Lecture 17

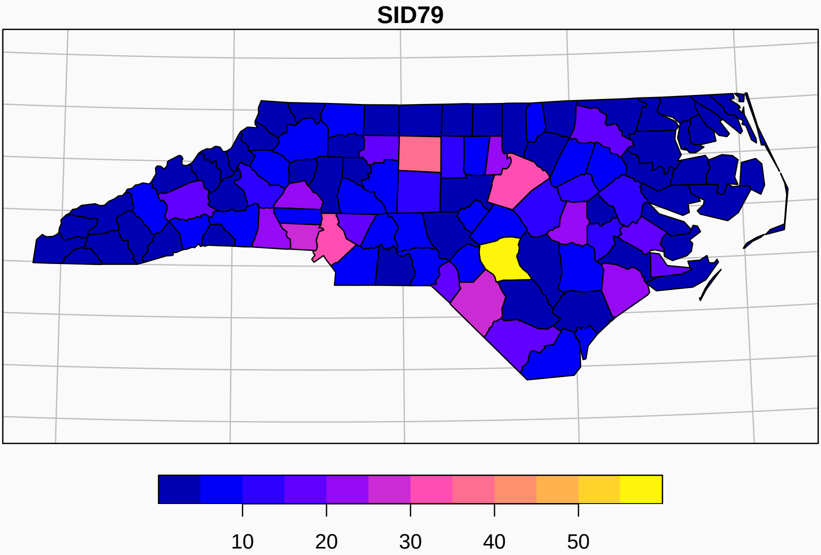
Models for areal data

Colin Rundel

11/02/2018

areal / lattice data

Example - NC SIDS



If we have observations at n spatial locations $(s_1, \ldots s_n)$

$$I = \frac{n}{\sum_{i=1}^n \sum_{j=1}^n w_{ij}} \frac{\sum_{i=1}^n \sum_{j=1}^n w_{ij} \big(y(s_i) - \bar{y}\big)\big(y(s_j) - \bar{y}\big)}{\sum_{i=1}^n \big(y(s_i) - \bar{y}\big)^2}$$

where $\mathbf{w} = \{w\}_{ij}$ is a spatial weights matrix.

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where $\mathbf{w} = \{w\}_{ij}$ is a spatial weights matrix.

Some properties of Moran's I when there is no spatial autocorrelation / dependence:

- $E(I) = -1/(n-1)$
- $Var(I) =$ Something ugly but closed form $- E(I)^2$
- $\lim\limits_{n\to\infty} \frac{I - E(I)}{\sqrt{Var(I)}} \sim \mathcal{N}(0,1)$

Adjacency Matrix

```r
1*st_touches(nc[1:12,], sparse=FALSE)
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
##  [1,]    0    1    0    0    0    0    0    0    0     0     0     0
##  [2,]    1    0    1    0    0    0    0    0    0     0     0     0
##  [3,]    0    1    0    0    0    0    0    0    0     1     0     0
##  [4,]    0    0    0    0    0    0    1    0    0     0     0     0
##  [5,]    0    0    0    0    0    1    0    0    1     0     0     0
##  [6,]    0    0    0    0    1    0    0    1    0     0     0     0
##  [7,]    0    0    0    1    0    0    0    1    0     0     0     0
##  [8,]    0    0    0    0    0    1    1    0    0     0     0     0
##  [9,]    0    0    0    0    1    0    0    0    0     0     0     0
## [10,]    0    0    1    0    0    0    0    0    0     0     0     1
## [11,]    0    0    0    0    0    0    0    0    0     0     0     1
## [12,]    0    0    0    0    0    0    0    0    0     1     1     0
```

Normalized Adjacency Matrix

```r
normalize_weights = function(w) {
  diag(w) = 0
  rs = rowSums(w)
  rs[rs == 0] = 1
  w/rs
}

normalize_weights( 1*st_touches(nc[1:12,], sparse=FALSE) )
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
##  [1,]  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   0.0
##  [2,]  0.5  0.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   0.0
##  [3,]  0.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.5   0.0   0.0
##  [4,]  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0   0.0   0.0   0.0
##  [5,]  0.0  0.0  0.0  0.0  0.0  0.5  0.0  0.0  0.5   0.0   0.0   0.0
##  [6,]  0.0  0.0  0.0  0.0  0.5  0.0  0.0  0.5  0.0   0.0   0.0   0.0
##  [7,]  0.0  0.0  0.0  0.5  0.0  0.0  0.0  0.5  0.0   0.0   0.0   0.0
##  [8,]  0.0  0.0  0.0  0.0  0.0  0.5  0.5  0.0  0.0   0.0   0.0   0.0
##  [9,]  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0   0.0   0.0   0.0
## [10,]  0.0  0.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   0.5
## [11,]  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   1.0
## [12,]  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.5   0.5   0.0
```

NC SIDS & Moran's I

Lets start by using a normalized adjacency matrix for $\mathbf{w}$ (shared county borders).

```
morans_I = function(y, w) {
  w = normalize_weights(w)
  n = length(y)
  y_bar = mean(y)
  num = sum(w * (y-y_bar) %*% t(y-y_bar))
  denom = sum( (y-y_bar)^2 )
  (n/sum(w)) * (num/denom)
}

w = 1*st_touches(nc, sparse=FALSE)

morans_I(y = nc$SID74, w)
## [1] 0.1477405

ape::Moran.I(nc$SID74, weight = w) %>% str()
## List of 4
##  $ observed: num 0.148
##  $ expected: num -0.0101
##  $ sd      : num 0.0627
##  $ p.value : num 0.0118
```

Like Moran's I, if we have observations at n spatial locations $(s_1, \dots s_n)$

$$C = \frac{n-1}{2\sum_{i=1}^n \sum_{j=1}^n w_{ij}} \frac{\sum_{i=1}^n \sum_{j=1}^n w_{ij}\big(y(s_i) - y(s_j)\big)^2}{\sum_{i=1}^n \big(y(s_i) - \bar{y}\big)^2}$$

where $\mathbf{w}$ is a spatial weights matrix.

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where $\mathbf{w}$ is a spatial weights matrix.

Some properties of Geary's C:

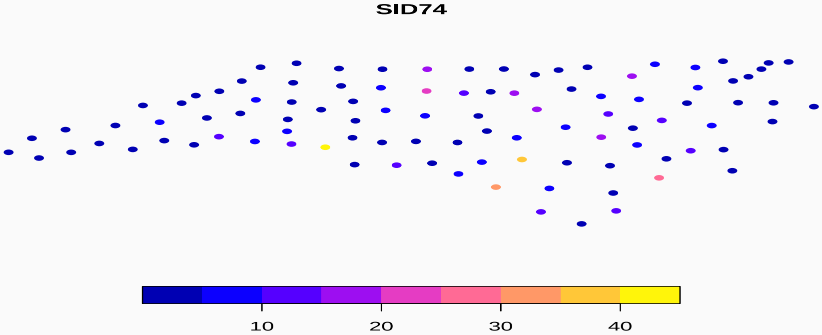
- $0 < C < 2$
  - If $C \approx 1$ then no spatial autocorrelation
  - If $C > 1$ then negative spatial autocorrelation
  - If $C < 1$ then positive spatial autocorrelation
- Geary's C is inversely related to Moran's I

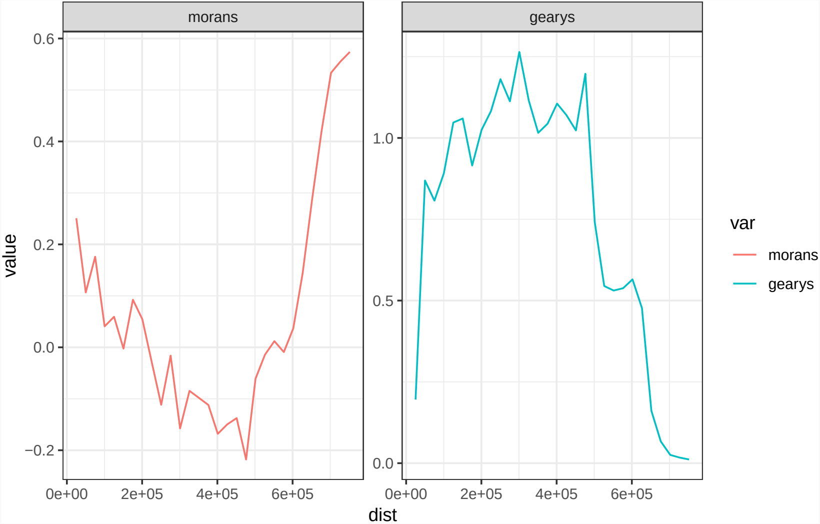
Again using an normalized adjacency matrix for $\mathbf{w}$ (shared county borders).

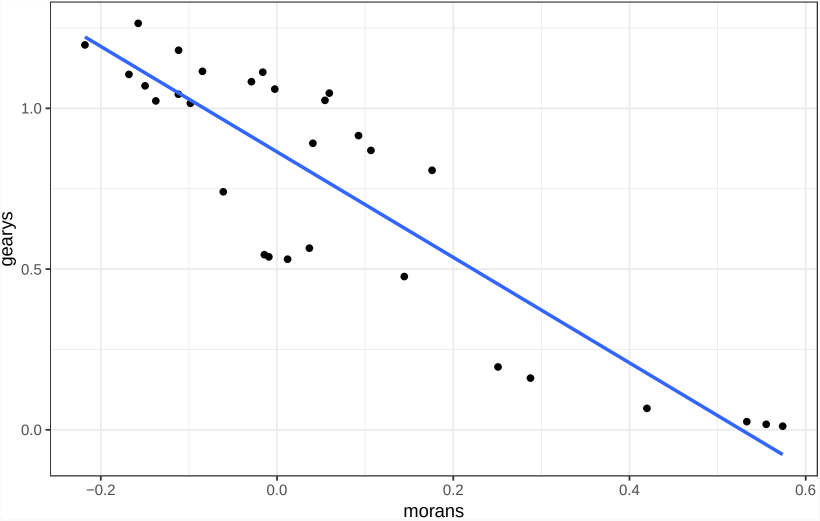
```
gearys_C = function(y, w) {
  w = normalize_weights(w)

  n = length(y)
  y_bar = mean(y)
  y_i = y %*% t(rep(1,n))
  y_j = t(y_i)
  num = sum(w * (y_i-y_j)^2)
  denom = sum( (y-y_bar)^2 )
  ((n-1)/(2*sum(w))) * (num/denom)
}

w = 1*st_touches(nc, sparse=FALSE)

gearys_C(y = nc$SID74, w = w)
## [1] 0.8438767
```

Spatial Correlogram

```
nc_pt = st_centroid(nc)
plot(nc_pt[,"SID74"], pch=16)
```




morans
gearys
value
dist
0.6
0.4
0.2
0.0
−0.2
1.0
0.5
0.0
0e+00
2e+05
4e+05
6e+05
var
morans
gearys



1.0
0.5
0.0
gearys
−0.2
0.0
0.2
0.4
0.6
morans

Autoregressive Models

Lets just focus on the simplest case, an $AR(1)$ process

$$y_t = \delta + \phi \, y_{t-1} + w_t$$

where $w_t \sim \mathcal{N}(0, \sigma^2)$ and $|\phi| < 1$, then

$$E(y_t) = \frac{\delta}{1-\phi}$$
$$Var(y_t) = \frac{\sigma^2}{1-\phi}$$
$$\rho(h) = \phi^h$$
$$\gamma(h) = \phi^h \frac{\sigma^2}{1-\phi}$$

AR Models - Time - Joint Distribution

Previously we saw that an $AR(1)$ model can be represented using a multivariate normal distribution

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \sim \mathcal{N}\left(\frac{\delta}{1-\phi} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},\ \frac{\sigma^2}{1-\phi} \begin{pmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & 1 & \cdots & \phi^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \cdots & 1 \end{pmatrix} \right)$$

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$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \sim \mathcal{N}\left(\frac{\delta}{1-\phi} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \; \frac{\sigma^2}{1-\phi} \begin{pmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & 1 & \cdots & \phi^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \cdots & 1 \end{pmatrix} \right)$$

In writing down the likelihood we also saw that an $AR(1)$ is 1st order Markovian,

$$\begin{aligned} f(y_1, \ldots, y_n) &= f(y_1)\, f(y_2|y_1)\, f(y_3|y_2, y_1) \, \cdots \, f(y_n|y_{n-1}, y_{n-2}, \ldots, y_1) \\ &= f(y_1)\, f(y_2|y_1)\, f(y_3|y_2) \, \cdots \, f(y_n|y_{n-1}) \end{aligned}$$

Alternative Definitions for y_t

$$y_t = \delta + \phi\, y_{t-1} + w_t$$

vs.

$$y_t | y_{t-1} \sim \mathcal{N}(\delta + \phi\, y_{t-1},\ \sigma^2)$$

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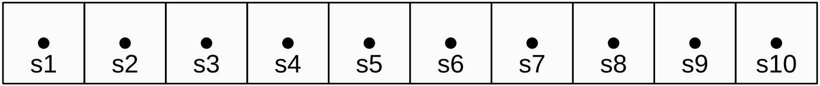
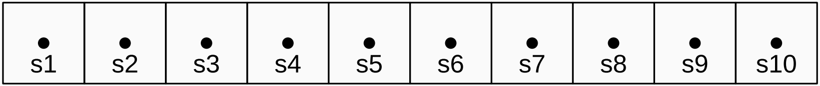
In the case of time, both of these definitions result in the same multivariate distribution for $\mathbf{y}$.

AR in Space

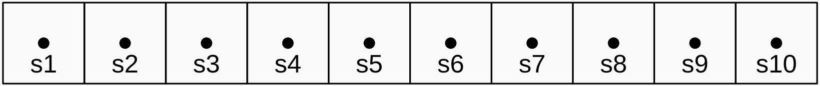




Even in the simplest spatial case there is no clear / unique ordering,

$$
\begin{aligned}
f(y(s_1), \dots, y(s_{10})) &= f(y(s_1))\, f(y(s_2)|y(s_1)) \,\cdots\, f(y(s_{10}|y(s_9), y(s_8), \dots, y(s_1)) \\
&= f(y(s_{10}))\, f(y(s_9)|y(s_{10})) \,\cdots\, f(y(s_1|y(s_2), y(s_3), \dots, y(s_{10})) \\
&= \; ?
\end{aligned}
$$

AR in Space



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&= f(y(s_{10}))\, f(y(s_9)|y(s_{10})) \cdots f(y(s_1|y(s_2), y(s_3), \dots, y(s_{10})) \\
&= \,?
\end{aligned}
$$

Instead we need to think about things in terms of their neighbors / neighborhoods. We define $N(s_i)$ to be the set of neighbors of location s_i.

- If we define the neighborhood based on “touching” then $N(s_3) = \{s_2, s_4\}$
- If we use distance within 2 units then $N(s_3) = \{s_1, s_2, s_3, s_4\}$

Defining the Spatial AR model

Here we will consider a simple average of neighboring observations, just like with the temporal AR model we have two options in terms of defining the autoregressive process,

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- Simultaneous Autogressve (SAR)

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$$y(s) = \delta + \phi \frac{1}{|N(s)|} \sum_{s' \in N(s)} y(s') + \mathcal{N}(0, \sigma^2)$$

- Conditional Autoregressive (CAR)

$$y(s)|\mathbf{y}(-s) \sim \mathcal{N}\left(\delta + \phi \frac{1}{|N(s)|} \sum_{s' \in N(s)} y(s'),\ \sigma^2\right)$$

Simultaneous Autogressve (SAR)

Using

$$y(s) = \phi \frac{1}{|N(s)|} \sum_{s' \in N(s)} y(s') + \mathcal{N}(0, \sigma^2)$$

we want to find the distribution of $\mathbf{y} = \Big(y(s_1),\, y(s_2),\, \ldots,\, y(s_n)\Big)^t$.

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we want to find the distribution of $\mathbf{y} = \Big(y(s_1),\, y(s_2),\, \ldots,\, y(s_n)\Big)^t$.

First we can define a weight matrix $\mathbf{W}$ where

$$\{\mathbf{W}\}_{ij} = \begin{cases} 1/|N(s_i)| & \text{if } j \in N(s_i) \\ 0 & \text{otherwise} \end{cases}$$

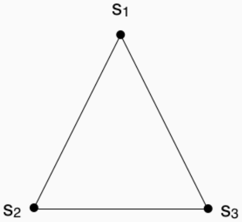
then we can write $\mathbf{y}$ as follows,

$$\mathbf{y} = \phi\, \mathbf{W}\, \mathbf{y} + \boldsymbol{\epsilon}$$

where

$$\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma^2\, \mathbf{I})$$

A toy example



Back to SAR

$$\mathbf{y} = \phi \, \mathbf{W} \, \mathbf{y} + \boldsymbol{\epsilon}$$

Conditional Autogressve (CAR)

This is a bit trickier, in the case of the temporal AR process we actually went from joint distribution $\rightarrow$ conditional distributions (which we were then able to simplify).

Since we don't have a natural ordering we can't get away with this (at least not easily).

Going the other way, conditional distributions $\rightarrow$ joint distribution is difficult because it is possible to specify conditional distributions that lead to an improper joint distribution.

Brooks' Lemma

For sets of observations $\mathbf{x}$ and $\mathbf{y}$ where $p(x) > 0 \;\; \forall\, x \in \mathbf{x}$ and $p(y) > 0 \;\; \forall\, y \in \mathbf{y}$ then

$$\begin{aligned}\frac{p(\mathbf{y})}{p(\mathbf{x})} &= \prod_{i=1}^{n} \frac{p(y_i \mid y_1, \dots, y_{i-1}, x_{i+1}, \dots, x_n)}{p(x_i \mid y_1, \dots, y_{i-1}, x_{i+1}, \dots, x_n)} \\ &= \prod_{i=1}^{n} \frac{p(y_i \mid x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_n)}{p(x_i \mid x_1, \dots, x_{i-1}, y_{i+1}, \dots, y_n)}\end{aligned}$$

A simplified example

Let $\mathbf{y} = (y_1, y_2)$ and $\mathbf{x} = (x_1, x_2)$ then we can derive Brook's Lemma for this case,

$$p(y_1, y_2) = p(y_1|y_2)p(y_2)$$

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\begin{aligned}
p(y_1, y_2) &= p(y_1|y_2)p(y_2) \\
&= p(y_1|y_2)\frac{p(y_2|x_1)}{p(x_1|y_2)}p(x_1)
\end{aligned}
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&= \frac{p(y_1|y_2)}{p(x_1|y_2)}p(y_2|x_1)\,p(x_1)\left(\frac{p(x_2|x_1)}{p(x_2|x_1)}\right)
\end{aligned}$$

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&= \frac{p(y_1|y_2)}{p(x_1|y_2)}p(y_2|x_1)\,p(x_1)\left(\frac{p(x_2|x_1)}{p(x_2|x_1)}\right) \\
&= \frac{p(y_1|y_2)}{p(x_1|y_2)}\frac{p(y_2|x_1)}{p(x_2|x_1)}\,p(x_1, x_2)
\end{aligned}
$$

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&= \frac{p(y_1|y_2)}{p(x_1|y_2)}p(y_2|x_1)\,p(x_1)\left(\frac{p(x_2|x_1)}{p(x_2|x_1)}\right) \\
&= \frac{p(y_1|y_2)}{p(x_1|y_2)}\,\frac{p(y_2|x_1)}{p(x_2|x_1)}\,p(x_1, x_2)
\end{aligned}
$$

$$
\frac{p(y_1, y_2)}{p(x_1, x_2)} = \frac{p(y_1|y_2)}{p(x_1|y_2)}\,\frac{p(y_2|x_1)}{p(x_2|x_1)}
$$

Utility?

Lets repeat that last example but consider the case where $\mathbf{y} = (y_1, y_2)$ but now we let $\mathbf{x} = (y_1 = 0, y_2 = 0)$

$$\frac{p(y_1, y_2)}{p(x_1, x_2)} = \frac{p(y_1, y_2)}{p(y_1 = 0, y_2 = 0)}$$

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$$\frac{p(y_1, y_2)}{p(x_1, x_2)} = \frac{p(y_1, y_2)}{p(y_1 = 0, y_2 = 0)}$$

$$p(y_1, y_2) = \frac{p(y_1|y_2)}{p(y_1 = 0|y_2)} \frac{p(y_2|y_1 = 0)}{p(y_2 = 0|y_1 = 0)} \, p(y_1 = 0, y_2 = 0)$$

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$$p(y_1, y_2) = \frac{p(y_1|y_2)}{p(y_1 = 0|y_2)} \frac{p(y_2|y_1 = 0)}{p(y_2 = 0|y_1 = 0)} p(y_1 = 0, y_2 = 0)$$

$$\begin{aligned} p(y_1, y_2) &\propto \frac{p(y_1|y_2)\, p(y_2|y_1 = 0)}{p(y_1 = 0|y_2)} \\ &\propto \frac{p(y_2|y_1)\, p(y_1|y_2 = 0)}{p(y_2 = 0|y_1)} \end{aligned}$$

As applied to a simple CAR

$$y(s_1)|y(s_2) \sim \mathcal{N}(\phi W_{12}\, y(s_2), \sigma^2)$$
$$y(s_2)|y(s_1) \sim \mathcal{N}(\phi W_{21}\, y(s_1), \sigma^2)$$

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$$y(s_2)|y(s_1) \sim \mathcal{N}(\phi W_{21}\, y(s_1), \sigma^2)$$

$$p(y(s_1), y(s_2)) \propto \frac{p(y(s_1)|y(s_2))\; p(y(s_2)|y(s_1) = 0)}{p(y(s_1) = 0|y(s_2))}$$

As applied to a simple CAR

$$y(s_1)|y(s_2) \sim \mathcal{N}(\phi W_{12}\, y(s_2), \sigma^2)$$
$$y(s_2)|y(s_1) \sim \mathcal{N}(\phi W_{21}\, y(s_1), \sigma^2)$$

$$\begin{aligned} p(y(s_1), y(s_2)) &\propto \frac{p(y(s_1)|y(s_2))\; p(y(s_2)|y(s_1) = 0)}{p(y(s_1) = 0|y(s_2))} \\ &\propto \frac{\exp\left(-\frac{1}{2\sigma^2}\left(y(s_1) - \phi\, W_{12}\, y(s_2)\right)^2\right) \exp\left(-\frac{1}{2\sigma^2}\left(y(s_2) - \phi\, W_{21}\, 0\right)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(0 - \phi W_{12} y(s_2)\right)^2\right)} \end{aligned}$$

As applied to a simple CAR

$$y(s_1)|y(s_2) \sim \mathcal{N}(\phi W_{12}\, y(s_2), \sigma^2)$$
$$y(s_2)|y(s_1) \sim \mathcal{N}(\phi W_{21}\, y(s_1), \sigma^2)$$

$$
\begin{aligned}
p(y(s_1), y(s_2)) &\propto \frac{p(y(s_1)|y(s_2))\, p(y(s_2)|y(s_1)=0)}{p(y(s_1)=0|y(s_2))} \\
&\propto \frac{\exp\left(-\frac{1}{2\sigma^2}\left(y(s_1) - \phi\, W_{12}\, y(s_2)\right)^2\right)\exp\left(-\frac{1}{2\sigma^2}\left(y(s_2) - \phi\, W_{21}\, 0\right)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(0 - \phi W_{12} y(s_2)\right)^2\right)} \\
&\propto \exp\left(-\frac{1}{2\sigma^2}\left(\left(y(s_1) - \phi\, W_{12}\, y(s_2)\right)^2 + y(s_2)^2 - (\phi W_{21} y(s_2))^2\right)\right)
\end{aligned}
$$

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&\propto \exp\left(-\frac{1}{2\sigma^2}\left((y(s_1) - \phi\, W_{12}\, y(s_2))^2 + y(s_2)^2 - (\phi W_{21} y(s_2))^2\right)\right) \\
&\propto \exp\left(-\frac{1}{2\sigma^2}\left(y(s_1)^2 - \phi\, W_{12}\, y(s_1)\, y(s_2) - \phi\, W_{21}\, y(s_1)\, y(s_2) + y(s_2)^2\right)\right)
\end{aligned}$$

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&\propto \exp\left(-\frac{1}{2\sigma^2}\left(\left(y(s_1)-\phi\, W_{12}\, y(s_2)\right)^2 + y(s_2)^2 - (\phi W_{21} y(s_2))^2\right)\right) \\
&\propto \exp\left(-\frac{1}{2\sigma^2}\left(y(s_1)^2 - \phi\, W_{12}\, y(s_1)\, y(s_2) - \phi\, W_{21}\, y(s_1)\, y(s_2) + y(s_2)^2\right)\right) \\
&\propto \exp\left(-\frac{1}{2\sigma^2}(\mathbf{y}-0)\begin{pmatrix} 1 & -\phi W_{12} \\ -\phi W_{21} & 1 \end{pmatrix}(\mathbf{y}-0)^t\right)
\end{aligned}$$

Implications for $\mathbf{y}$

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we can then conclude that for $\mathbf{y} = (y(s_1),\ y(s_2))^t$,

$$\mathbf{y} \sim \mathcal{N}\left(\mathbf{0},\ \sigma^2(\mathbf{I} - \phi\,\mathbf{W})^{-1}\right)$$

which generalizes for all mean 0 CAR models.

General Proof

Let $\mathbf{y} = (y(s_1), \ldots, y(s_n))$ and $\mathbf{0} = (y(s_1) = 0, \ldots, y(s_n) = 0)$ then by Brook's lemma,

$$\frac{p(\mathbf{y})}{p(\mathbf{0})} = \prod_{i=1}^{n} \frac{p(y_i | y_1, \ldots, y_{i-1}, 0_{i+1}, \ldots, 0_n)}{p(0_i | y_1, \ldots, y_{i-1}, 0_{i+1}, \ldots, 0_n)}$$

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$$= \prod_{i=1}^{n} \frac{\exp\left(-\frac{1}{2\sigma^2}\left(y_i - \phi \sum_{j<i} W_{ij}\, y_j - \phi \sum_{j>i} 0_j\right)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(0_i - \phi \sum_{j<i} W_{ij}\, y_j - \phi \sum_{j>i} 0_j\right)^2\right)}$$

$$= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(y_i - \phi \sum_{j<i} W_{ij}\, y_j\right)^2 + \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(\phi \sum_{j<i} W_{ij}\, y_j\right)^2\right)$$

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&= \prod_{i=1}^{n} \frac{\exp\left(-\frac{1}{2\sigma^2}\left(y_i - \phi \sum_{j<i} W_{ij}\, y_j - \phi \sum_{j>i} 0_j\right)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(0_i - \phi \sum_{j<i} W_{ij}\, y_j - \phi \sum_{j>i} 0_j\right)^2\right)} \\
&= \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \phi \sum_{j<i} W_{ij}\, y_j\right)^2 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\phi \sum_{j<i} W_{ij}\, y_j\right)^2\right) \\
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&= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(y_i - \phi \sum_{j<i} W_{ij}\, y_j\right)^2 + \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(\phi \sum_{j<i} W_{ij}\, y_j\right)^2\right) \\
&= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 - 2\phi y_i \sum_{j<i} W_{ij}\, y_j\right) \\
&= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 - \phi \sum_{i=1}^{n} \sum_{j=1}^{n} y_i\, W_{ij}\, y_j\right) \quad (\text{if } W_{ij} = W_{ji})
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&= \exp\left(-\frac{1}{2\sigma^2} (\mathbf{y} - 0)^t (\mathbf{I} - \phi \mathbf{W})(\mathbf{y} - 0)\right)
\end{aligned}
$$

- Simultaneous Autogressve (SAR)

$$y(s) = \phi \sum_{s'} W_{s\,s'}\, y(s') + \epsilon$$

$$\mathbf{y} \sim \mathcal{N}(0,\ \sigma^2\, ((\mathbf{I} - \phi \mathbf{W})^{-1})((\mathbf{I} - \phi \mathbf{W})^{-1})^t)$$

- Conditional Autoregressive (CAR)

$$y(s)|\mathbf{y}(-s) \sim \mathcal{N}\left(\sum_{s'} W_{s\,s'}\, y(s'),\ \sigma^2\right)$$

$$\mathbf{y} \sim \mathcal{N}(0,\ \sigma^2\, (\mathbf{I} - \phi \mathbf{W})^{-1})$$

Generalization

- Adopting different weight matrices, $\mathbf{W}$
  - Between SAR and CAR model we move to a generic weight matrix definition (beyond average of nearest neighbors)
  - In time we varied p in the $AR(p)$ model, in space we adjust the weight matrix.
  - In general having a symmetric W is helpful, but not required

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  - Between SAR and CAR model we move to a generic weight matrix definition (beyond average of nearest neighbors)
  - In time we varied p in the $AR(p)$ model, in space we adjust the weight matrix.
  - In general having a symmetric W is helpful, but not required
- More complex Variance (beyond $\sigma^2\, I$)
  - σ^2 can be a vector (differences between areal locations)
  - i.e. since areal data tends to be aggregated - adjust variance based on sample size
  - i.e. scale based on the number of neighbors

Some specific generalizations

Generally speaking we will want to work with a scaled / normalized version of the weight matrix,

$$\frac{W_{ij}}{W_{i\cdot}}$$

When W is derived from an adjacency matrix, $\mathbf{A}$, we can express this as

$$\mathbf{W} = \mathbf{D}^{-1}\mathbf{A}$$

where $\mathbf{D}^{-1} = \text{diag}(1/|N(s_i)|)$.

We can also allow σ^2 to vary between locations, we can define this as $\mathbf{D}_{\sigma^2} = \text{diag}(\sigma_i^2)$ and most often we use

$$\mathbf{D}_{\sigma^2}^{-1} = \text{diag}\left(\frac{\sigma^2}{|N(s_i)|}\right) = \sigma^2\mathbf{D}^{-1}.$$

Revised SAR Model

- Formula Model

$$y(s_i) = X_{i\cdot}\beta + \phi \sum_{j=1}^{n} D_{jj}^{-1} \, A_{ij} \left(y(s_j) - X_{j\cdot}\beta\right) + \epsilon_i$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0},\, \mathbf{D}_{\sigma^2}^{-1}) = \mathcal{N}(\mathbf{0},\, \sigma^2\mathbf{D}^{-1})$$

- Joint Model

Revised SAR Model

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$$\mathbf{y} \sim \mathcal{N}\left(\mathbf{X}\boldsymbol{\beta}, (\mathbf{I} - \phi\mathbf{D}^{-1}\mathbf{A})^{-1}\sigma^2\mathbf{D}^{-1}\left((\mathbf{I} - \phi\mathbf{D}^{-1}\mathbf{W})^{-1}\right)^{t}\right)$$

Revised CAR Model

- Conditional Model

$$y(s_i)|\mathbf{y}_{-s_i} \sim \mathcal{N}\left(X_{i\cdot}\beta + \phi \sum_{j=1}^{n} \frac{W_{ij}}{D_{ii}}\,(y(s_j) - X_{j\cdot}\beta),\; \sigma^2 D_{ii}^{-1}\right)$$

- Joint Model

Revised CAR Model

- Conditional Model

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- Joint Model

$$\mathbf{y} \sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta},\ \Sigma_{CAR})$$

Revised CAR Model

- Conditional Model

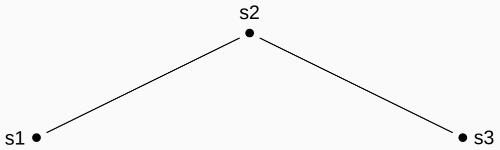
$$y(s_i)|\mathbf{y}_{-s_i} \sim \mathcal{N}\left(X_{i\cdot}\beta + \phi\sum_{j=1}^{n}\frac{W_{ij}}{D_{ii}}\,(y(s_j) - X_{j\cdot}\beta),\; \sigma^2 D_{ii}^{-1}\right)$$

- Joint Model

$$\mathbf{y} \sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta},\; \Sigma_{CAR})$$

$$\begin{aligned}\Sigma_{CAR} &= (\mathbf{D}_\sigma\,(\mathbf{I} - \phi\mathbf{D}^{-1}\mathbf{W}))^{-1}\\ &= (1/\sigma^2\mathbf{D}\,(\mathbf{I} - \phi\mathbf{D}^{-1}\mathbf{W}))^{-1}\\ &= (1/\sigma^2(\mathbf{D} - \phi\mathbf{W}))^{-1}\\ &= \sigma^2(\mathbf{D} - \phi\mathbf{W})^{-1}\end{aligned}$$

Toy CAR Example



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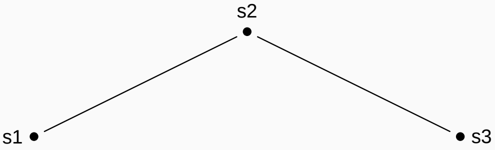

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

. . .

$$\mathbf{\Sigma} = \sigma^2 \left(\mathbf{D} - \phi\,\mathbf{W}\right) = \sigma^2 \begin{pmatrix} 1 & -\phi & 0 \\ -\phi & 2 & -\phi \\ 0 & -\phi & 1 \end{pmatrix}^{-1}$$

When does Σ exist?

```
check_sigma = function(phi) {
  Sigma_inv = matrix(c(1,-phi,0,-phi,2,-phi,0,-phi,1), ncol=3, byrow=TRUE)
  solve(Sigma_inv)
}

check_sigma(phi=0)
##      [,1] [,2] [,3]
## [1,]    1  0.0    0
## [2,]    0  0.5    0
## [3,]    0  0.0    1

check_sigma(phi=0.5)
##           [,1]      [,2]      [,3]
## [1,] 1.1666667 0.3333333 0.1666667
## [2,] 0.3333333 0.6666667 0.3333333
## [3,] 0.1666667 0.3333333 1.1666667

check_sigma(phi=-0.6)
##          [,1]     [,2]     [,3]
## [1,]  1.28125 -0.46875  0.28125
## [2,] -0.46875  0.78125 -0.46875
## [3,]  0.28125 -0.46875  1.28125
```

```r
check_sigma(phi=1)
## Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactl

check_sigma(phi=-1)
## Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactl

check_sigma(phi=1.2)
##            [,1]      [,2]       [,3]
## [1,] -0.6363636 -1.363636 -1.6363636
## [2,] -1.3636364 -1.136364 -1.3636364
## [3,] -1.6363636 -1.363636 -0.6363636

check_sigma(phi=-1.2)
##            [,1]      [,2]       [,3]
## [1,] -0.6363636  1.363636 -1.6363636
## [2,]  1.3636364 -1.136364  1.3636364
## [3,] -1.6363636  1.363636 -0.6363636
```

When is Σ positive semidefinite?

```
check_sigma_pd = function(phi) {
  Sigma_inv = matrix(c(1,-phi,0,-phi,2,-phi,0,-phi,1), ncol=3, byrow=TRUE)
  chol(solve(Sigma_inv))
}

check_sigma_pd(phi=0)
##      [,1]      [,2] [,3]
## [1,]    1 0.0000000    0
## [2,]    0 0.7071068    0
## [3,]    0 0.0000000    1

check_sigma_pd(phi=0.5)
##          [,1]      [,2]      [,3]
## [1,] 1.080123 0.3086067 0.1543033
## [2,] 0.000000 0.7559289 0.3779645
## [3,] 0.000000 0.0000000 1.0000000

check_sigma_pd(phi=-0.6)
##          [,1]       [,2]       [,3]
## [1,] 1.131923 -0.4141182  0.2484709
## [2,] 0.000000  0.7808688 -0.4685213
## [3,] 0.000000  0.0000000  1.0000000
```

```r
check_sigma_pd(phi=1)
## Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactl

check_sigma_pd(phi=-1)
## Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactl

check_sigma_pd(phi=1.2)
## Error in chol.default(solve(Sigma_inv)): the leading minor of order 1 is

check_sigma_pd(phi=-1.2)
## Error in chol.default(solve(Sigma_inv)): the leading minor of order 1 is
```

Generally speaking just like the AR(1) model for time series we require that $|\phi| < 1$ for the CAR model to be proper.

These results for ϕ also apply in the context where σ_i^2 is constant across locations (i.e. $\mathbf{\Sigma} = \left(\sigma^2 \left(\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{W}\right)\right)^{-1}$)

As a side note, the special case where $\phi = 1$ is known as an intrinsic autoregressive (IAR) model and they are popular as an *improper* prior for spatial random effects. An additional sum constraint is necessary for identifiability ($\sum_{i=1}^{n} y(s_i) = 0$).