Lecture 7

AR Models

[redacted]/2018

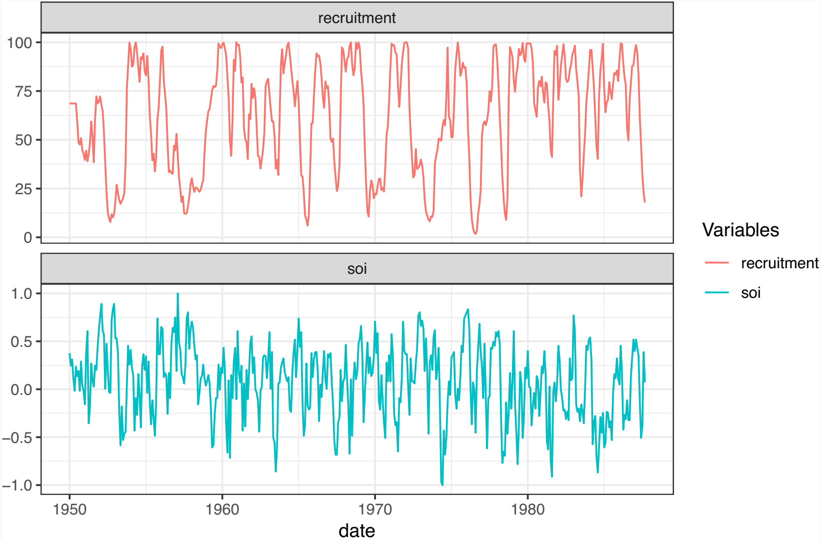
Lagged Predictors and CCFs

Southern Oscillation Index & Recruitment

The Southern Oscillation Index (SOI) is an indicator of the development and intensity of El Niño (negative SOI) or La Niña (positive SOI) events in the Pacific Ocean. These data also included the estimate of "recruitment", which indicate fish population sizes in the southern hemisphere.

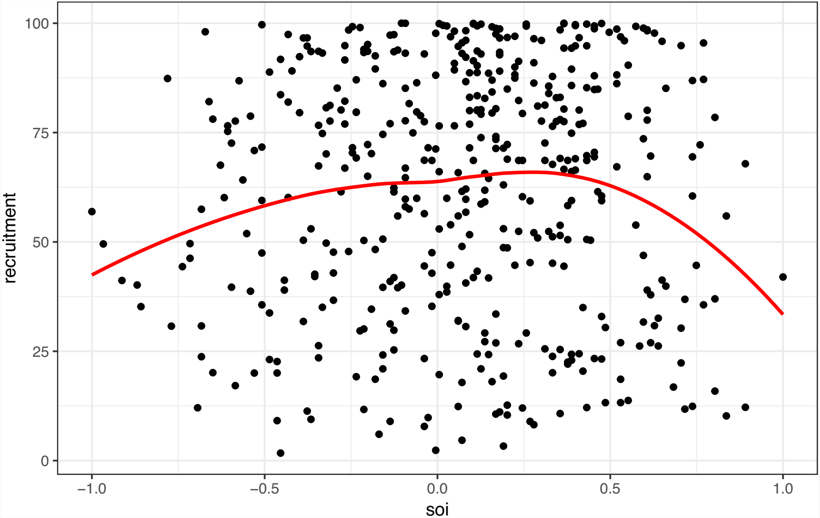
```
##
## Attaching package: 'astsa'
## The following object is masked from 'package:forecast':
##
##     gas
## # A tibble: 453 x 3
##     date    soi recruitment
##    <dbl>  <dbl>       <dbl>
##  1 1950   0.377        68.6
##  2 1950.  0.246        68.6
##  3 1950.  0.311        68.6
##  4 1950.  0.104        68.6
##  5 1950. -0.016        68.6
##  6 1950.  0.235        68.6
##  7 1950.  0.137        59.2
##  8 1951.  0.191        48.7
##  9 1951. -0.016        47.5
## 10 1951.  0.290        50.9
## # ... with 443 more rows
```

Time series



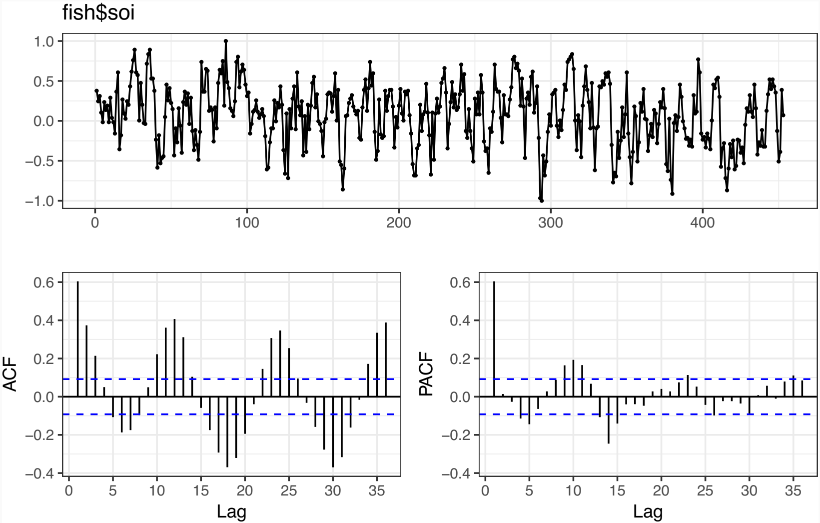
recruitment
100
75
50
25
0
soi
1.0
0.5
0.0
−0.5
−1.0
1950
1960
1970
1980
date
Variables
recruitment
soi

Relationship?



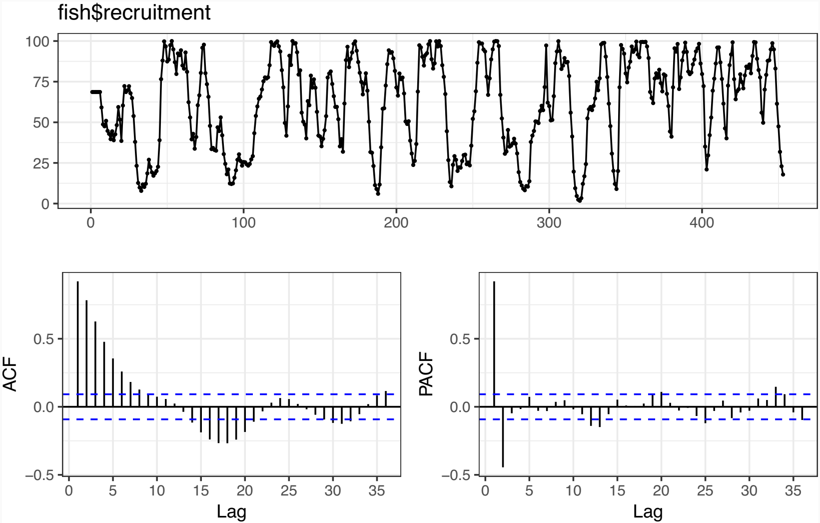
sois ACF & PACF

```
forecast::ggtsdisplay(fish$soi, lag.max = 36)
```



recruitment

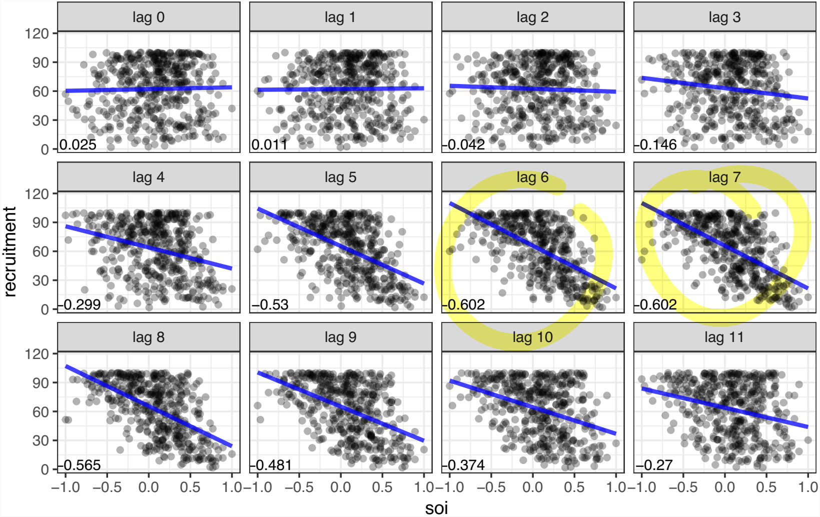
```
forecast::ggtsdisplay(fish$recruitment, lag.max = 36)
```



Cross correlation function

```
with(fish, forecast::ggCcf(soi, recruitment))
```

Cross correlation function - Scatter plots

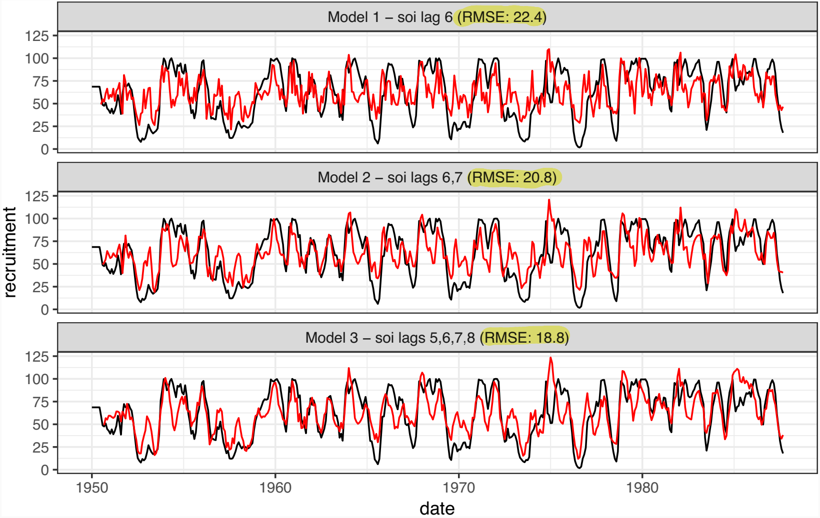


Model

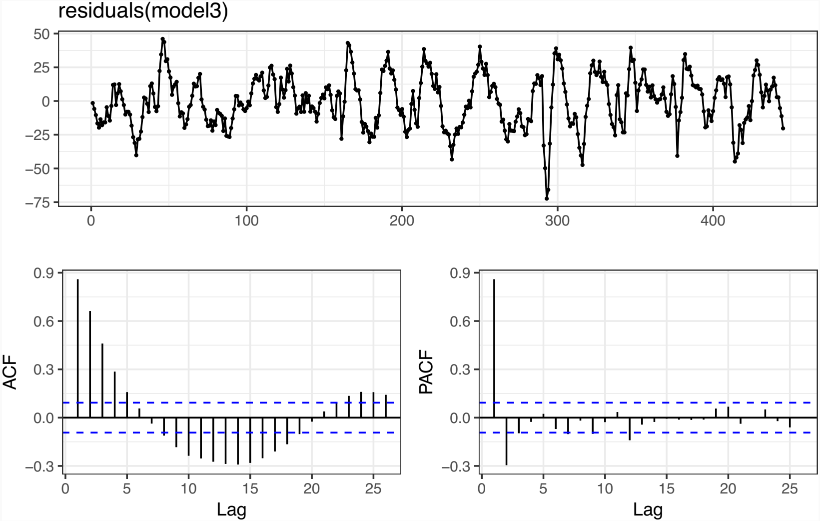
```
model1 = lm(recruitment~lag(soi,6), data=fish)
model2 = lm(recruitment~lag(soi,6)+lag(soi,7), data=fish)
model3 = lm(recruitment~lag(soi,5)+lag(soi,6)+lag(soi,7)+lag(soi,8), data=fish)

summary(model3)
##
## Call:
## lm(formula = recruitment ~ lag(soi, 5) + lag(soi, 6) + lag(soi,
##     7) + lag(soi, 8), data = fish)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -72.409 -13.527   0.191  12.851  46.040
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  67.9438     0.9306  73.007  < 2e-16 ***
## lag(soi, 5) -19.1502     2.9508  -6.490 2.32e-10 ***
## lag(soi, 6) -15.6894     3.4334  -4.570 6.36e-06 ***
## lag(soi, 7) -13.4041     3.4332  -3.904 0.000109 ***
## lag(soi, 8) -23.1480     2.9530  -7.839 3.46e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 18.93 on 440 degrees of freedom
##   (8 observations deleted due to missingness)
## Multiple R-squared:  0.5539, Adjusted R-squared:  0.5498
## F-statistic: 136.6 on 4 and 440 DF,  p-value: < 2.2e-16
```

Prediction


Model 1 – soi lag 6 (RMSE: 22.4)
Model 2 – soi lags 6,7 (RMSE: 20.8)
Model 3 – soi lags 5,6,7,8 (RMSE: 18.8)
recruitment
125
100
75
50
25
0
1950
1960
1970
1980
date

Residual ACF - Model 3



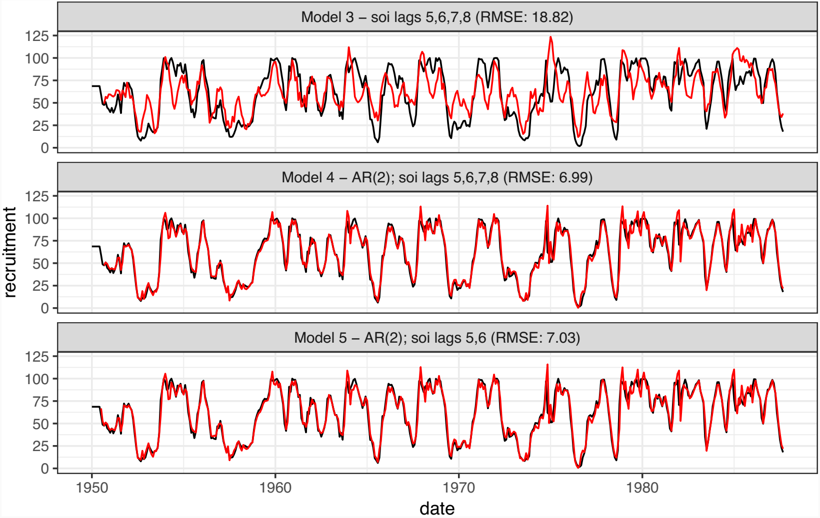
Autoregessive model 1

```
model4 = lm(recruitment~lag(recruitment,1) + lag(recruitment,2) +
                         lag(soi,5)+lag(soi,6)+lag(soi,7)+lag(soi,8),
            data=fish)
summary(model4)
##
## Call:
## lm(formula = recruitment ~ lag(recruitment, 1) + lag(recruitment,
##     2) + lag(soi, 5) + lag(soi, 6) + lag(soi, 7) + lag(soi, 8),
##     data = fish)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -51.996  -2.892   0.103   3.117  28.579
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)         10.25007    1.17081   8.755  < 2e-16 ***
## lag(recruitment, 1)  1.25301    0.04312  29.061  < 2e-16 ***
## lag(recruitment, 2) -0.39961    0.03998  -9.995  < 2e-16 ***
## lag(soi, 5)        -20.76309    1.09906 -18.892  < 2e-16 ***
## lag(soi, 6)          9.71918    1.56265   6.220 1.16e-09 ***
## lag(soi, 7)         -1.01131    1.31912  -0.767   0.4437
## lag(soi, 8)         -2.29814    1.20730  -1.904   0.0576 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 7.042 on 438 degrees of freedom
##   (8 observations deleted due to missingness)
```

Autoregessive model 2

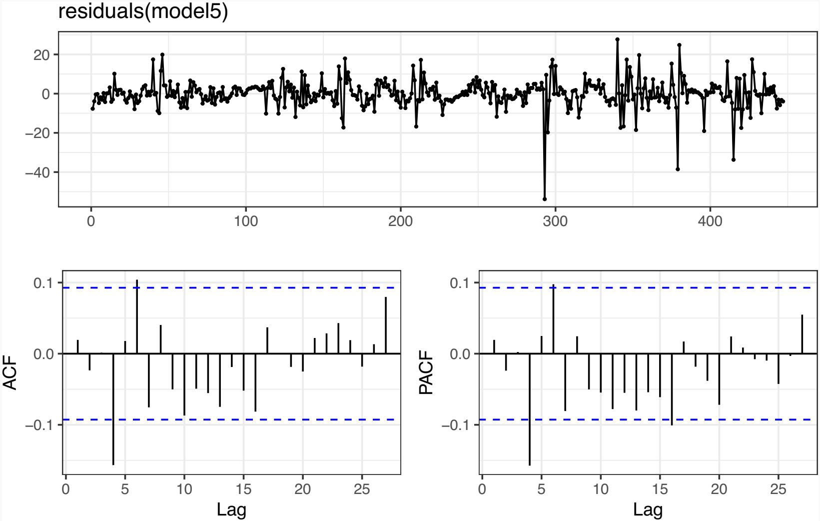
```
model5 = lm(recruitment~lag(recruitment,1) + lag(recruitment,2) +
                        lag(soi,5) + lag(soi,6),
            data=fish)
summary(model5)
##
## Call:
## lm(formula = recruitment ~ lag(recruitment, 1) + lag(recruitment,
##     2) + lag(soi, 5) + lag(soi, 6), data = fish)
##
## Residuals:
##     Min       1Q   Median       3Q      Max
## -53.786  -2.999  -0.035   3.031  27.669
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)           8.78498    1.00171   8.770  < 2e-16 ***
## lag(recruitment, 1)   1.24575    0.04314  28.879  < 2e-16 ***
## lag(recruitment, 2)  -0.37193    0.03846  -9.670  < 2e-16 ***
## lag(soi, 5)         -20.83776    1.10208 -18.908  < 2e-16 ***
## lag(soi, 6)           8.55600    1.43146   5.977 4.68e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 7.069 on 442 degrees of freedom
##   (6 observations deleted due to missingness)
## Multiple R-squared:  0.9375, Adjusted R-squared:  0.937
## F-statistic:  1658 on 4 and 442 DF,  p-value: < 2.2e-16
```

Prediction



Model 3 – soi lags 5,6,7,8 (RMSE: 18.82)
Model 4 – AR(2); soi lags 5,6,7,8 (RMSE: 6.99)
Model 5 – AR(2); soi lags 5,6 (RMSE: 7.03)
recruitment
125
100
75
50
25
0
1950
1960
1970
1980
date

```
forecast::ggtsdisplay(residuals(model5))
```



Non-stationarity

Non-stationary models

> *All happy families are alike; each unhappy family is unhappy in its own way.*
>
> - *Tolstoy, Anna Karenina*

This applies to time series models as well, just replace happy family with stationary model.

All happy families are alike; each unhappy family is unhappy in its own way.

- *Tolstoy, Anna Karenina*

This applies to time series models as well, just replace happy family with stationary model.

A simple example of a non-stationary time series is a trend stationary model

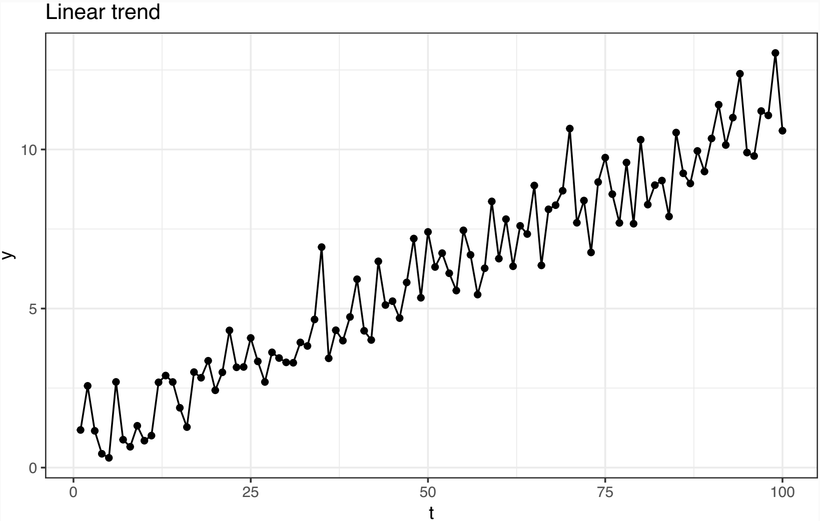
$w_t \overset{iid}{\sim} N(0, \sigma_w^2)$

$$y_t = \mu(t) + w_t$$

where $\mu(t)$ denotes a time dependent trend and w_t is a white noise (stationary) process.

Linear trend model

Lets imagine a simple model where $y_t = \delta + \beta t + x_t$ where δ and β are constants and x_t is a stationary process.



An simple approach to remove trend is to difference your response variable, specifically examine $y_t - y_{t-1}$ instead of y_t. $= \delta + \beta t + w_t$

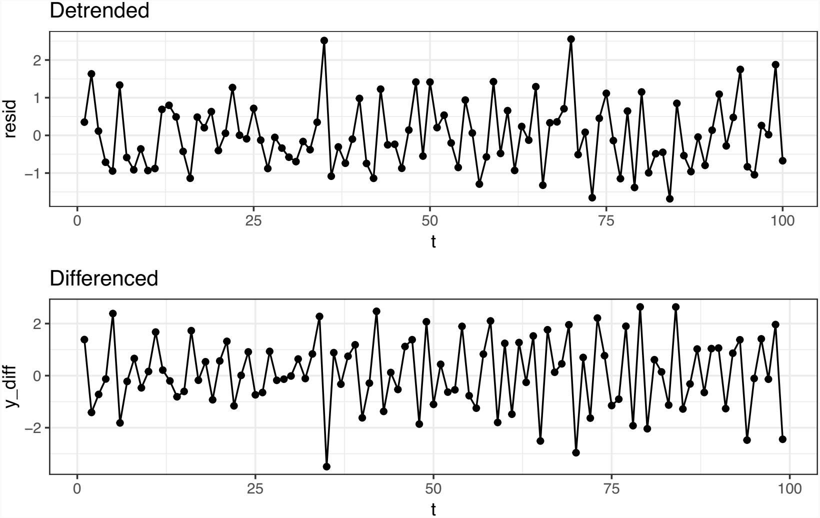
$d_t = y_t - y_{t-1}$

$= (\delta + \beta t + w_t) - (\delta + \beta(t-1) + w_{t-1})$

$= w_t - w_{t-1} + \beta$

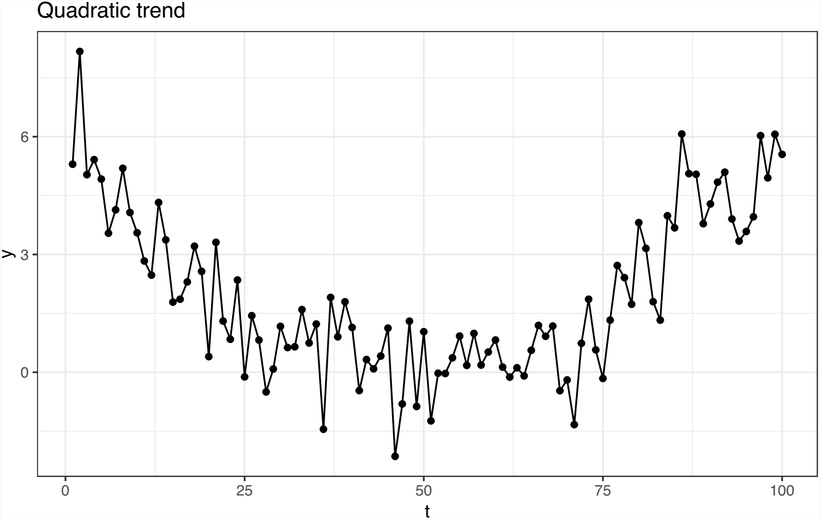
$\Rightarrow$ is stationary

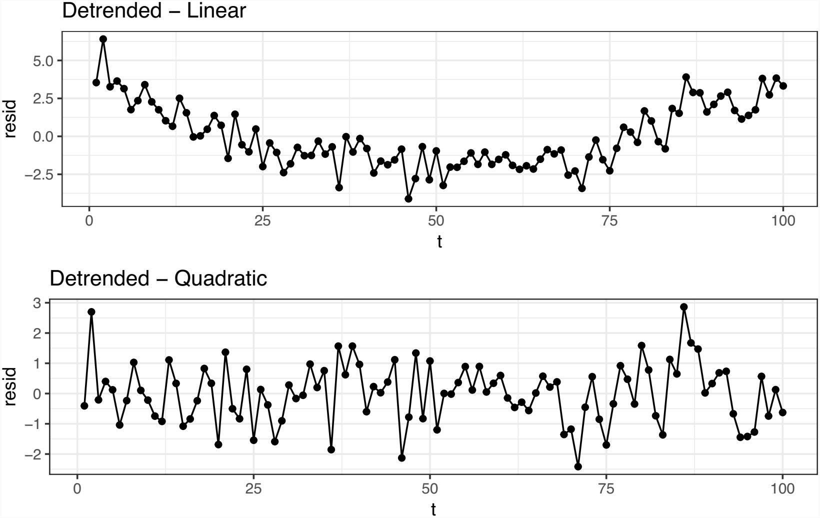
Detrending vs Difference



Quadratic trend model

Lets imagine another simple model where $y_t = \delta + \beta t + \gamma t^2 + x_t$ where δ, β, and γ are constants and x_t is a stationary process.




Detrended – Linear
5.0
2.5
0.0
–2.5
resid
0
25
50
75
100
t
Detrended – Quadratic
3
2
1
0
–1
–2
resid
0
25
50
75
100
t

Let $d_t = y_t - y_{t-1}$ be a first order difference then $d_t - d_{t-1}$ is a 2nd order difference.

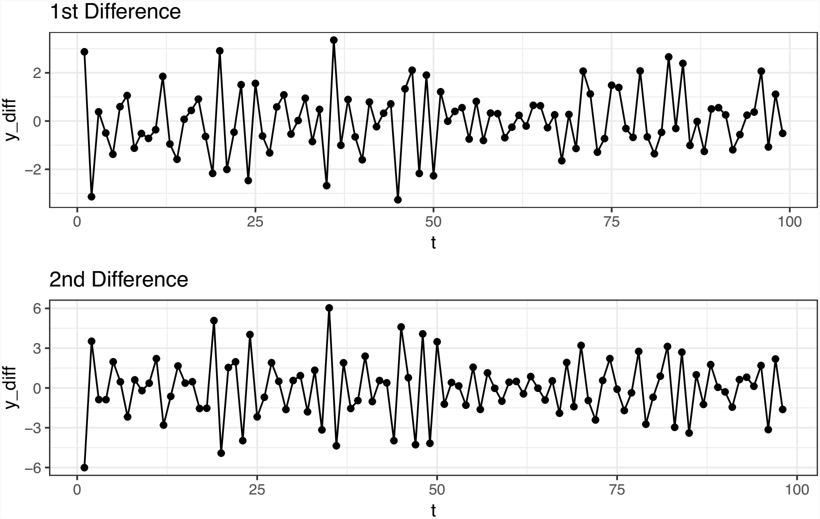
$$d_t^2 = d_t^1 - d_{t-1}^1 = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

$$= \Big((\delta + \beta t + \gamma t^2 + w_t) - (\delta + \beta(t-1) + \gamma(t-1)^2 + w_{t-1}) - (\delta + \beta(t-1) + \gamma(t-1)^2 + w_{t-1}) + (\delta + \beta(t-2) + \gamma(t-2)^2 + w_{t-2}) \Big)$$

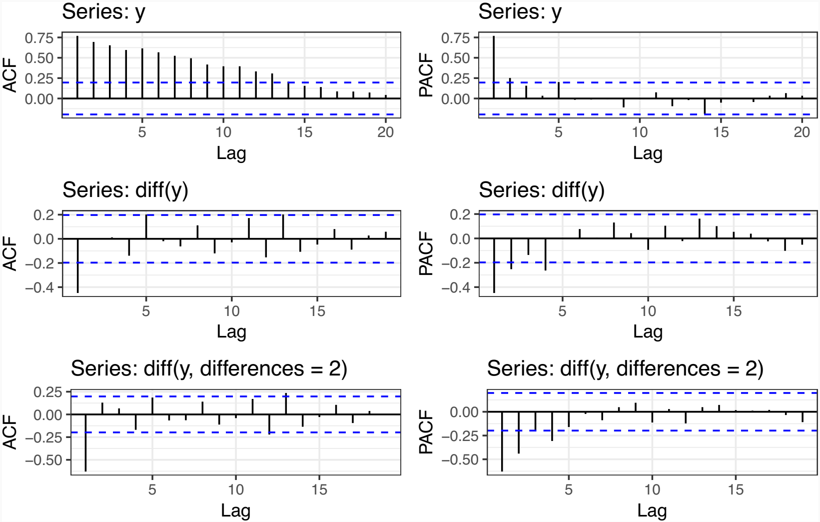
$$= w_t - 2w_{t-1} + w_{t-2} + \gamma t^2 - 2\gamma(t^2 - 2t + 1) + \gamma(t^2 - 4t + 4)$$

$$= w_t - 2w_{t-1} + w_{t-2} + 2\gamma$$

Differencing



Differencing - ACF



AR Models

AR(1)

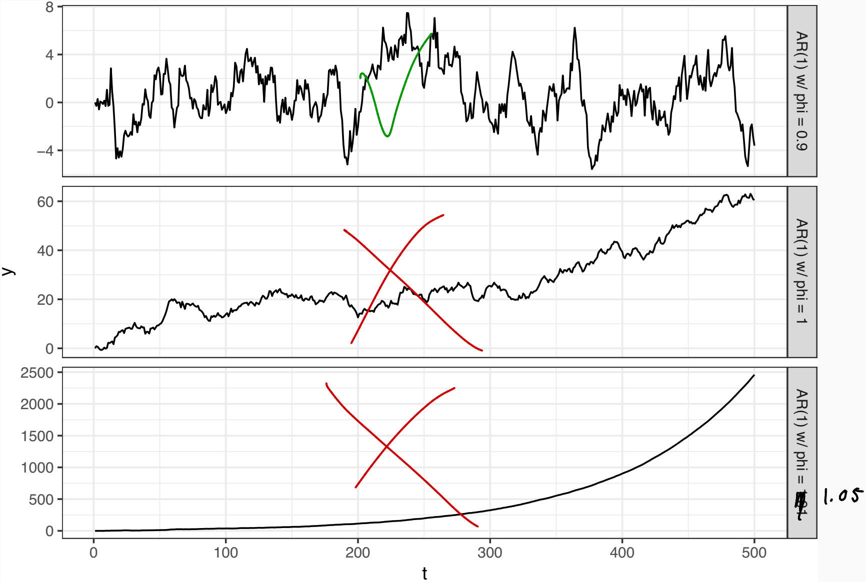
Last time we mentioned a random walk with trend process where $y_t = \delta + y_{t-1} + w_t$.

The AR(1) process is a generalization of this where we include a coefficient in front of the y_{t-1} term.

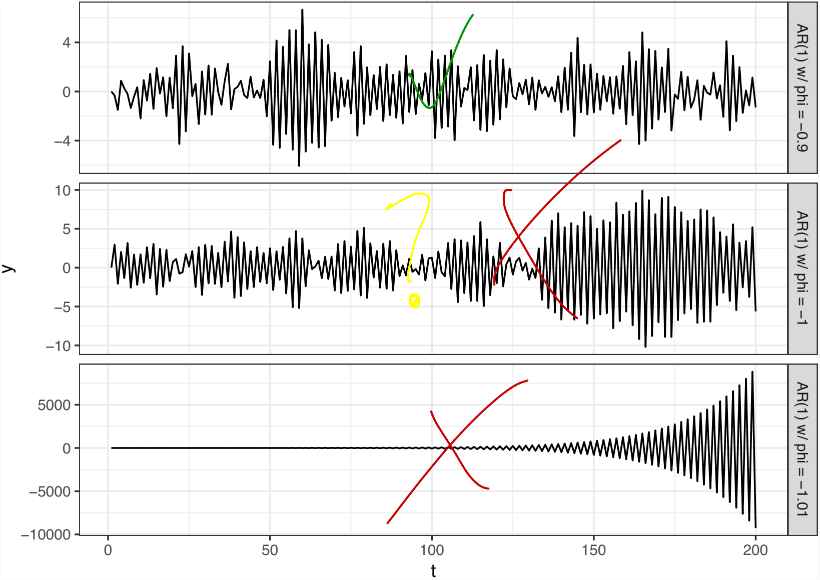
$$AR(1): \quad y_t = \delta + \phi\, y_{t-1} + w_t$$

$$w_t \sim N(0, \sigma_w^2)$$

AR(1) - Positive ϕ



AR(1) - Negative ϕ



Lets rewrite the AR(1) without any autoregressive terms

$$y_t = \delta + \phi y_{t-1} + v_t$$

$$= \delta + \phi(\delta + \phi y_{t-2} + v_{t-1}) + v_t$$

$$= \delta + \phi\delta + \phi^2 y_{t-2} + v_t + \phi v_{t-1}$$

$$= \delta + \phi\delta + \phi^2(\delta + \phi y_{t-3} + w_{t-2}) + v_t + \phi w_{t-1}$$

$$= \delta + \phi\delta + \phi^2\delta + \phi^3 y_{t-3} + w_t + \phi w_{t-1} + \phi^2 w_{t-2}$$

$$= \sum_{i=0}^{\infty} \phi^i \delta + \sum_{i=0}^{\infty} \phi^i v_{t-i}$$

Stationarity of $AR(1)$ processes

Under what conditions will an AR(1) process be stationary?

(1) $E(Y_t) = E\left(\sum_{i=0}^{\infty} \phi^i \delta\right) + E\left(\sum_{i=0}^{\infty} \phi^i v_{t-i}\right)$

$= \delta + \phi\delta + \phi^2\delta + \phi^3\delta + \dots$

$= \delta(1 + \phi + \phi^2 + \phi^3 + \dots)$

$= \begin{cases} \delta \frac{1}{1-\phi} & |\phi| < 1 \\ \infty & \text{otherwise} \end{cases}$

Under what conditions will an AR(1) process be stationary?

$$Var(Y_t) = Var\left(\sum_{i=0}^{\infty} \phi^i \delta\right) + Var\left(\sum_{i=0}^{\infty} \phi^i v_{t-i}\right)$$

$$= \sum_{i=0}^{\infty} Var\left(\phi^i v_{t-i}\right) = \sum_{i=0}^{\infty} \phi^{2i} \, Var(v_{t-i})$$

$$= \sum_{i=0}^{\infty} \phi^{2i} \sigma_v^2 = \sigma_v^2 \left(1 + \phi^2 + \phi^4 + \phi^6 + \phi^8 + \dots\right)$$

$$= \begin{cases} \sigma_v^2 \left(\frac{1}{1-\phi^2}\right) & \text{if } \phi^2 < 1 \\ \infty & \text{otherwise} \end{cases}$$

Stationarity of $AR(1)$ processes

Under what conditions will an AR(1) process be stationary?

$$\gamma(h) = Cov(Y_t, Y_{t-h})$$

$$\gamma(0) = Cov(Y_t, Y_t) = Var(Y_t) = \frac{\sigma_v^2}{1-\phi^2} = \sigma_v^2(1+\phi^2+\phi^4+\phi^6+\dots)$$

$$\gamma(1) = E\left((Y_t - E(Y_t))(Y_{t-1} - E(Y_{t-1}))\right) = \phi\,\gamma(0)$$

$$= E\left(\left(\sum_{i=0}^{\infty} \phi^i v_{t-i}\right)\left(\sum_{i=0}^{\infty} \phi^i v_{t-1-i}\right)\right)$$

$$= E\left((v_t + \phi w_{t-1} + \phi^2 w_{t-2} + \dots)(w_{t-1} + \phi v_{t-2} + \phi^2 v_{t-3} + \dots)\right)$$

$$= E\left(\phi v_{t-1}^2 + \phi^3 v_{t-2}^2 + \phi^5 v_{t-3}^2 + \dots\right)$$

$$= \phi\sigma_v^2 + \phi^3\sigma_v^2 + \phi^5\sigma_v^2 + \dots = \phi\sigma_v^2(1+\phi^2+\phi^4+\phi^6+\dots)$$

Stationarity of $AR(1)$ processes

Under what conditions will an AR(1) process be stationary?

$$\gamma(2) = E\Big(\big(v_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 v_{t-3} + \dots \big)\big(v_{t-2} + \phi v_{t-3} + \phi^2 v_{t-4} + \dots \big) \Big)$$

$$= E\big(\phi^2 w_{t-2} + \phi^4 w_{t-3} + \phi^6 w_{t-4} + \dots \big)$$

$$= \phi^2 \sigma_v^2 + \phi^4 \sigma_v^2 + \phi^6 \sigma_v^2 + \phi^8 \sigma_v^2 + \dots$$

$$= \phi^2 \sigma_v^2 \big(1 + \phi^2 + \phi^4 + \phi^6 + \dots \big)$$

$$= \phi^2 \gamma(0)$$

Properties of a stationary $AR(1)$ processes

$$E(y_t) = \frac{\delta}{1-\phi}$$

$$Var(y_t) = \frac{\sigma_w^2}{1-\phi^2} = \gamma(0)$$

$$\gamma(h) = \phi^{|h|} \gamma(0)$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^{|h|}$$

Properties of a stationary $AR(1)$ processes

Assume stationary

$$E(y_t) = E(\delta + \phi y_{t-1} + v_t)$$

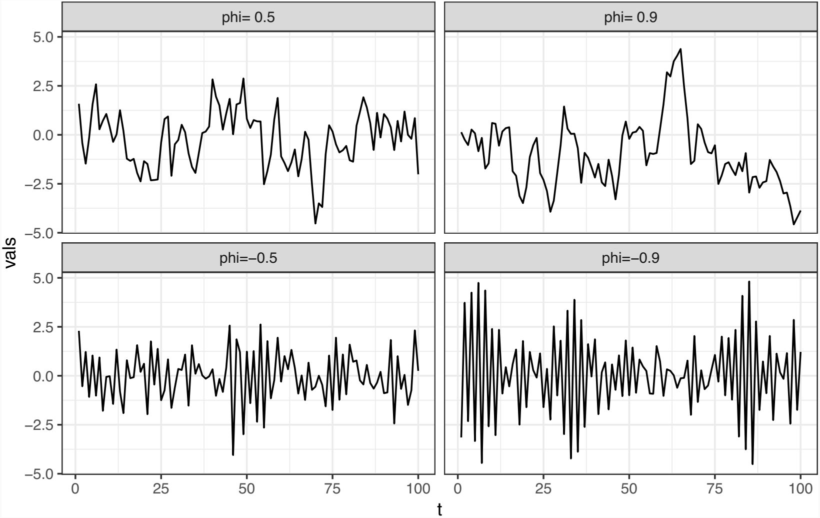
$$E(y_t) = \delta + \phi E(y_{t-1})$$

$$\mu = \delta + \phi \mu$$

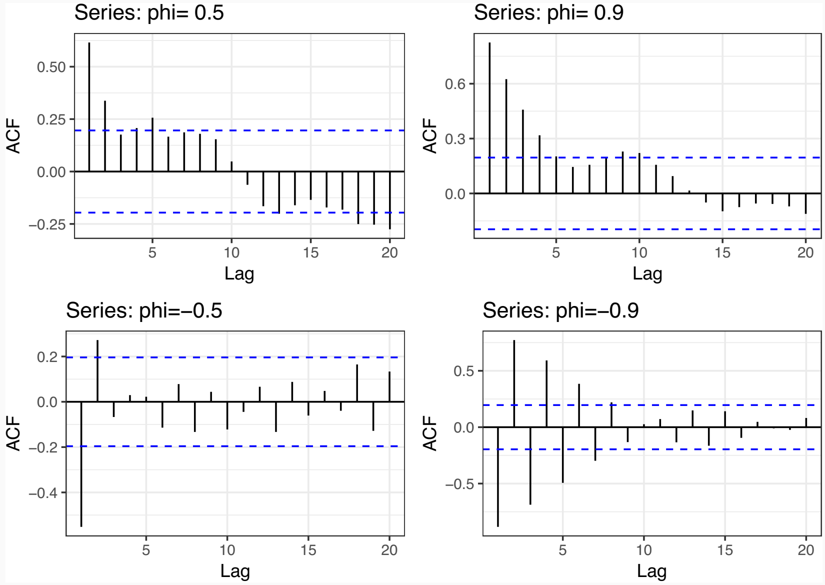
$$\mu - \phi \mu = \delta$$

$$\mu = \frac{\delta}{1-\phi}$$

Identifying AR(1) Processes



Identifying AR(1) Processes - ACFs



Identifying AR(1) Processes - PACFs