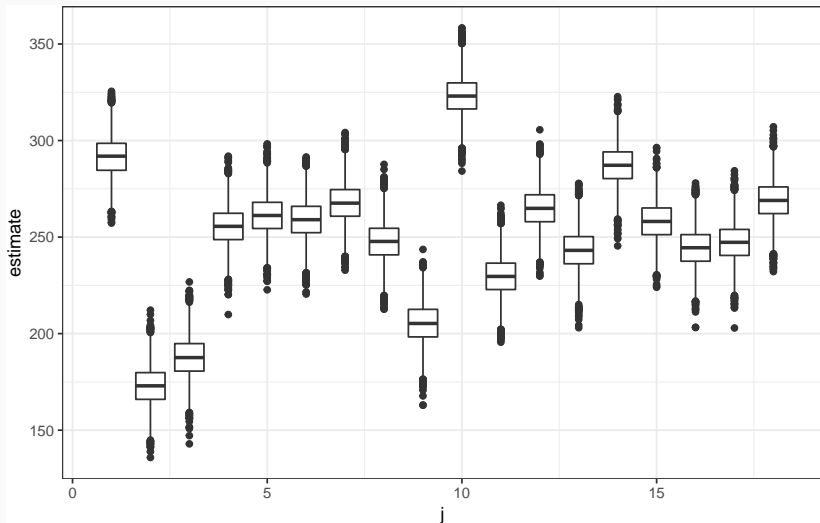
Model fit



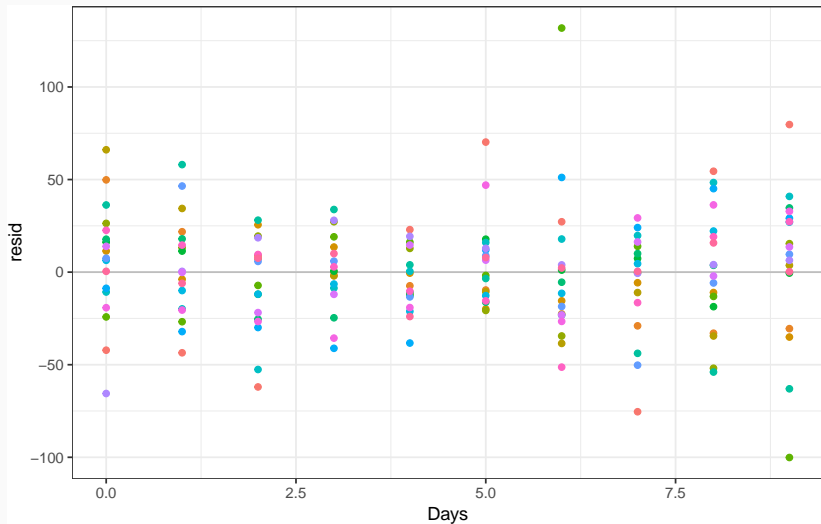
Model fit - lines



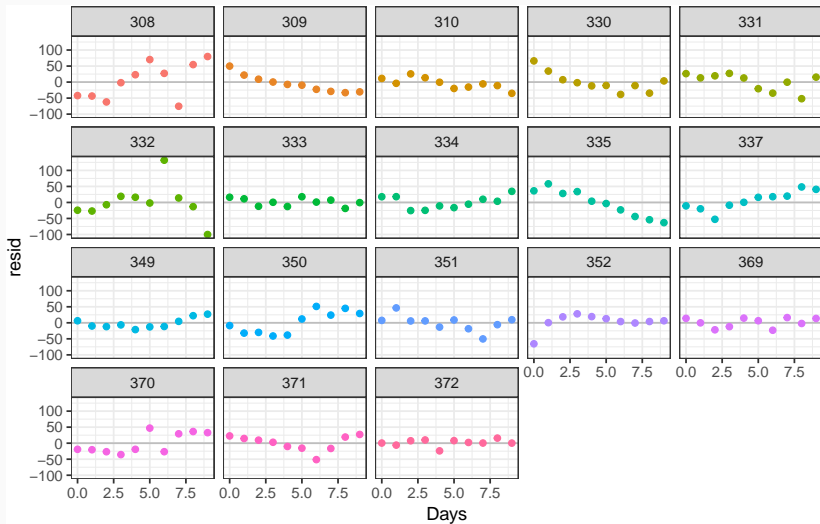
Random Effects



Residuals



Residuals by subject



Why not a fixed effect for Subject?

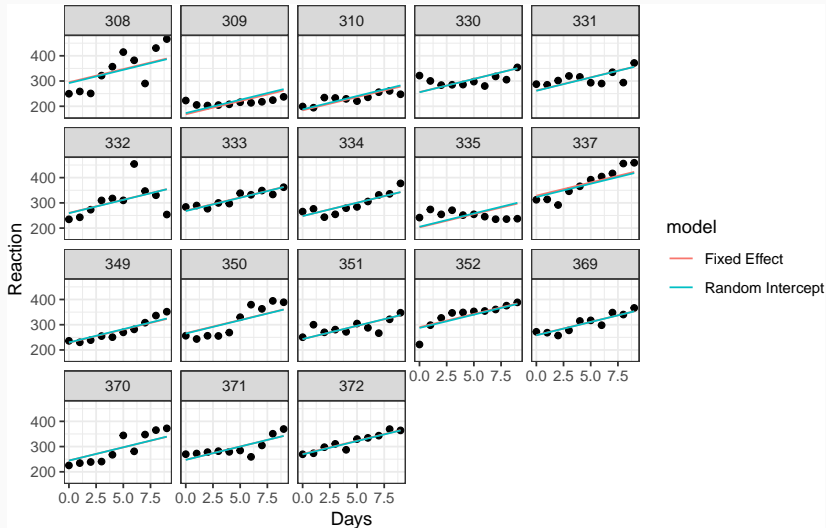
We are not going to bother with the Bayesian model here to avoid the headache of dummy coding and the resulting β s.

Why not a fixed effect for Subject?

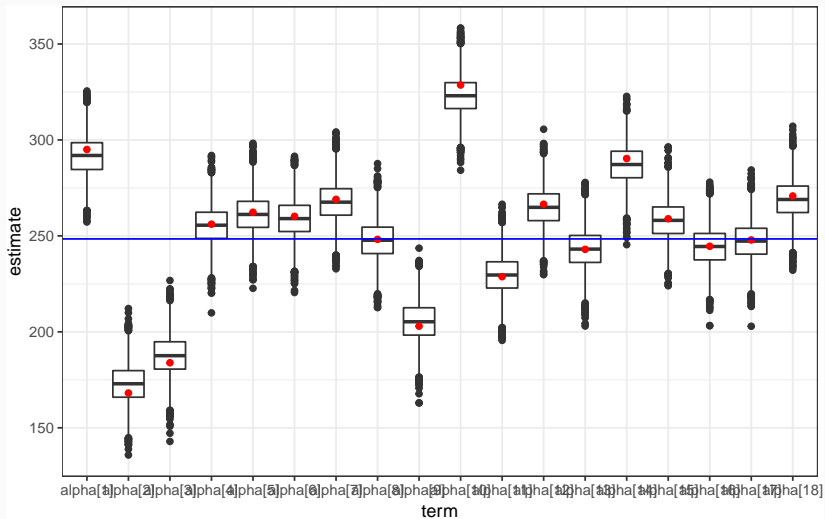
We are not going to bother with the Bayesian model here to avoid the headache of dummy coding and the resulting β s.

```
l = lm(Reaction ~ Days + Subject - 1, data=sleep)
summary(l)
##
## Call:
## lm(formula = Reaction ~ Days + Subject - 1, data = sleep)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -100.540  -16.389   -0.341   15.215  131.159
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## Days              10.4673     0.88042   13.02 <2e-16 ***
## Subject308       295.0310    10.4471  28.24 <2e-16 ***
## Subject309       168.1302    10.4471  16.09 <2e-16 ***
## Subject310       183.8985    10.4471  17.60 <2e-16 ***
## Subject330       256.1186    10.4471  24.52 <2e-16 ***
## Subject331       262.3333    10.4471  25.11 <2e-16 ***
## Subject332       260.1993    10.4471  24.91 <2e-16 ***
## Subject333       269.0555    10.4471  25.75 <2e-16 ***
## Subject334       248.1993    10.4471  23.76 <2e-16 ***
## Subject335       202.9673    10.4471  19.43 <2e-16 ***
## Subject337       328.6182    10.4471  31.45 <2e-16 ***
## Subject349       228.7317    10.4471  21.89 <2e-16 ***
## Subject350       266.4999    10.4471  25.51 <2e-16 ***
## Subject351       242.9950    10.4471  23.26 <2e-16 ***
## Subject352       290.3188    10.4471  27.79 <2e-16 ***
## Subject369       258.9319    10.4471  24.79 <2e-16 ***
## Subject370       244.5990    10.4471  23.41 <2e-16 ***
## Subject371       247.8813    10.4471  23.73 <2e-16 ***
## Subject372       270.7833    10.4471  25.92 <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Comparing Model fit



Random effects vs fixed effects



Random Intercept Model (strong prior for σ_α)

```
sleep_ri2 = "model{
  for(i in 1:length(Reaction)) {
    Reaction[i] ~ dnorm(mu[i],tau)
    mu[i] = alpha[Subject_index[i]] + beta*Days[i]

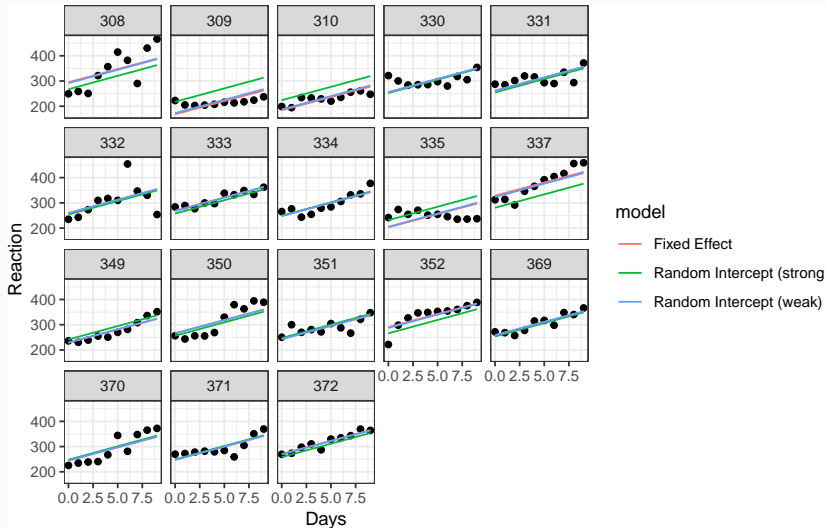
    y_pred[i] ~ dnorm(mu[i],tau)
  }

  for(j in 1:18) {
    alpha[j] ~ dnorm(beta_alpha, tau_alpha)
  }

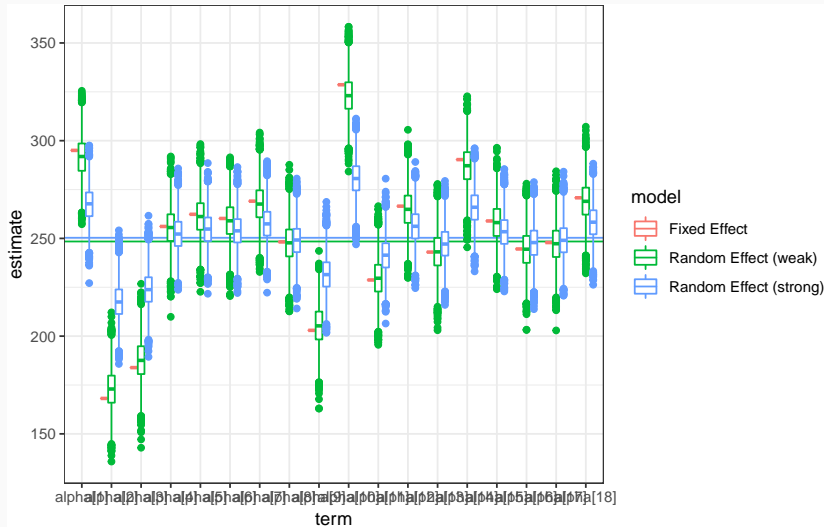
  beta_alpha ~ dnorm(0,1/10000)
  sigma_alpha ~ dunif(0, 10)
  tau_alpha = 1/(sigma_alpha*sigma_alpha)

  beta ~ dnorm(0,1/10000)
  sigma ~ dunif(0, 100)
  tau = 1/(sigma*sigma)
}"
```

Comparing Model fit



Prior Effect on α



Random intercept and slope model

Let i represent each observation and $j(i)$ be the subject in observation i then

$$Y_i = \alpha_{j(i)} + \beta_{j(i)} \times \text{Days} + \epsilon_i$$

$$\alpha_j \sim \mathcal{N}(\beta_0, \sigma_\alpha^2)$$

$$\beta_j \sim \mathcal{N}(\beta_1, \sigma_\beta^2)$$

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

$$\beta_\alpha, \beta_\beta \sim \mathcal{N}(0, 10000)$$

$$\sigma, \sigma_\alpha, \sigma_\beta \sim \text{Unif}(0, 100)$$

Model - JAGS

```
sleep_ris = "model{
  for(i in 1:length(Reaction)) {
    Reaction[i] ~ dnorm(mu[i],tau)
    mu[i] = alpha[Subject_index[i]] + beta[Subject_index[i]] * Days[i]
    y_pred[i] ~ dnorm(mu[i], tau)
  }

  sigma ~ dunif(0, 100)
  tau = 1/(sigma*sigma)

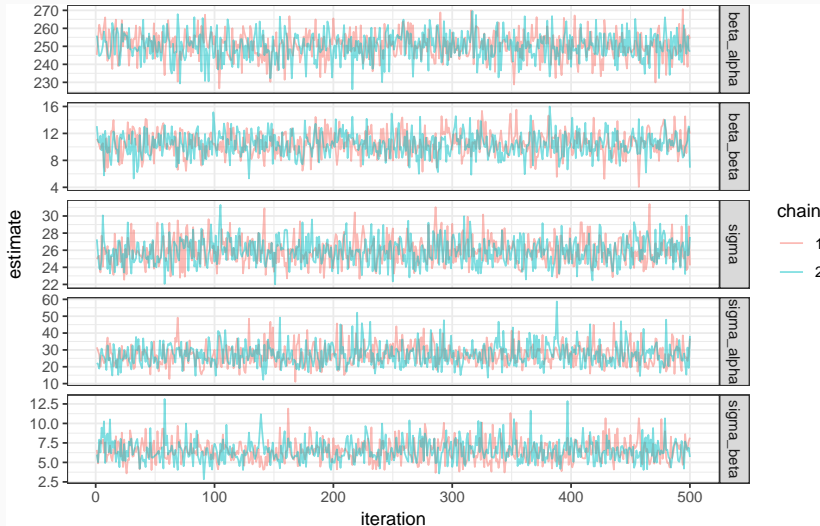
  for(j in 1:18) {
    alpha[j] ~ dnorm(beta_alpha, tau_alpha)
    beta[j] ~ dnorm(beta_beta, tau_beta)
  }

  beta_alpha ~ dnorm(0,1/10000)
  beta_beta ~ dnorm(0,1/10000)

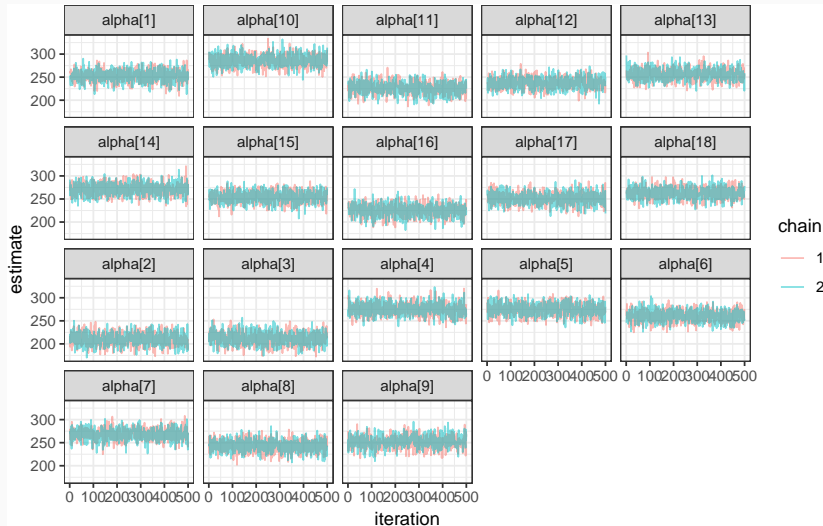
  sigma_alpha ~ dunif(0, 100)
  tau_alpha = 1/(sigma_alpha*sigma_alpha)

  sigma_beta ~ dunif(0, 100)
  tau_beta = 1/(sigma_beta*sigma_beta)
}"
```

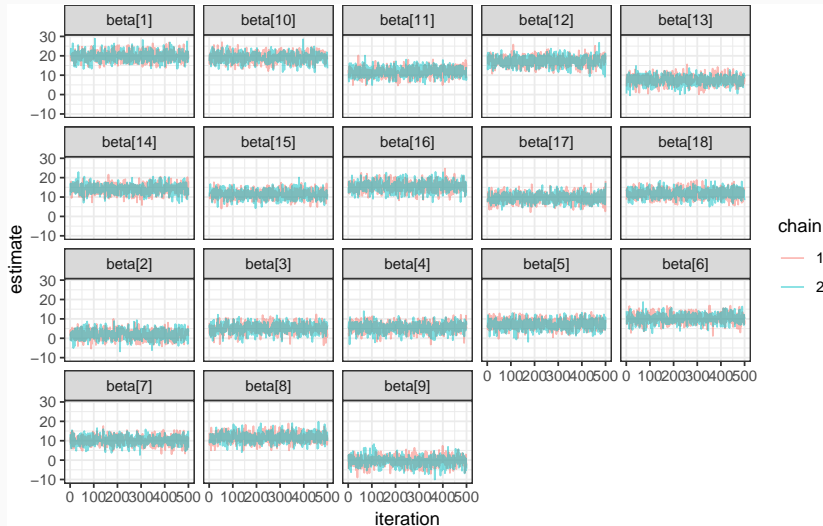
MCMC Diagnostics



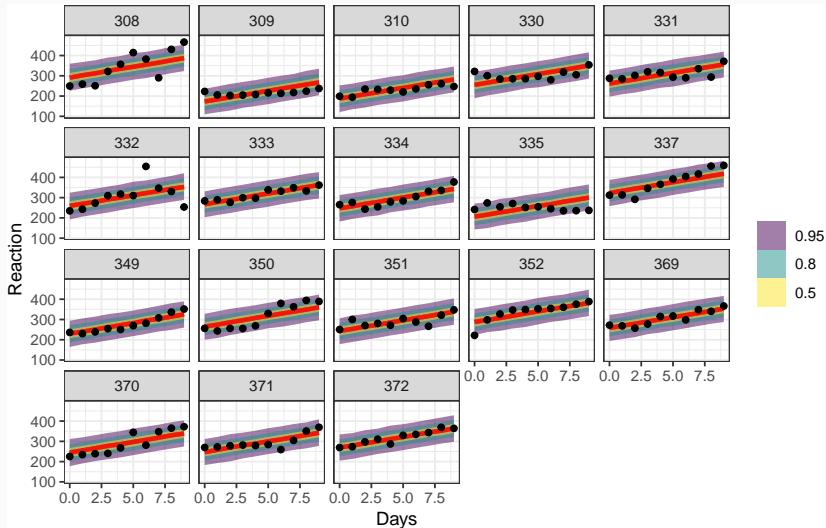
MCMC Diagnostics - α



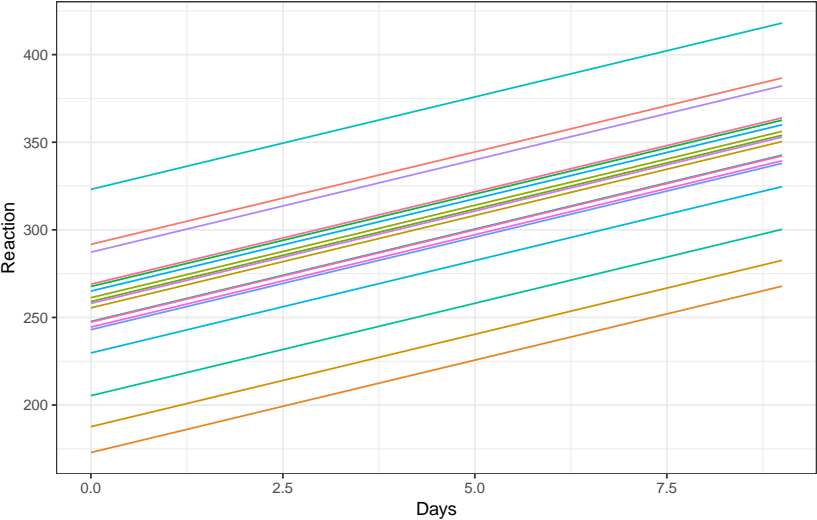
MCMC Diagnostics - β



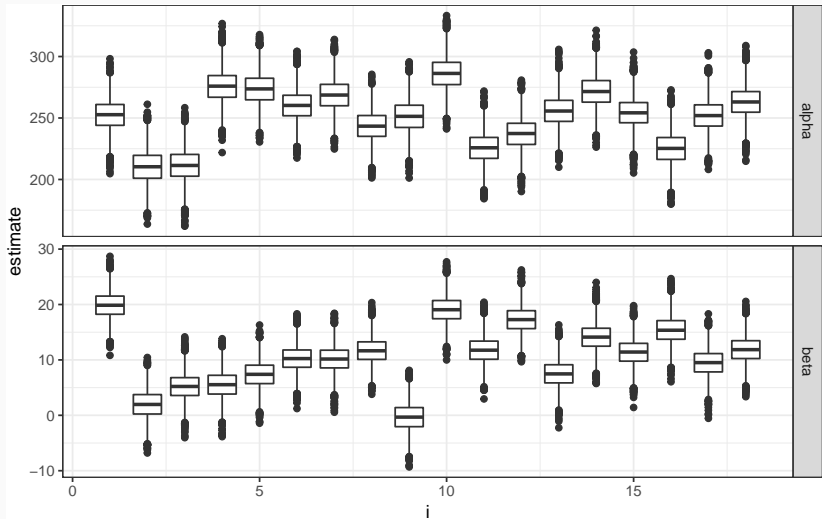
Model fit



Model fit - lines



Random Effects



Residuals by subject

