Lecture 18

Fitting CAR and SAR Models

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Fitting areal models

Revised SAR Model

- Formula Model

$$y(s_i) = X_{i.}\beta + \phi \sum_{j=1}^{n} D_{jj}^{-1} A_{ij} \left(y(s_j) - X_{j.}\beta \right) + \epsilon_i$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0},\, \sigma^2 \mathbf{D}^{-1})$$

- Joint Model

$$\mathbf{y} \sim \mathcal{N}\left(\mathbf{X}\boldsymbol{\beta}, (\mathbf{I} - \phi \mathbf{D}^{-1}\mathbf{A})^{-1} \sigma^2 \mathbf{D}^{-1} \left((\mathbf{I} - \phi \mathbf{D}^{-1}\mathbf{A})^{-1} \right)^t \right)$$

Revised CAR Model

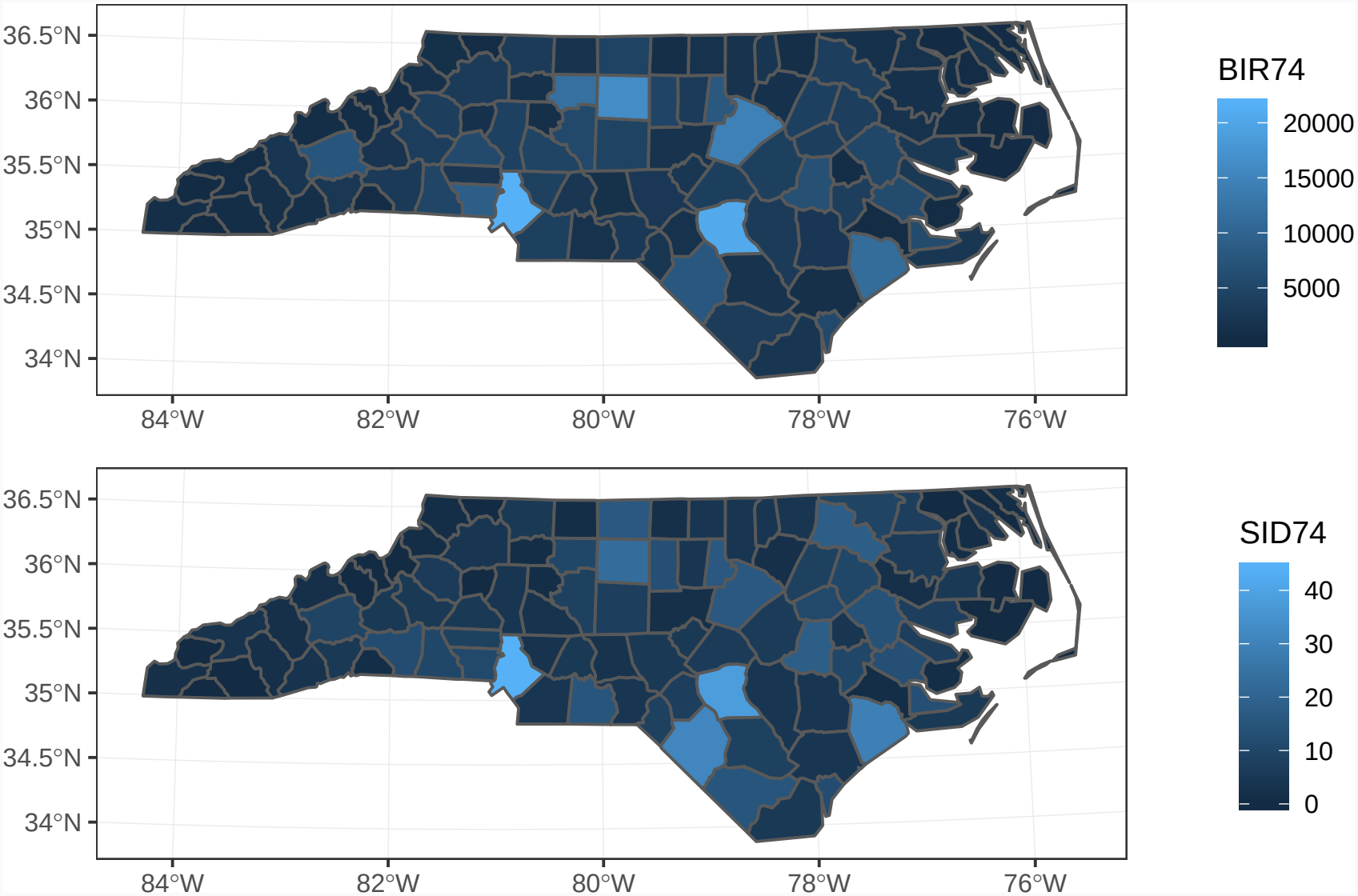
- Conditional Model

$$y(s_i)|\mathbf{y}_{-s_i} \sim \mathcal{N}\left(X_{i\cdot}\beta + \phi \sum_{j=1}^{n} \frac{A_{ij}}{D_{ii}} \left(y(s_j) - X_{j\cdot}\beta\right),\ \sigma^2 D_{ii}^{-1}\right)$$

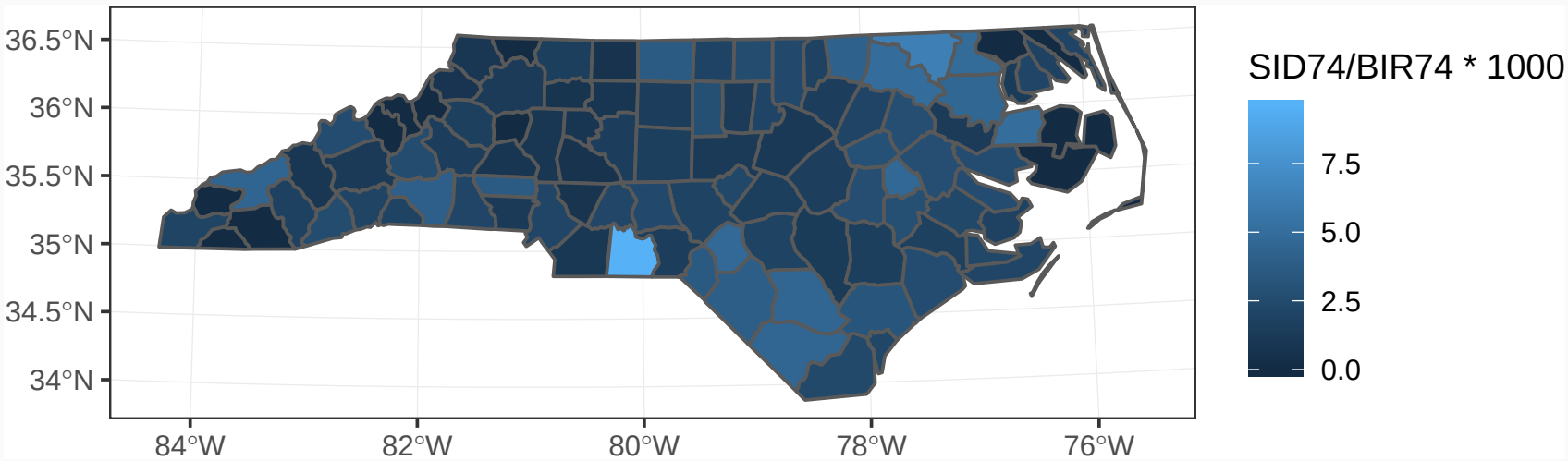
- Joint Model

$$\mathbf{y} \sim \mathcal{N}(\mathbf{X}\boldsymbol{\beta},\ \sigma^2(\mathbf{D} - \phi\mathbf{A})^{-1})$$

Example - NC SIDS



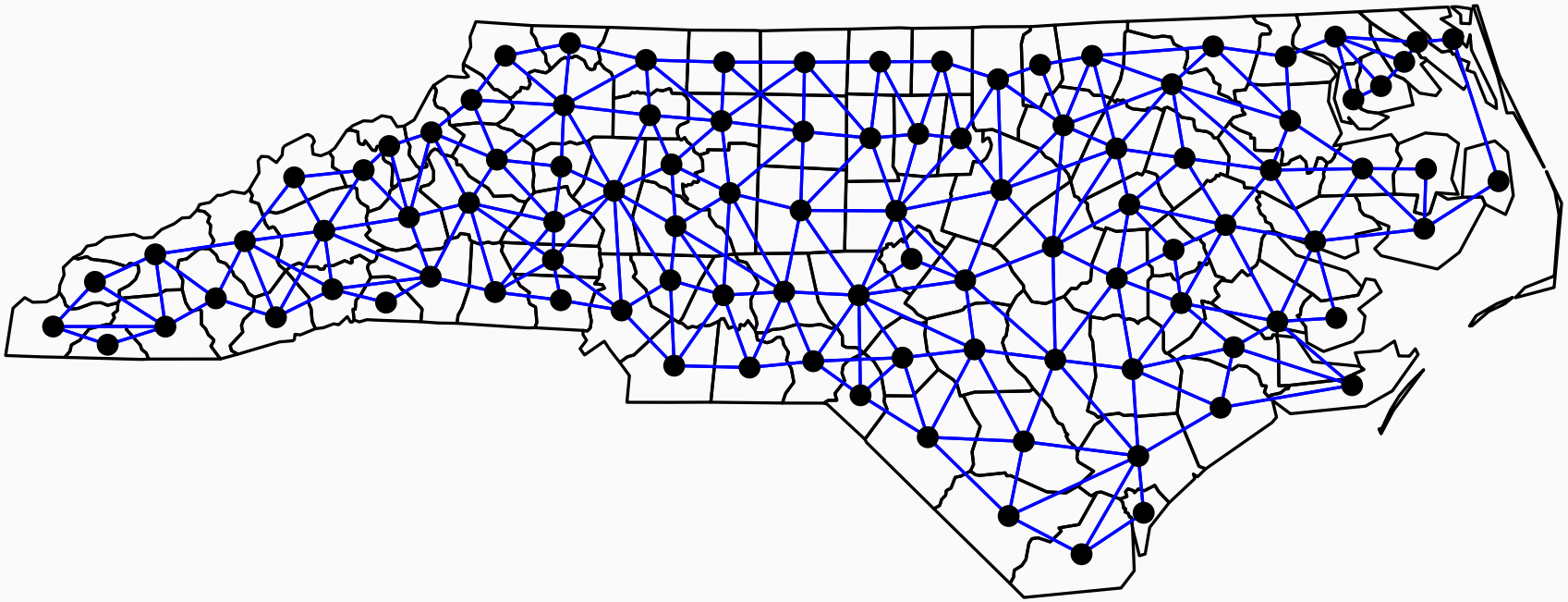
```
ggplot() + geom_sf(data=nc, aes(fill=SID74/BIR74*1000))
```



Using spautolm from spdep

```
library(spdep)

A = st_touches(nc, sparse=FALSE)
listW = mat2listw(A)

listW
## Characteristics of weights list object:
## Neighbour list object:
## Number of regions: 100
## Number of nonzero links: 490
## Percentage nonzero weights: 4.9
## Average number of links: 4.9
##
## Weights style: M
## Weights constants summary:
##     n    nn  S0  S1    S2
## M 100 10000 490 980 10696
```

```
nc_coords = nc %>% st_centroid() %>% st_coordinates()

plot(st_geometry(nc))
plot(listW, nc_coords, add=TRUE, col="blue", pch=16)
```



Moran's I

```
spdep::moran.test(nc$SID74, listW)
##
##  Moran I test under randomisation
##
## data:  nc$SID74
## weights: listW
##
## Moran I statistic standard deviate = 2.1707, p-value = 0.01498
## alternative hypothesis: greater
## sample estimates:
## Moran I statistic       Expectation          Variance
##      0.119089049      -0.010101010       0.003542176
spdep::moran.test(1000*nc$SID74/nc$BIR74, listW)
##
##  Moran I test under randomisation
##
## data:  1000 * nc$SID74/nc$BIR74
## weights: listW
##
## Moran I statistic standard deviate = 3.6355, p-value = 0.0001387
## alternative hypothesis: greater
## sample estimates:
## Moran I statistic       Expectation          Variance
##      0.210046454      -0.010101010       0.003666802
```

Geary's C

```
spdep::geary.test(nc$SID74, listW)
##
##  Geary C test under randomisation
##
## data:  nc$SID74
## weights: listW
##
## Geary C statistic standard deviate = 0.91949, p-value = 0.1789
## alternative hypothesis: Expectation greater than statistic
## sample estimates:
## Geary C statistic       Expectation          Variance
##        0.88988684         1.00000000        0.01434105
spdep::geary.test(nc$SID74/nc$BIR74, listW)
##
##  Geary C test under randomisation
##
## data:  nc$SID74/nc$BIR74
## weights: listW
##
## Geary C statistic standard deviate = 3.0989, p-value = 0.0009711
## alternative hypothesis: Expectation greater than statistic
## sample estimates:
## Geary C statistic       Expectation          Variance
##        0.67796679         1.00000000        0.01079878
```

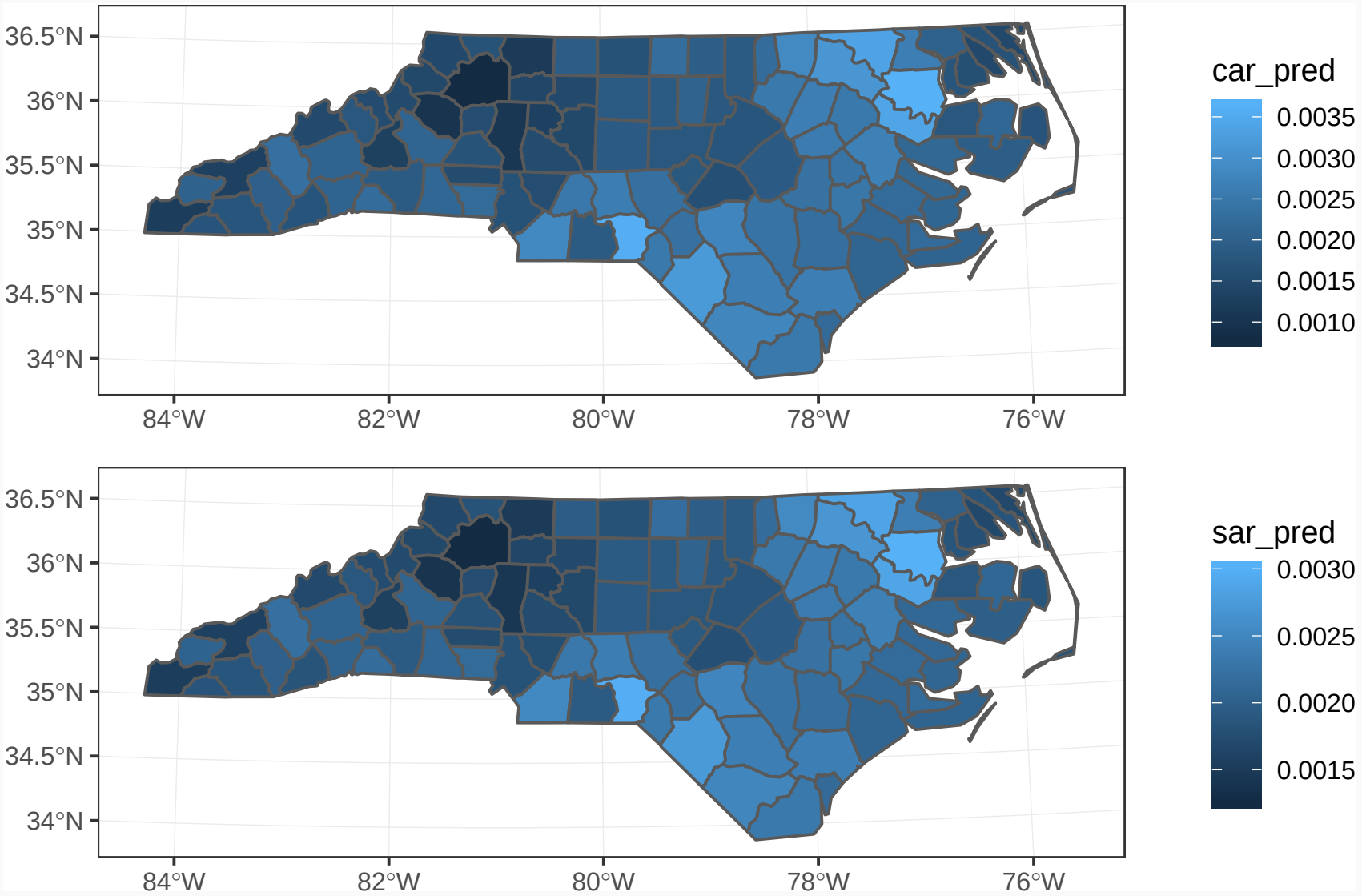
CAR Model

```
nc_car = spautolm(formula = SID74/BIR74 ~ 1, data = nc,
                  listw = listW, family = "CAR")

summary(nc_car)
##
## Call: spautolm(formula = SID74/BIR74 ~ 1, data = nc, listw = listW,
##     family = "CAR")
##
## Residuals:
##         Min          1Q      Median          3Q         Max
## -0.00213872 -0.00083535 -0.00022355  0.00055014  0.00768640
##
## Coefficients:
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept) 0.00200203 0.00024272  8.2484 2.22e-16
##
## Lambda: 0.13062 LR test value: 8.6251 p-value: 0.0033157
## Numerical Hessian standard error of lambda: 0.030472
##
## Log likelihood: 508.3767
## ML residual variance (sigma squared): 2.1205e-06, (sigma: 0.0014562)
## Number of observations: 100
## Number of parameters estimated: 3
## AIC: -1010.8
```

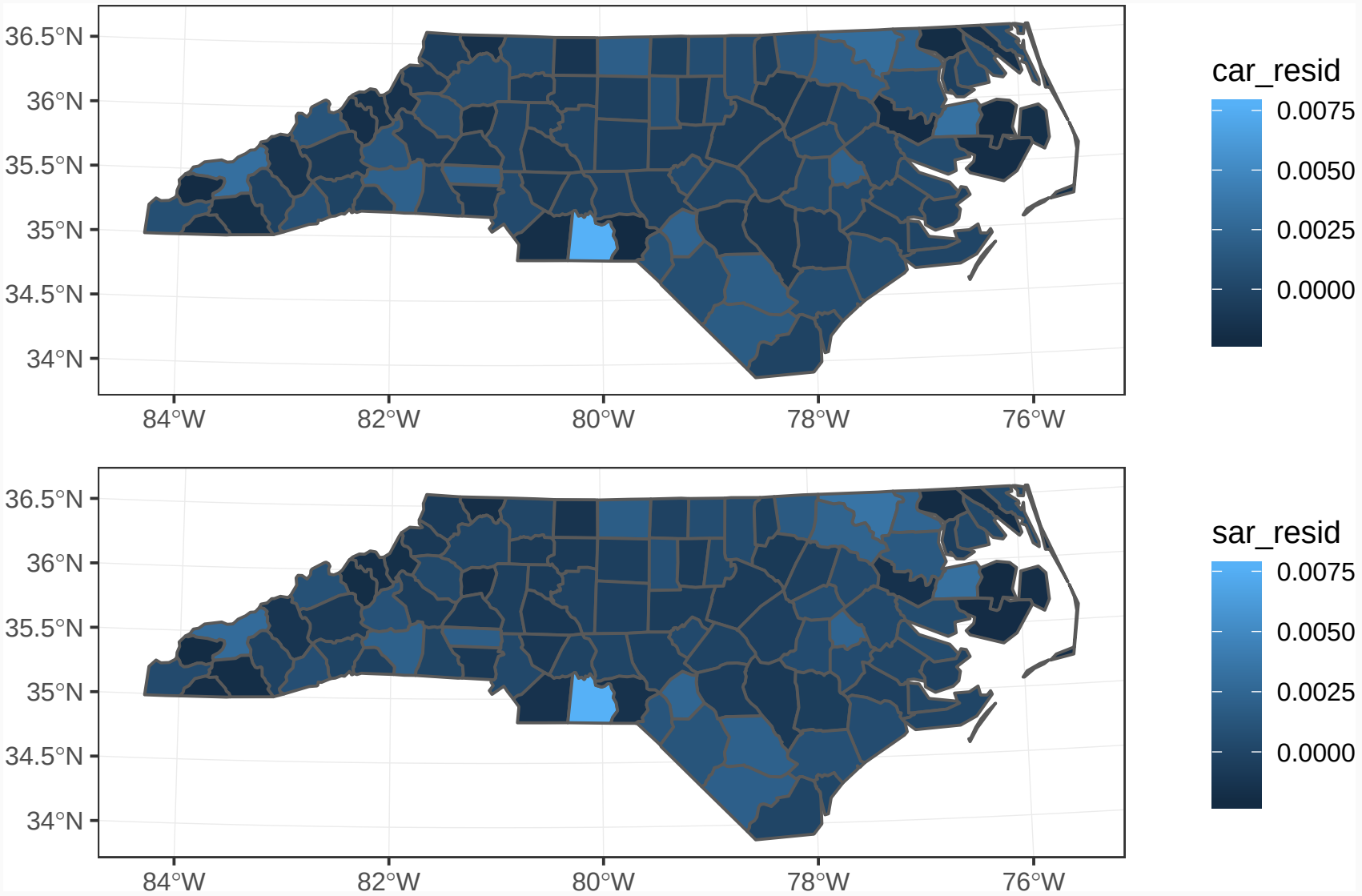
SAR Model

```
nc_sar = spautolm(formula = SID74/BIR74 ~ 1, data = nc,
                  listw = listW, family = "SAR")

summary(nc_sar)
##
## Call: spautolm(formula = SID74/BIR74 ~ 1, data = nc, listw = listW,
##     family = "SAR")
##
## Residuals:
##         Min          1Q      Median          3Q         Max
## -0.00209307 -0.00087039 -0.00020274  0.00051156  0.00762830
##
## Coefficients:
##               Estimate Std. Error z value  Pr(>|z|)
## (Intercept) 0.00201084 0.00023622  8.5127 < 2.2e-16
##
## Lambda: 0.079934 LR test value: 8.8911 p-value: 0.0028657
## Numerical Hessian standard error of lambda: 0.0246
##
## Log likelihood: 508.5097
## ML residual variance (sigma squared): 2.1622e-06, (sigma: 0.0014704)
## Number of observations: 100
## Number of parameters estimated: 3
## AIC: -1011
```

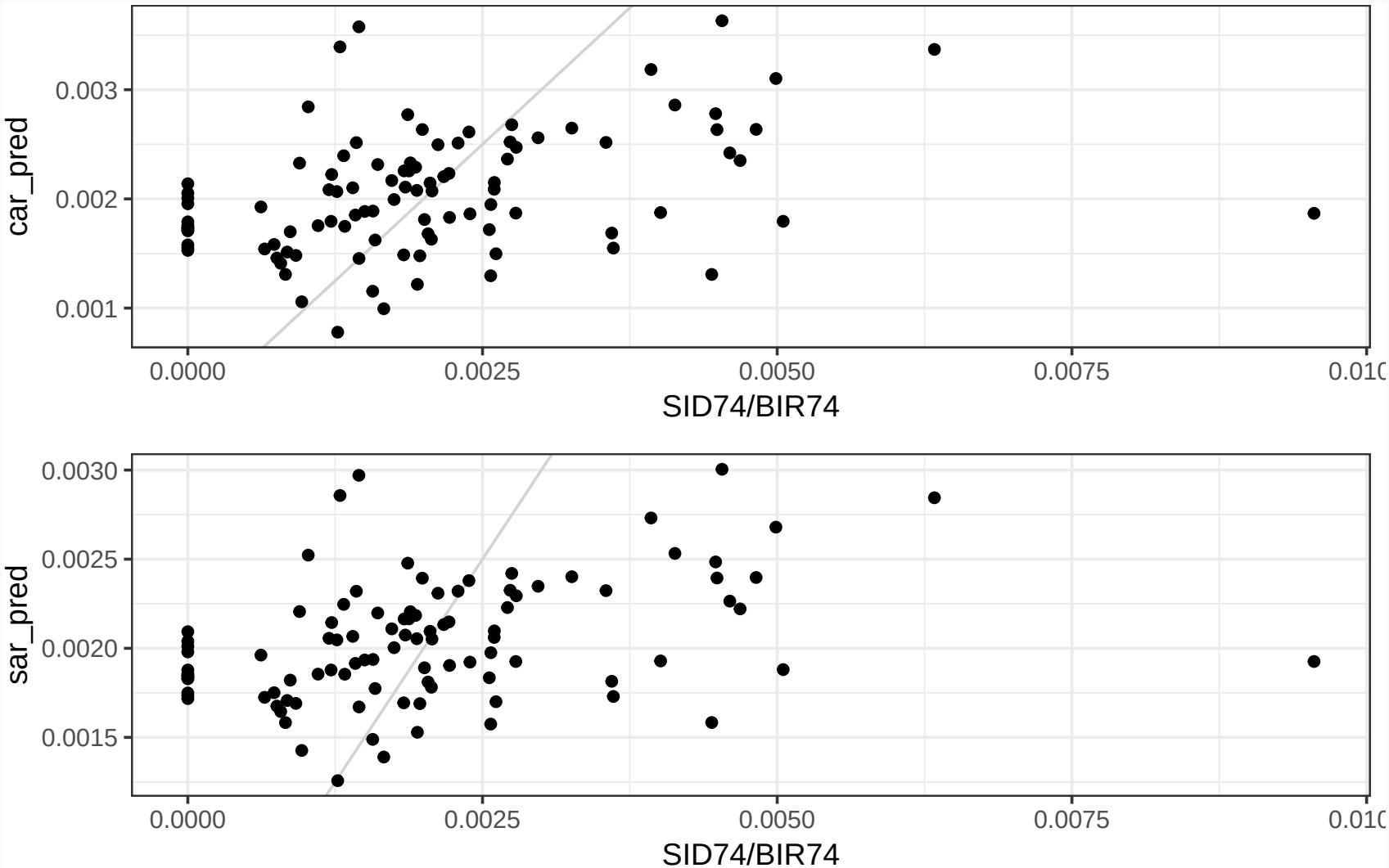
Predictions



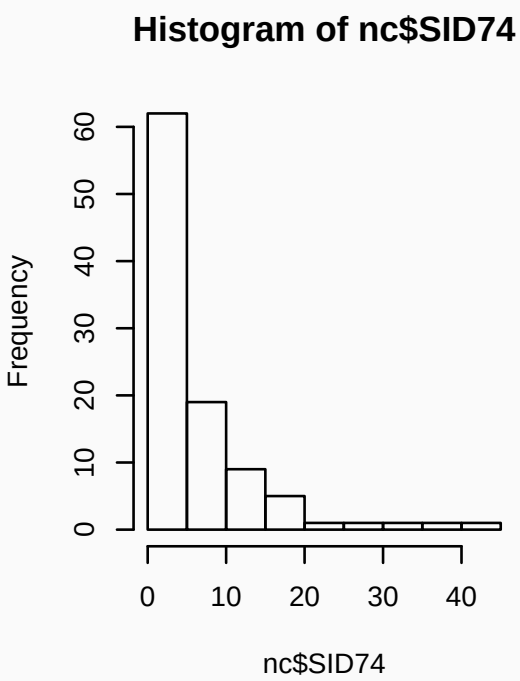
Residuals

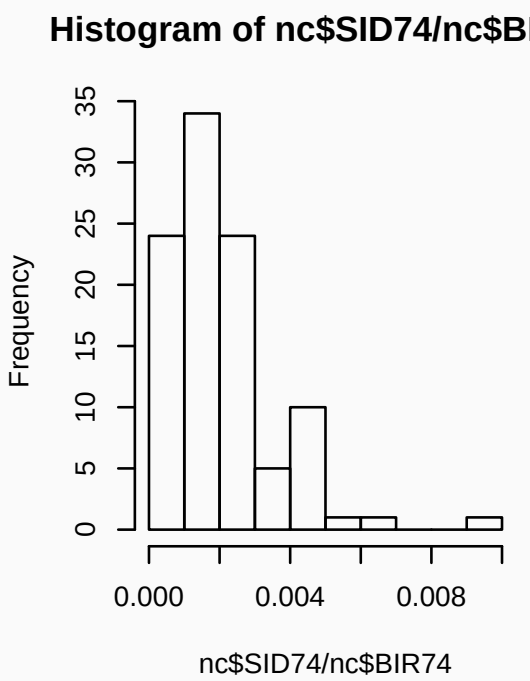

36.5°N
36°N
35.5°N
35°N
34.5°N
34°N
84°W
82°W
80°W
78°W
76°W
car_resid
0.0075
0.0050
0.0025
0.0000
sar_resid
0.0075
0.0050
0.0025
0.0000

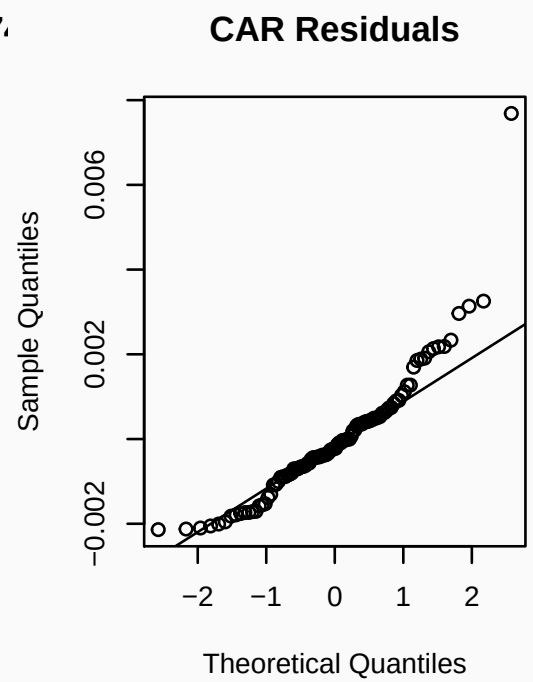
Predicted vs Observed



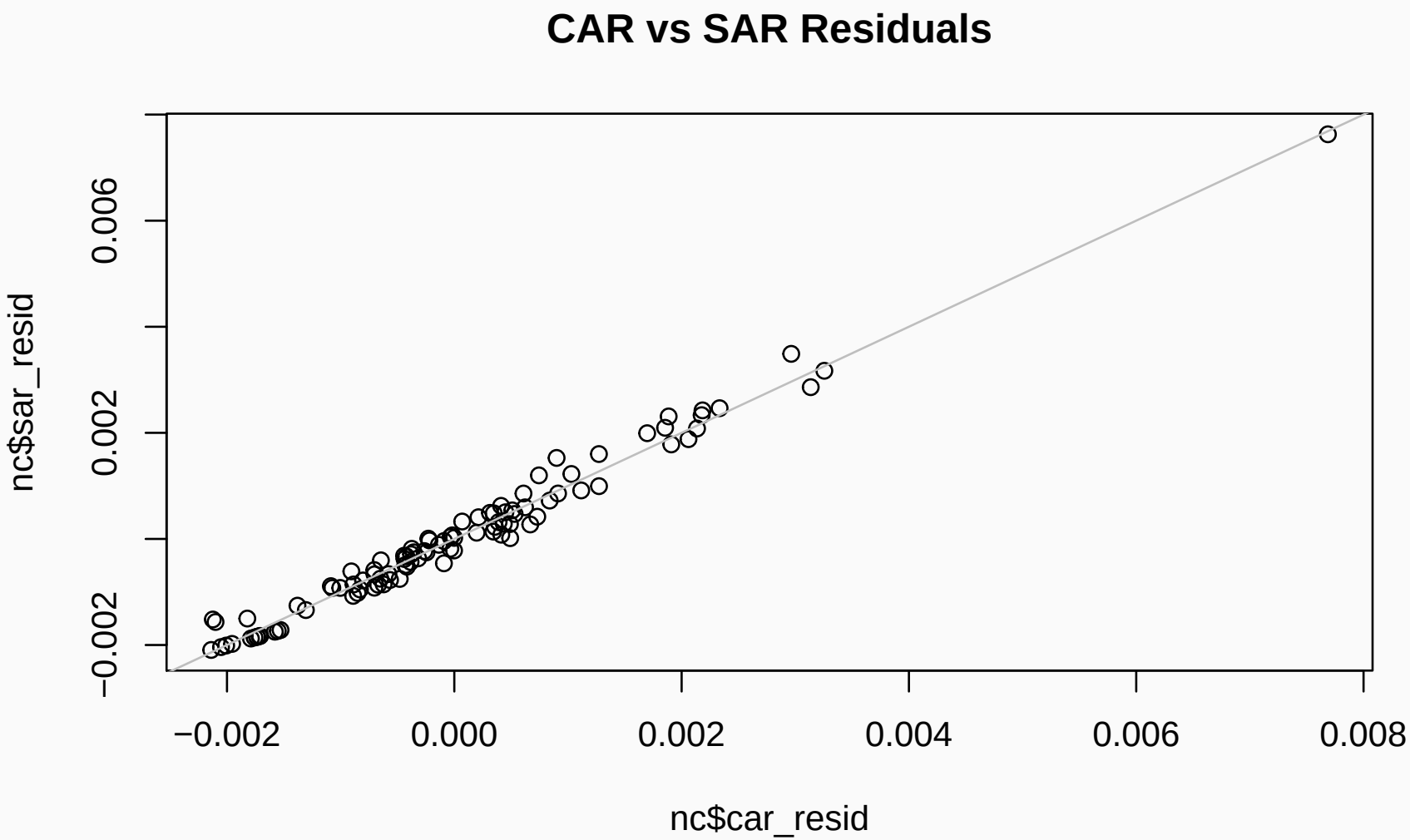
What's wrong?


Histogram of nc$SID74
Frequency
0
10
20
30
40
50
60
0
10
20
30
40
nc$SID74


Histogram of nc$SID74/nc$BIR7
Frequency
0
5
10
15
20
25
30
35
0.000
0.004
0.008
nc$SID74/nc$BIR74


CAR Residuals
Sample Quantiles
−0.002
0.002
0.006
−2
−1
0
1
2
Theoretical Quantiles

Comparing CAR vs SAR.



Stan CAR Model

```
car_model = ”
data {
  int<lower=0> N;
  vector[N] y;
  matrix[N,N] A;
  matrix[N,N] D;
}
parameters {
  vector[N] w_s;
  real beta;
  real<lower=0> sigma2;
  real<lower=0> sigma2_w;
  real<lower=0,upper=1> phi;
}
transformed parameters {
  vector[N] y_pred = beta + w_s;
}
model {
  matrix[N,N] Sigma_inv = (D - phi * A) / sigma2;
  w_s ~ multi_normal_prec(rep_vector(0,N), Sigma_inv);

  beta ~ normal(0,10);
  sigma2 ~ cauchy(0,5);
  sigma2_w ~ cauchy(0,5);

  y ~ normal(beta+w_s, sigma2_w);
}
”
```

```r
data = list(
  N = nrow(nc),
  y = nc$SID74 / nc$BIR74,
  A = A * 1,
  D = diag(rowSums(A))
)
```

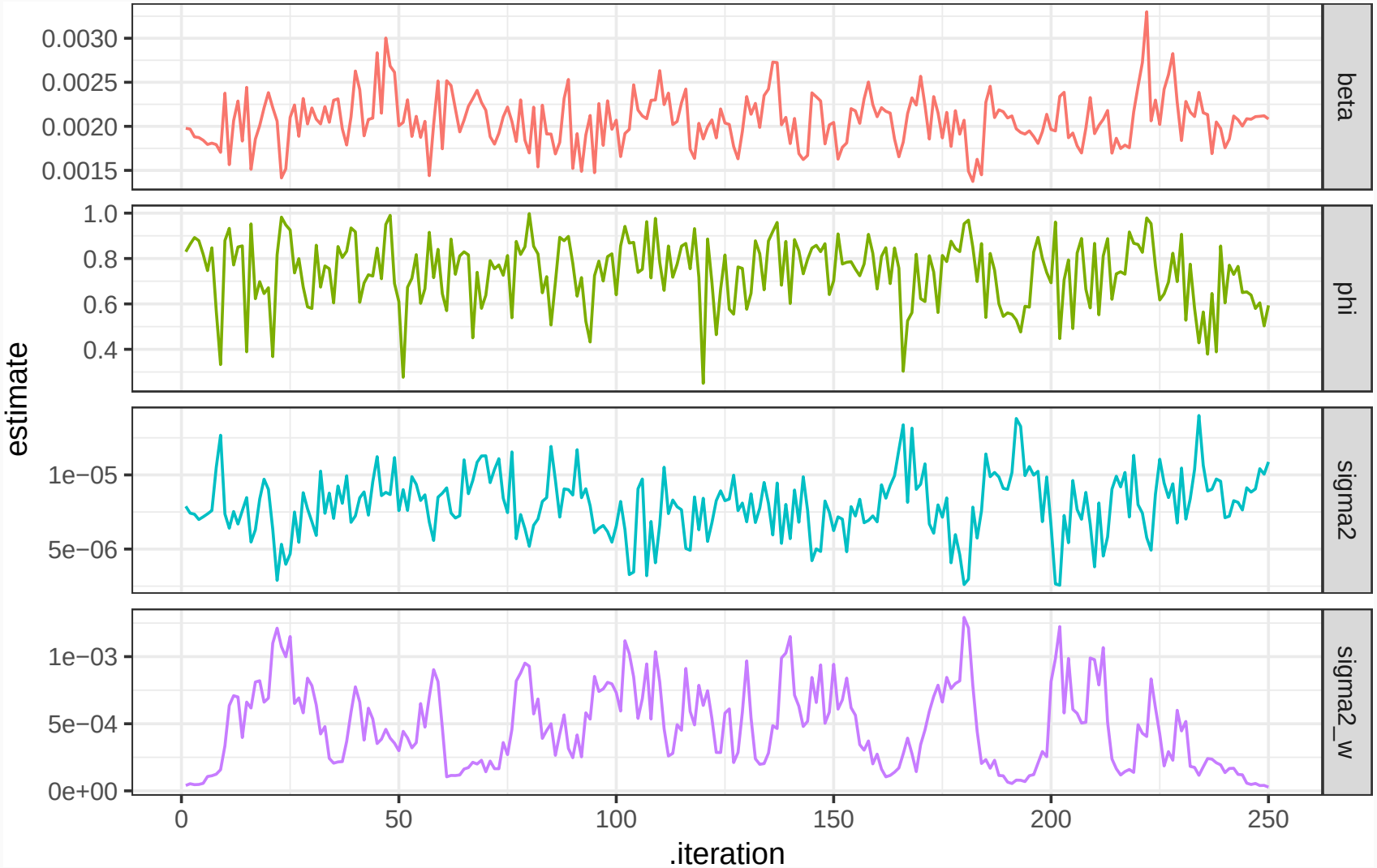
```r
car_fit = rstan::stan(
  model_code = car_model, data = data,
  iter = 10000, chains = 1, thin=20
)
```

```
data = list(
  N = nrow(nc),
  y = nc$SID74 / nc$BIR74,
  A = A * 1,
  D = diag(rowSums(A))
)
```

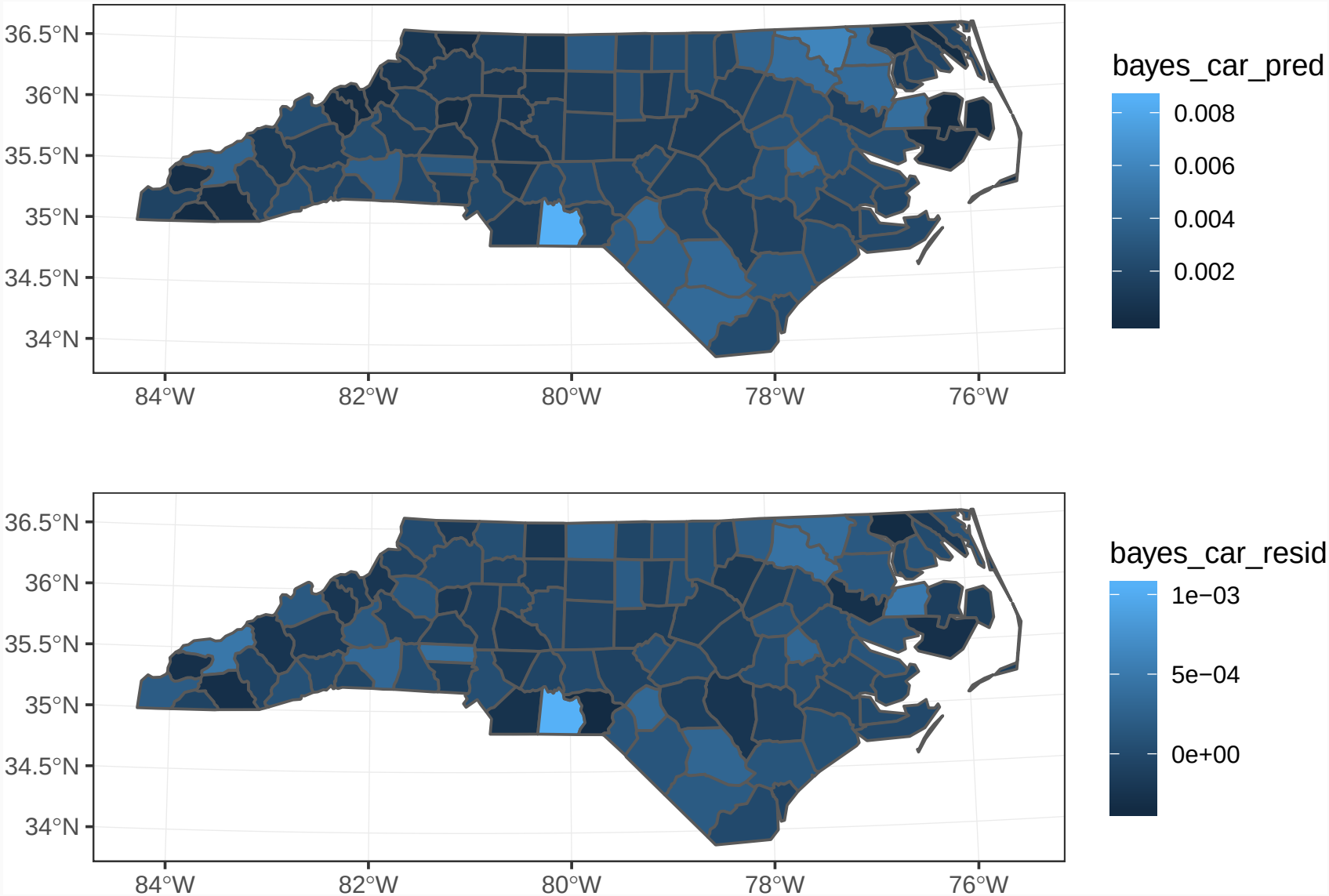
```
car_fit = rstan::stan(
  model_code = car_model, data = data,
  iter = 10000, chains = 1, thin=20
)
```

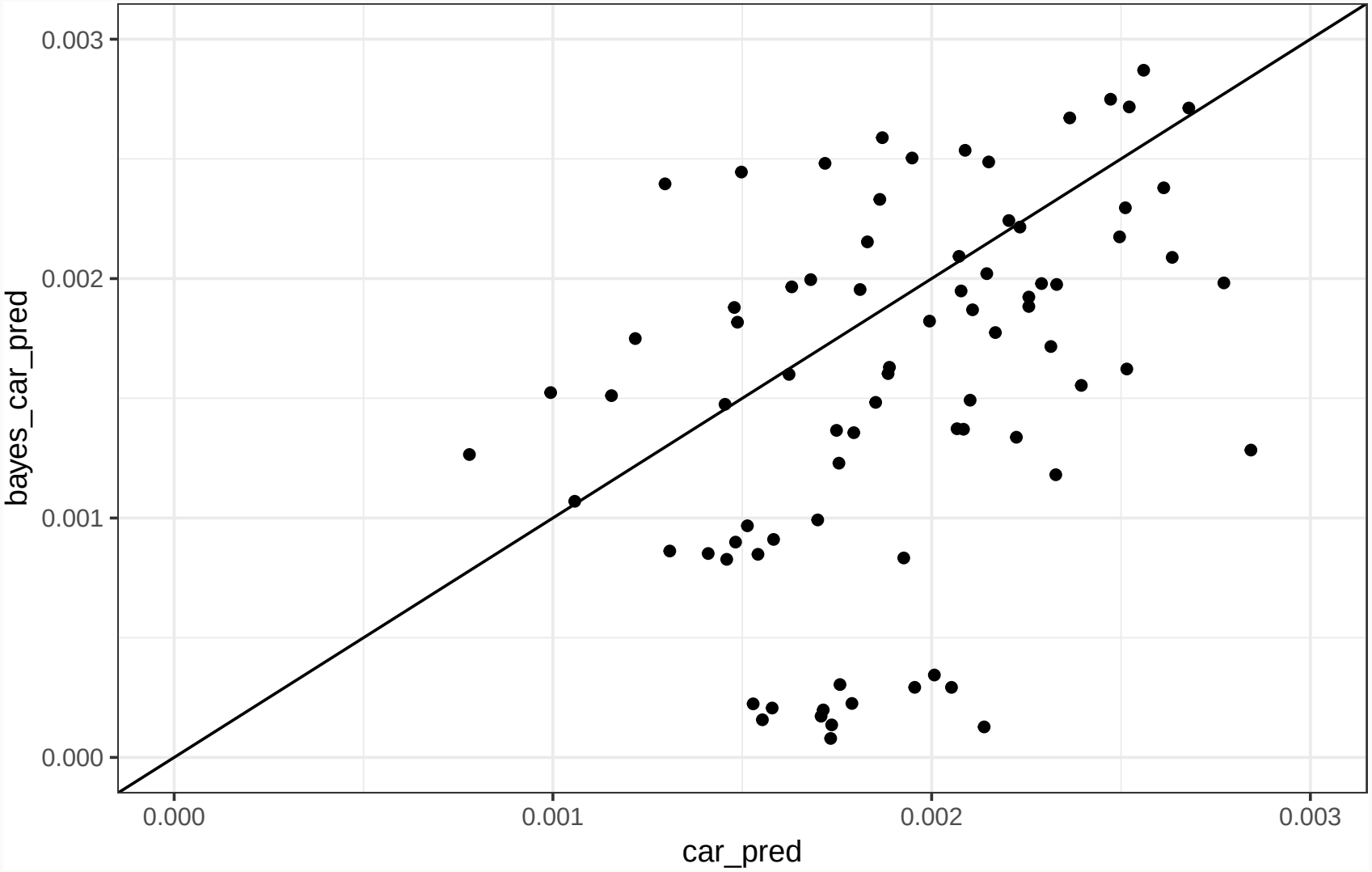
Why don't we use the conditional definition for the y's?

Model Results



Predictions




0.003
0.002
0.001
0.000
bayes_car_pred
0.000
0.001
0.002
0.003
car_pred

Brief Aside - SAR Precision Matrix

$$\Sigma_{SAR} = (\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})^{-1} \sigma^2\, \mathbf{D}^{-1} \left((\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})^{-1}\right)^t$$

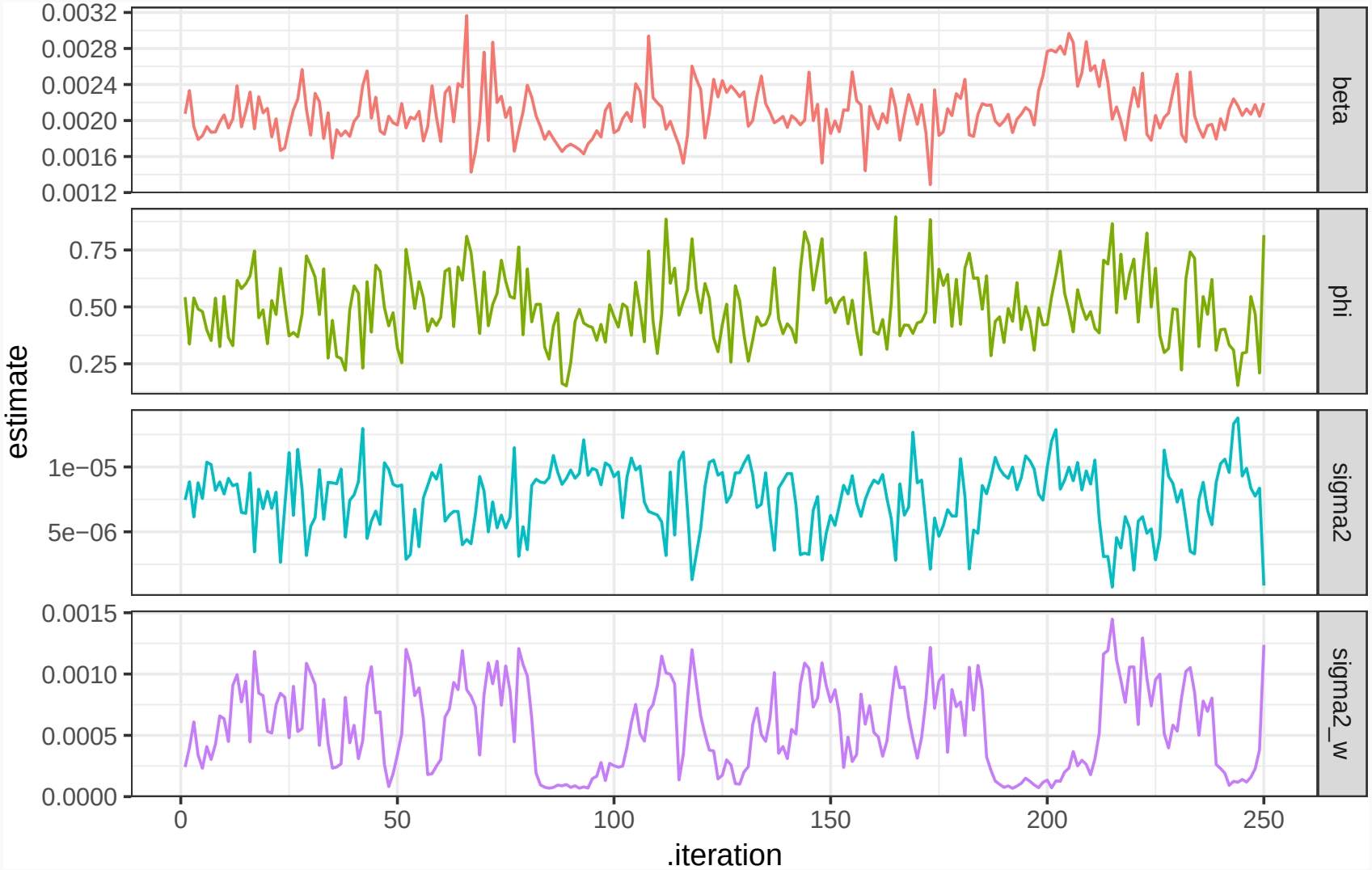
$$\begin{aligned}
\Sigma_{SAR}^{-1} &= \left((\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})^{-1} \sigma^2\, \mathbf{D}^{-1} \left((\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})^{-1}\right)^t\right)^{-1} \\
&= \left(\left((\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})^{-1}\right)^t\right)^{-1} \frac{1}{\sigma^2}\, \mathbf{D}\, (\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A}) \\
&= \frac{1}{\sigma^2}\, (\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})^t\, \mathbf{D}\, (\mathbf{I} - \phi \mathbf{D}^{-1}\,\mathbf{A})
\end{aligned}$$

Jags SAR Model

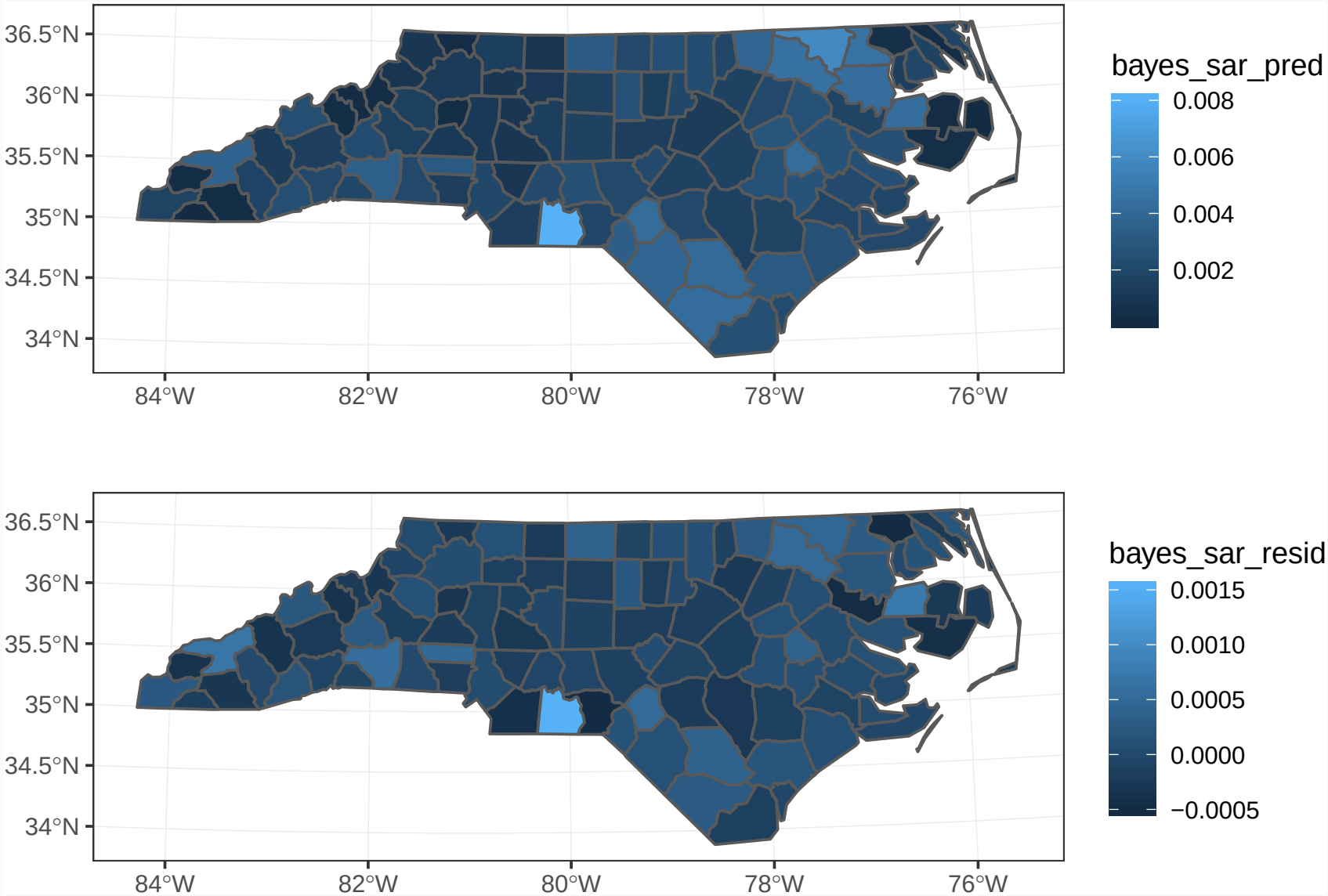
```
sar_model = "
data {
  int<lower=0> N;
  vector[N] y;
  matrix[N,N] W_tilde;
  matrix[N,N] D;
}
transformed data {
  matrix[N,N] I = diag_matrix(rep_vector(1, N));
}
parameters {
  vector[N] w_s;
  real beta;
  real<lower=0> sigma2;
  real<lower=0> sigma2_w;
  real<lower=0,upper=1> phi;
}
transformed parameters {
  vector[N] y_pred = beta + w_s;
}
model {
  matrix[N,N] C = I - phi * W_tilde;
  matrix[N,N] Sigma_inv = C' * D * C / sigma2;

  w_s ~ multi_normal_prec(rep_vector(0,N), Sigma_inv);

  beta ~ normal(0,10);
  sigma2 ~ cauchy(0,5);
  sigma2_w ~ cauchy(0,5);

  y ~ normal(beta + w_s, sigma2_w);
}
"
```

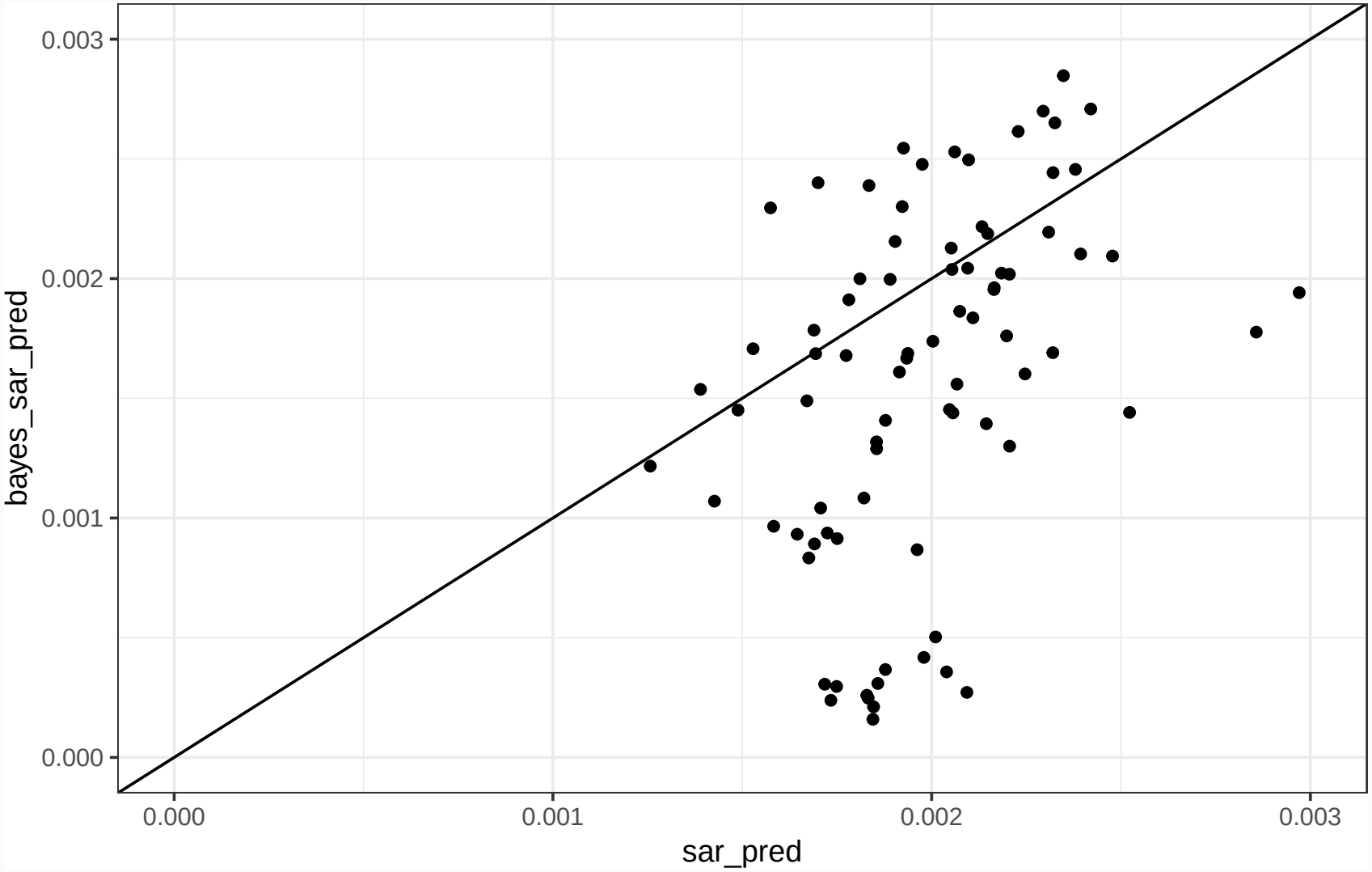
```
D = diag(rowSums(A))
D_inv = diag(1/diag(D))
data = list(
  N = nrow(nc),
  y = nc$SID74 / nc$BIR74,
  x = rep(1, nrow(nc)),
  D_inv = D_inv,
  W_tilde = D_inv %*% A
)

sar_fit = rstan::stan(
  model_code = sar_model, data = data,
  iter = 10000, chains = 1, thin=20
)
```

Model Results



Predictions

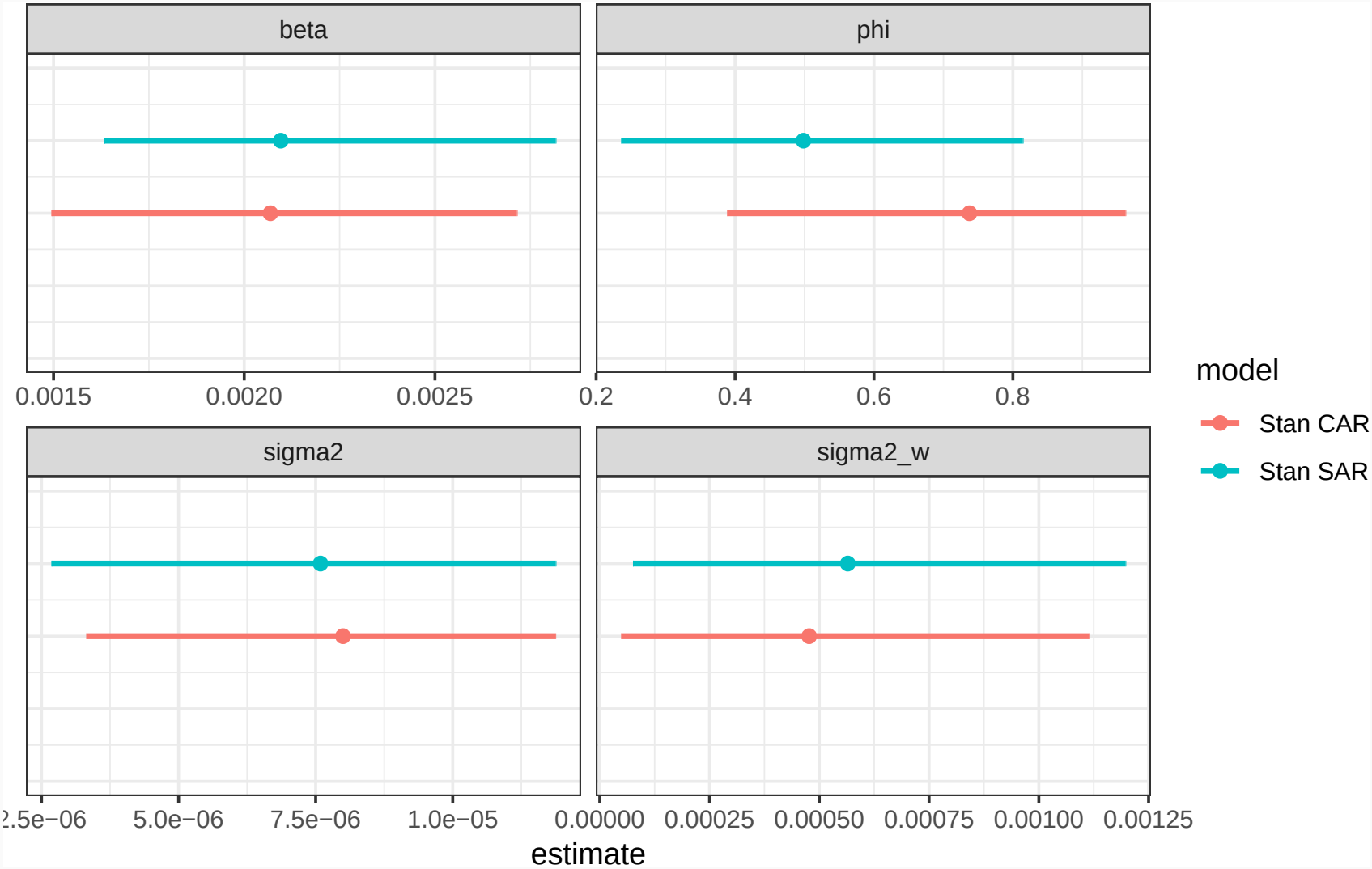



0.003
0.002
0.001
0.000
bayes_sar_pred
0.000
0.001
0.002
0.003
sar_pred

Comparing Predictions

```
# RMSE
sqrt(mean(nc$bayes_car_resid^2))
## [1] 0.0002092447

sqrt(mean(nc$bayes_sar_resid^2))
## [1] 0.0002983034

sqrt(mean(nc$car_resid^2))
## [1] 0.001448564

sqrt(mean(nc$sar_resid^2))
## [1] 0.001470432
```

Comparing Parameters



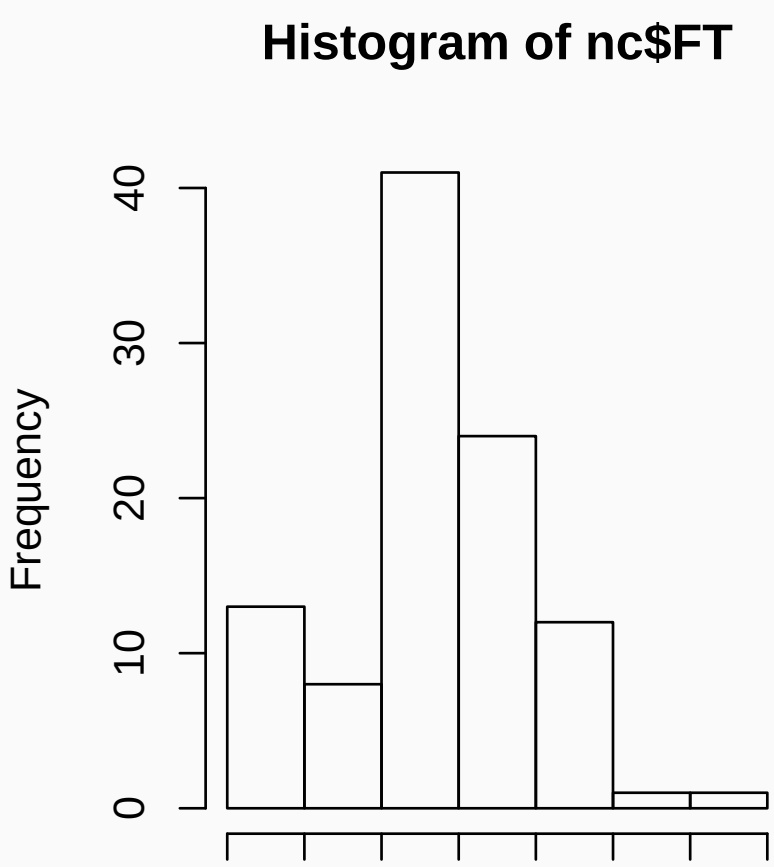
CAR vs SAR?

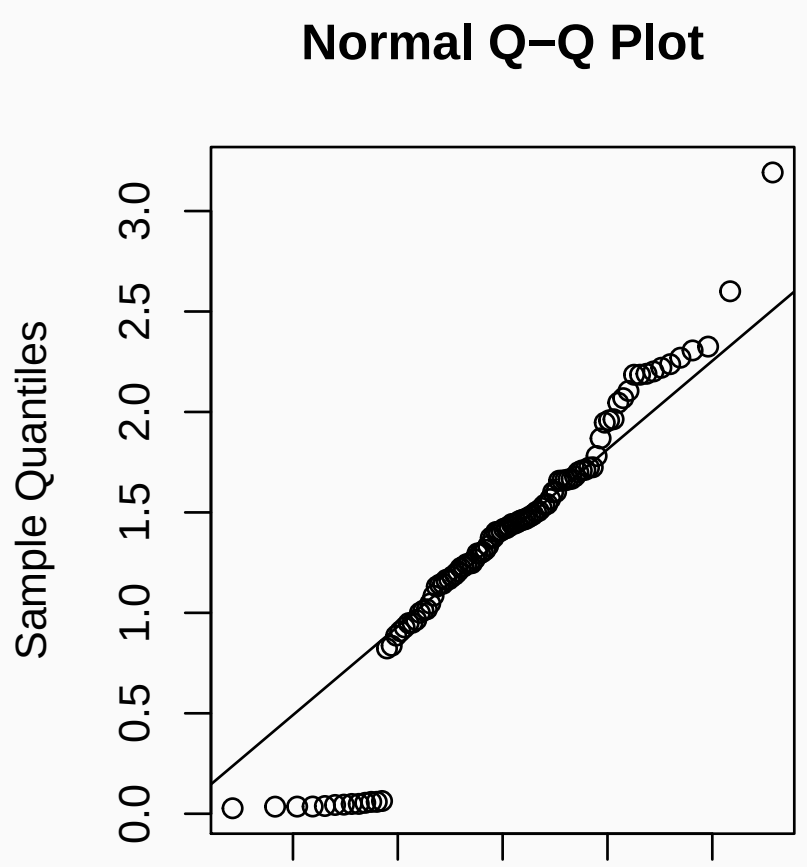
Transforming the data

Freeman-Tukey's transformation

This is the transformation used by Cressie and Road in Spatial Data Analysis of Regional Counts (1989).

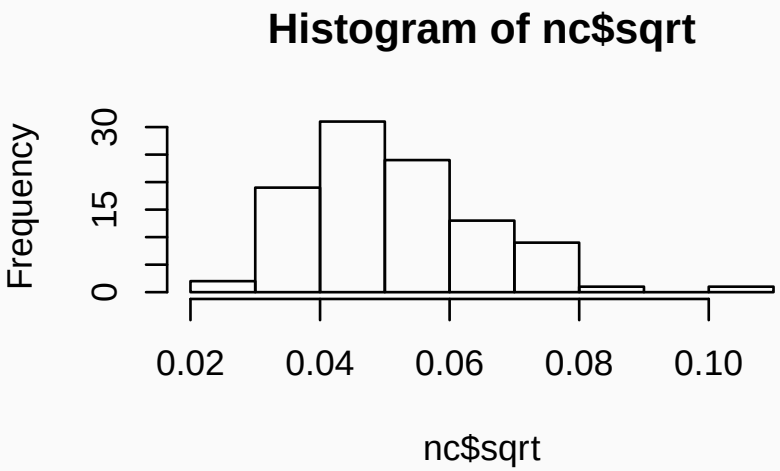
$$FT = \sqrt{1000}\left(\sqrt{\frac{SID74}{BIR74}} + \sqrt{\frac{SID74+1}{BIR74}}\right)$$

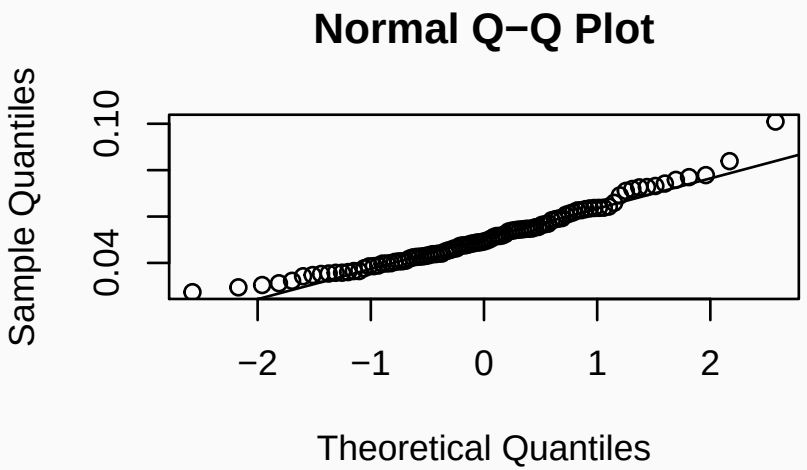


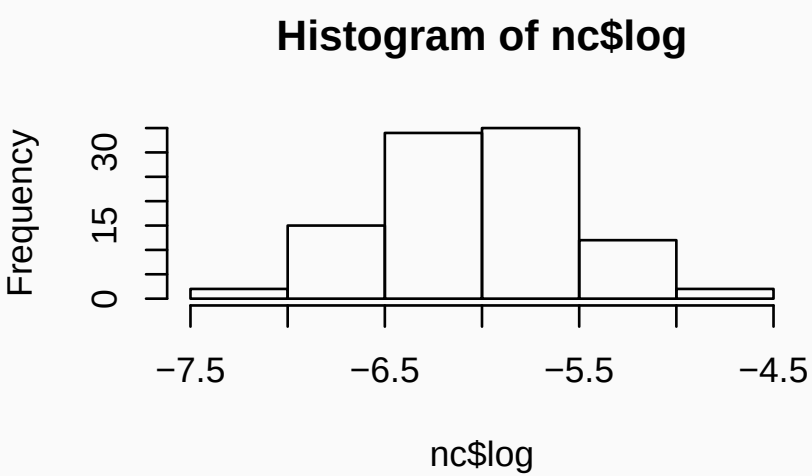


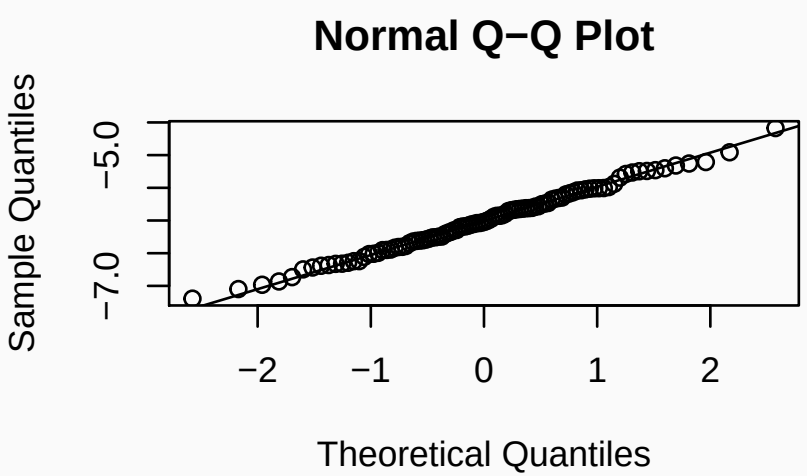
Other options

```
nc = mutate(nc,
  sqrt = sqrt((SID74+1)/BIR74),
  log  = log((SID74+1)/BIR74),
)
```

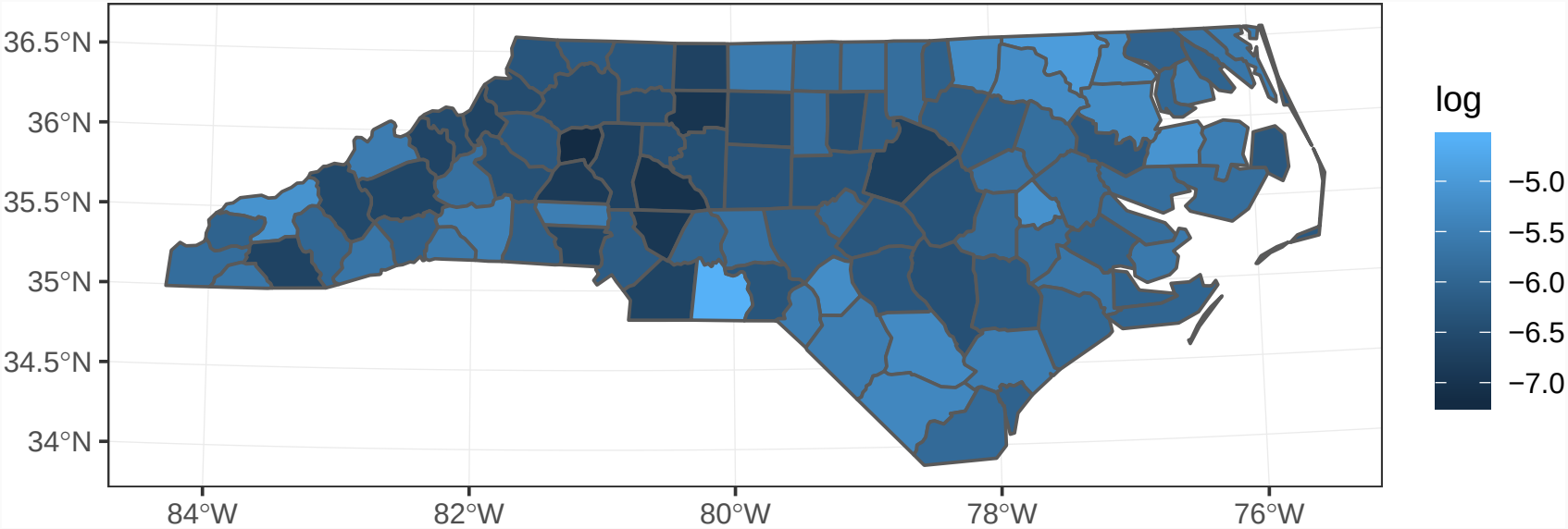








log transformation



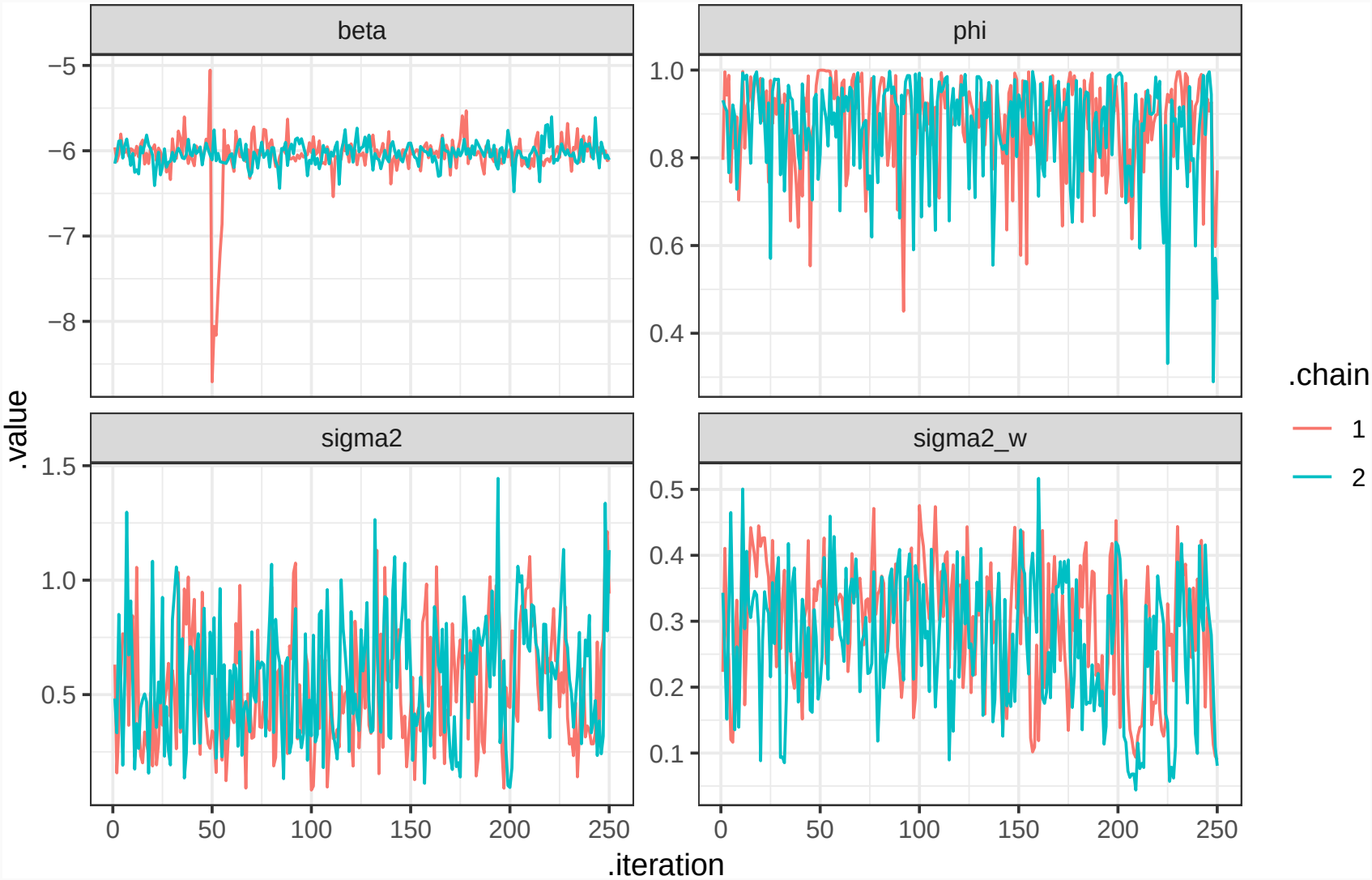

```
##
##  Moran I test under randomisation
##
## data:  nc$log
## weights: listW
##
## Moran I statistic standard deviate = 4.9895, p-value = 3.027e-07
## alternative hypothesis: greater
## sample estimates:
## Moran I statistic       Expectation          Variance
##       0.299245438      -0.010101010       0.003843927
```

log CAR Model

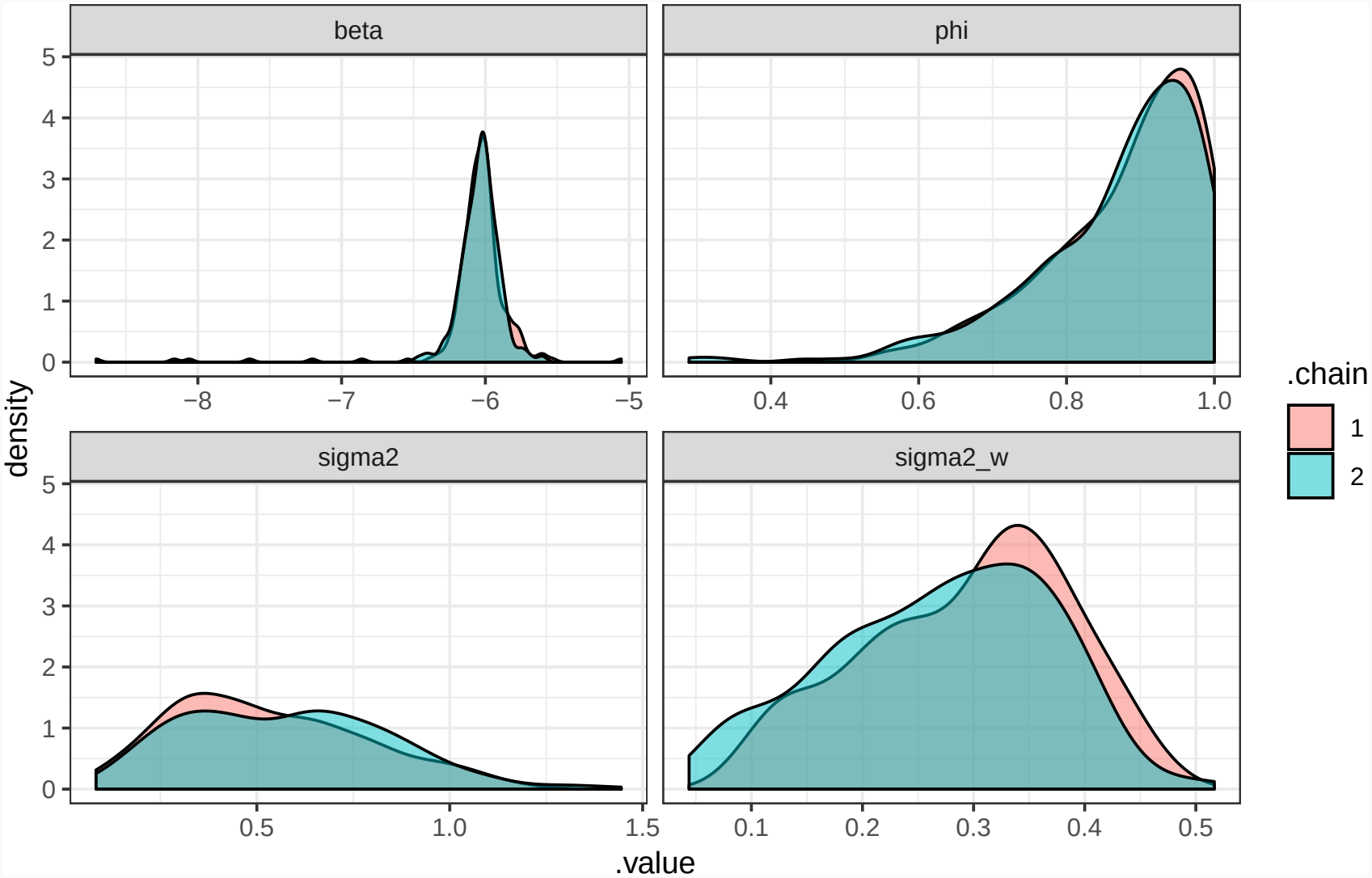
```
car_model = "
data {
  int<lower=0> N;
  vector[N] y;
  matrix[N,N] A;
  matrix[N,N] D;
}
parameters {
  vector[N] w_s;
  real beta;
  real<lower=0> sigma2;
  real<lower=0> sigma2_w;
  real<lower=0,upper=1> phi;
}
transformed parameters {
  vector[N] y_pred = beta + w_s;
}
model {
  matrix[N,N] Sigma_inv = (D - phi * A) / sigma2;
  w_s ~ multi_normal_prec(rep_vector(0,N), Sigma_inv);

  beta ~ normal(0,10);
  sigma2 ~ cauchy(0,5);
  sigma2_w ~ cauchy(0,5);

  y ~ normal(beta+w_s, sigma2_w);
}
"
```

```
data = list(
  N = nrow(nc),
  y = nc$log,
  x = rep(1, nrow(nc)),
  A = A * 1,
  D = diag(rowSums(A))
)

car_log_fit = rstan::stan(
  model_code = car_model, data = data,
  iter = 10000, thin=20, chains = 2, cores = 2
)
```

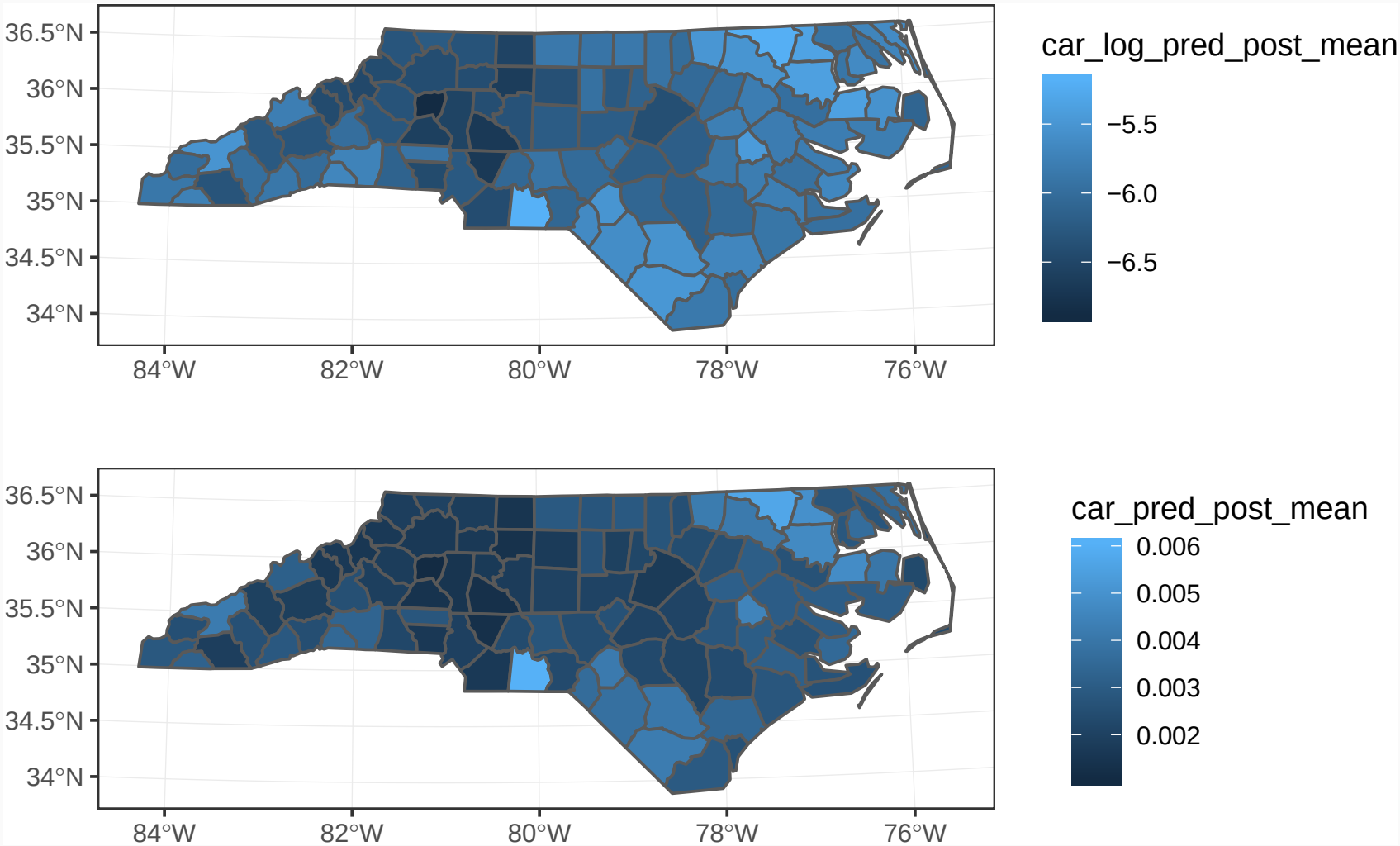
Chains


beta
phi
sigma2
sigma2_w
.value
.iteration
.chain
1
2
−5
−6
−7
−8
1.0
0.8
0.6
0.4
1.5
1.0
0.5
0.5
0.4
0.3
0.2
0.1
0
50
100
150
200
250

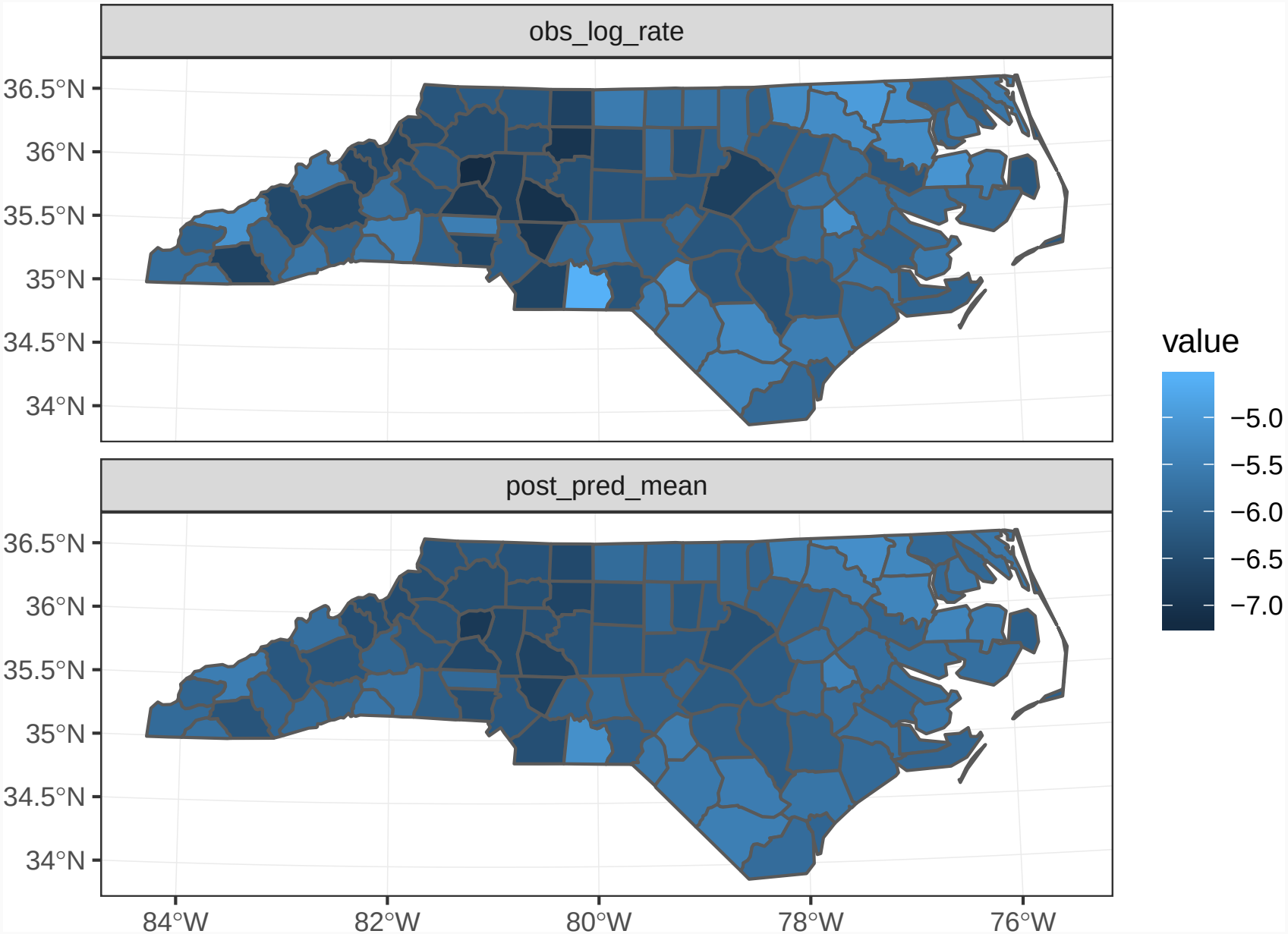
Posteriors



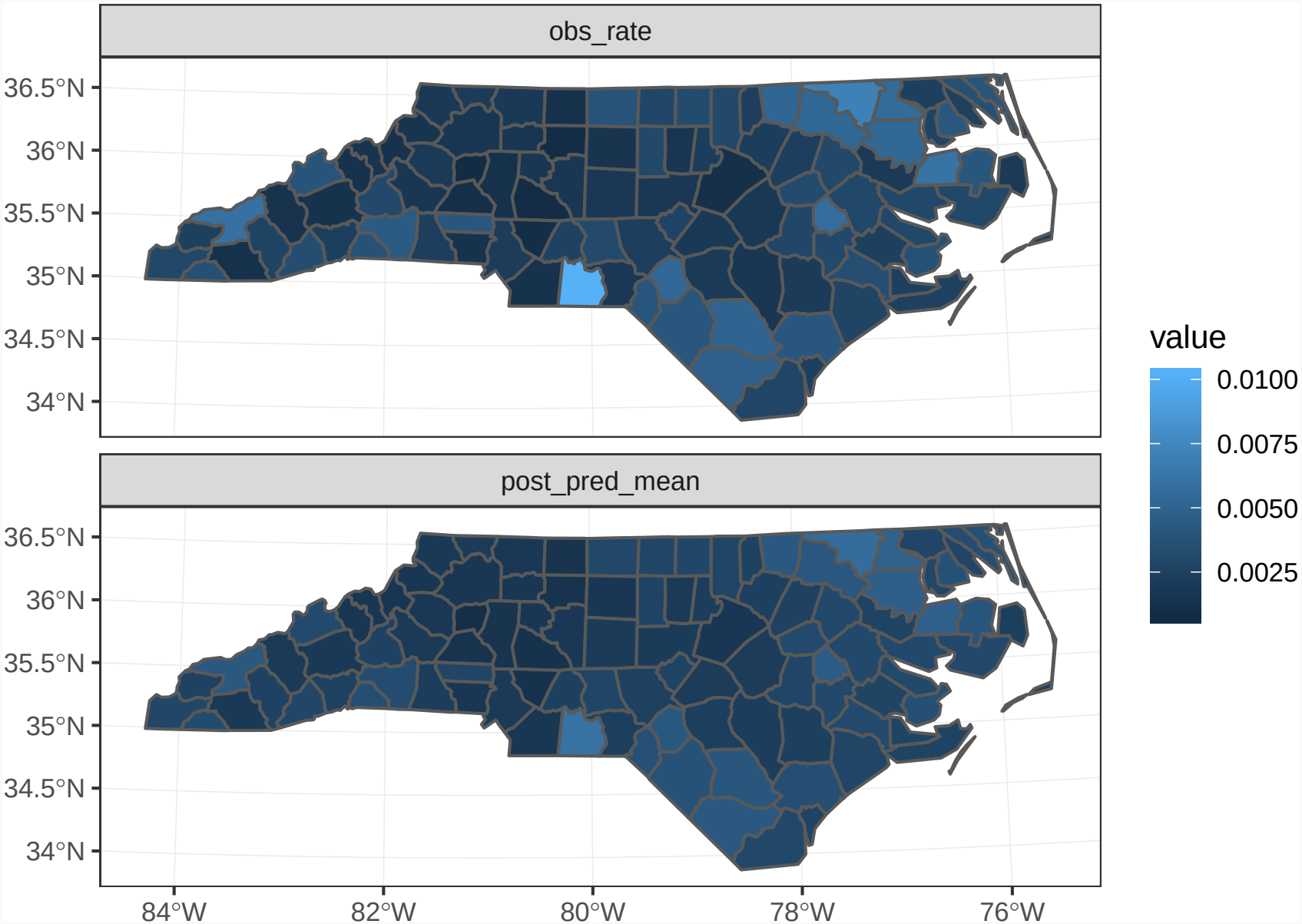
Predictions



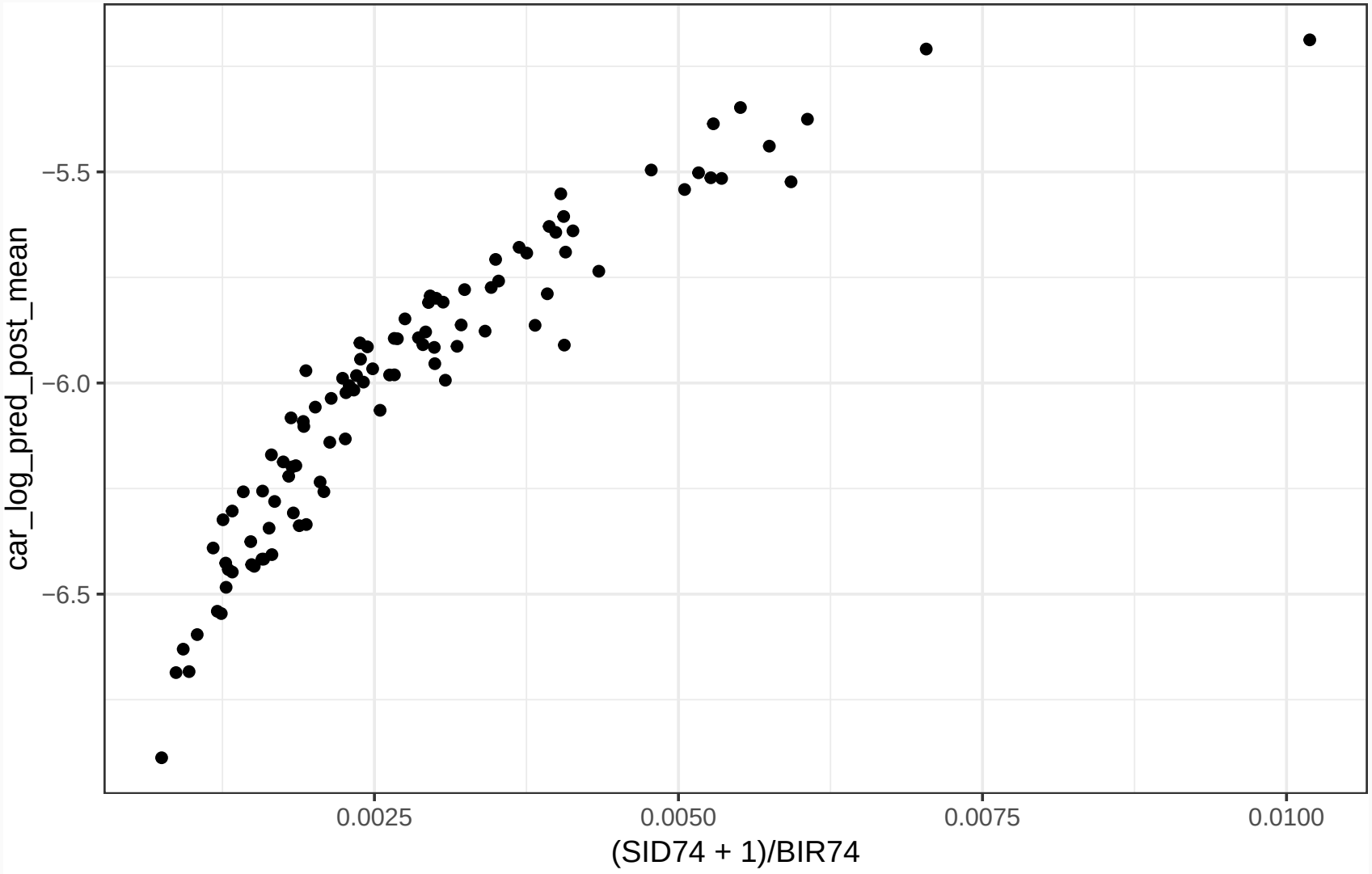
Predicted vs Observed



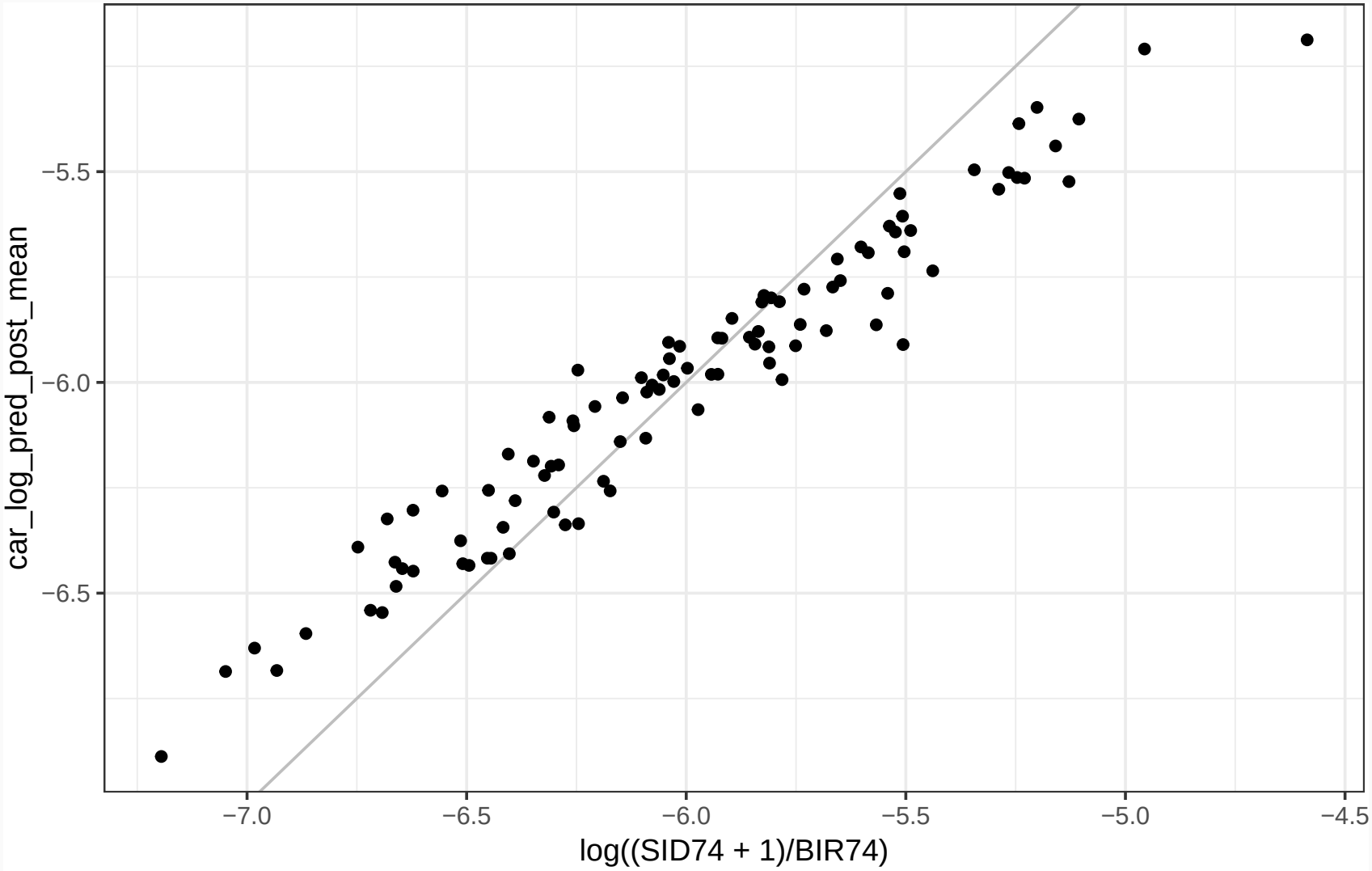
Predicted vs Observed (cont)



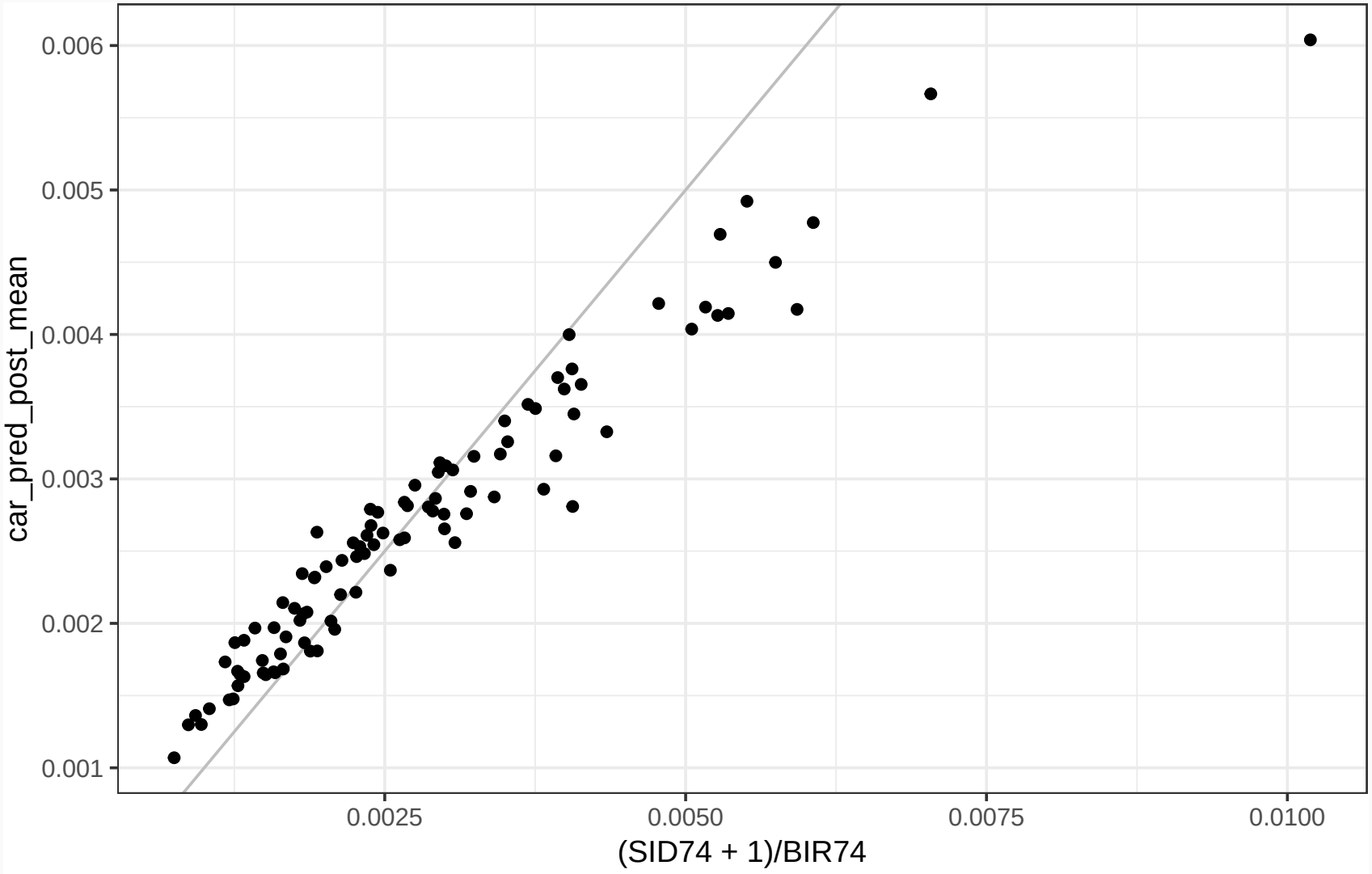
Model Fit


car_log_pred_post_mean
−5.5
−6.0
−6.5
0.0025
0.0050
0.0075
0.0100
(SID74 + 1)/BIR74

Model Fit (cont.)

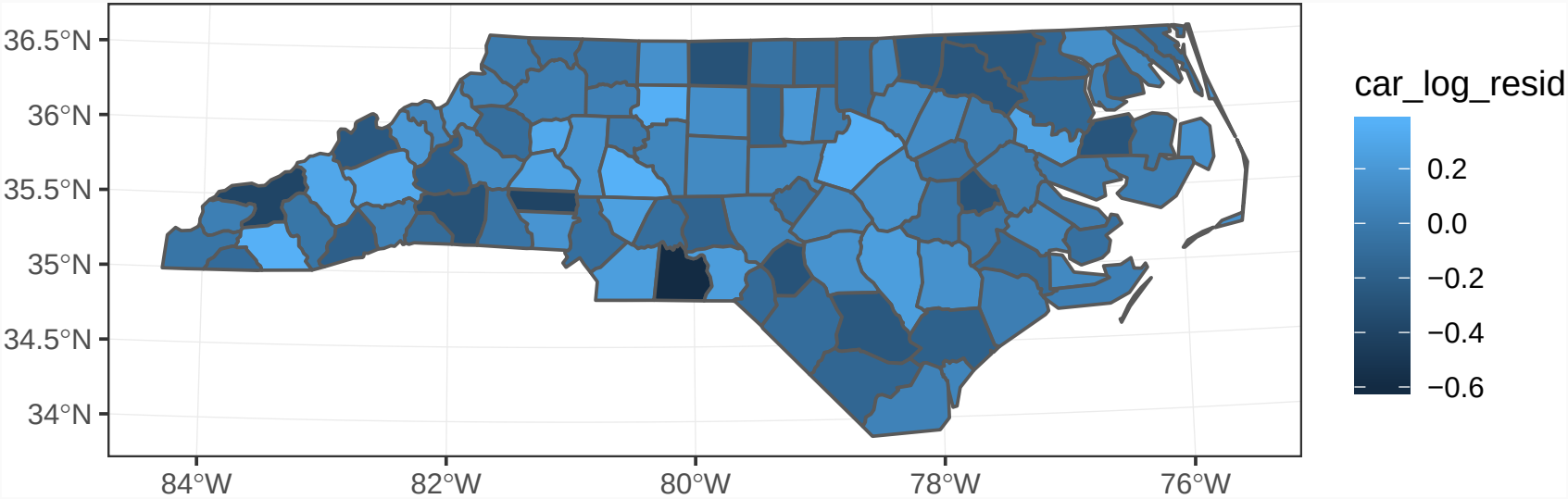


Model Fit (cont.)



Residuals

```
nc = mutate(nc, car_log_resid = car_log_pred_post_mean - log((SID74+1)/BIR74))

ggplot(nc, aes(fill=car_log_resid)) + geom_sf()
```




```
spdep::moran.test(nc$car_log_resid, listW)
##
##  Moran I test under randomisation
##
## data:  nc$car_log_resid
## weights: listW
##
## Moran I statistic standard deviate = -1.1361, p-value = 0.872
## alternative hypothesis: greater
## sample estimates:
## Moran I statistic       Expectation          Variance
```