Lecture 8

AR, MA, and ARMA Models

9/27/2018

AR models

AR(p) models

We can generalize from an AR(1) to an AR(p) model by simply adding additional autoregressive terms to the model.

$$
\begin{aligned}
AR(p): \quad y_t &= \delta + \phi_1 \, y_{t-1} + \phi_2 \, y_{t-2} + \cdots + \phi_p \, y_{t-p} + w_t \\
&= \delta + w_t + \sum_{i=1}^{p} \phi_i \, y_{t-i}
\end{aligned}
$$

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&= \delta + w_t + \sum_{i=1}^{p} \phi_i \, y_{t-i}
\end{aligned}
$$

What are the properities of $AR(p)$,

1. Expected value?
2. Autocovariance / autocorrelation?
3. Stationarity conditions?

Lag operator

The lag operator is convenience notation for writing out AR (and other) time series models.

We define the lag operator L as follows,

$$L\,y_t = y_{t-1}$$

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We define the lag operator L as follows,

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this can be generalized where,

$$\begin{aligned} L^2 y_t &= L\,(L\, y_t) \\ &= L\, y_{t-1} \\ &= y_{t-2} \end{aligned}$$

therefore,

$$L^k\, y_t = y_{t-k}$$

Lag polynomial

Lets rewrite the $AR(p)$ model using the lag operator,

$$y_t = \delta + \phi_1 \, y_{t-1} + \phi_2 \, y_{t-2} + ... + \phi_p \, y_{t-p} + w_t$$
$$y_t = \delta + \phi_1 \, L \, y_t + \phi_2 \, L^2 \, y_t + ... + \phi_p \, L^p \, y_t + w_t$$

Lag polynomial

Lets rewrite the $AR(p)$ model using the lag operator,

$$y_t = \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + ... + \phi_p\, y_{t-p} + w_t$$
$$y_t = \delta + \phi_1\, L\, y_t + \phi_2\, L^2\, y_t + ... + \phi_p\, L^p\, y_t + w_t$$

$$y_t - \phi_1\, L\, y_t - \phi_2\, L^2\, y_t - ... - \phi_p\, L^p\, y_t = \delta + w_t$$
$$(1 - \phi_1\, L - \phi_2\, L^2 - ... - \phi_p\, L^p)\, y_t = \delta + w_t$$

Lag polynomial

Lets rewrite the $AR(p)$ model using the lag operator,

$$
\begin{aligned}
y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + ... + \phi_p\, y_{t-p} + w_t \\
y_t &= \delta + \phi_1\, L\, y_t + \phi_2\, L^2\, y_t + ... + \phi_p\, L^p\, y_t + w_t
\end{aligned}
$$

$$
\begin{aligned}
y_t - \phi_1\, L\, y_t - \phi_2\, L^2\, y_t - ... - \phi_p\, L^p\, y_t &= \delta + w_t \\
(1 - \phi_1\, L - \phi_2\, L^2 - ... - \phi_p\, L^p)\, y_t &= \delta + w_t
\end{aligned}
$$

This polynomial of lags

$$\phi_p(L) = (1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)$$

is called the characteristic polynomial of the AR process.

Stationarity of $AR(p)$ processes

Claim: An $AR(p)$ process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

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Claim: An $AR(p)$ process is stationary if the roots of the characteristic polynomial lay *outside* the complex unit circle

If we define $\lambda = 1/L$ then we can rewrite the characteristic polynomial as

$$(\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p)$$

then as a corollary of our claim the $AR(p)$ process is stationary if the roots of this new polynomial are *inside* the complex unit circle (i.e. $|\lambda| < 1$).

Example AR(1)

$$y_t = \delta + \phi y_{t-1} + v_t$$

$$y_t = \delta + \phi L y_t + w_t$$

$$(1 - \phi L) y_t = \delta + w_t$$

Char. Poly

$$1 - \phi L = 0$$

$$\lambda - \phi = 0$$

$$\lambda = \phi$$

Claim:

$$|\lambda| < 1$$

$$|\phi| < 1$$

$$-1 < \phi < 1$$

Example AR(2)

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + w_t$$

$$(1 - \phi_1 L - \phi_2 L^2) Y_t = \delta + w_t$$

$$1 - \phi_1 L - \phi_2 L^2 = 0$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

$$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

If $\phi_1^2 + 4\phi_2 > 0$ (real root)

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

$$\phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2$$

$$\phi_1 + \phi_2 < 1$$

$$\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} > -1$$

$$\phi_1 + 2 > \sqrt{\phi_1^2 + 4\phi_2}$$

$$\phi_1^2 + 4\phi_1 + 4 > \phi_1^2 + 4\phi_2$$

$$\phi_1 + 1 > \phi_2$$

$$1 > \phi_2 - \phi_1$$

Example AR(2)

$$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

if $\phi_1^2 + 4\phi_2 < 0$

(imag root)

$$= \frac{\phi_1 \pm \left(\sqrt{-(\phi_1^2 + 4\phi_2)}\right) i}{2}$$

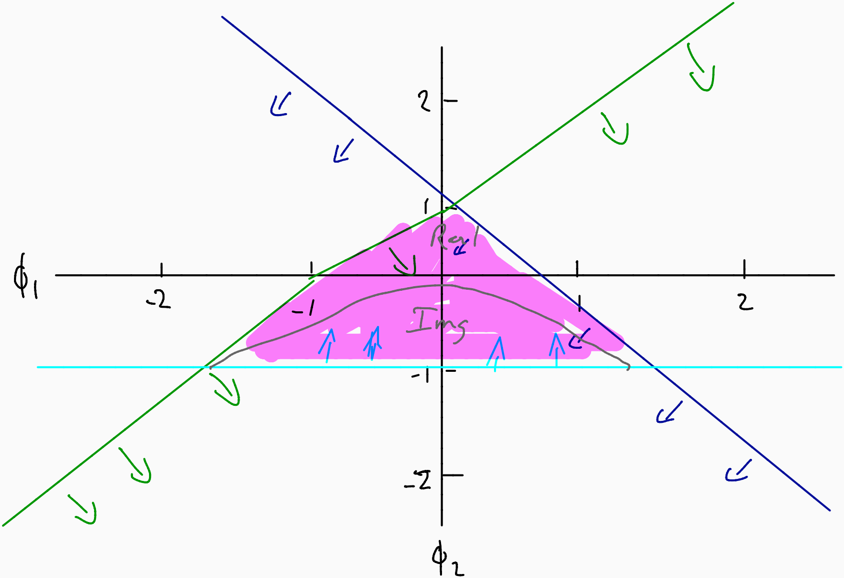
$$|\lambda| < 1$$

$$\left(\left(\frac{\phi_1^2}{4}\right) + \left(-\frac{\phi_1^2 + 4\phi_2}{4}\right)\right)^{1/2} < 1$$

$$\sqrt{-\phi_2} < 1$$

$$-1 < \phi_2$$

Example AR(2)



AR(2) Stationarity Conditions

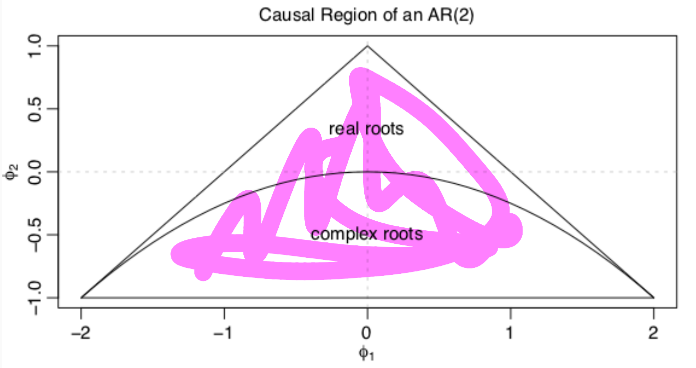


Fig. 3.3. *Causal region for an AR(2) in terms of the parameters.*

From Shumway&Stofer4thed.

Proof Sketch

We can rewrite the $AR(p)$ model into an $AR(1)$ form using matrix notation

$$
\begin{aligned}
y_t &= \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t \\
\boldsymbol{\xi}_t &= \boldsymbol{\delta} + \mathbf{F}\,\boldsymbol{\xi}_{t-1} + \mathbf{w}_t
\end{aligned}
$$

where

$$
\begin{aligned}
\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}
&= \begin{bmatrix} \delta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
+ \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}
\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix}
+ \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \delta + w_t + \sum_{i=1}^{p} \phi_i\, y_{t-i} \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}
\end{aligned}
$$

Proof sketch (cont.)

So just like the original $AR(1)$ we can expand out the autoregressive equation

$$\begin{aligned}
\boldsymbol{\xi}_t &= \boldsymbol{\delta} + \mathbf{w}_t + \mathbf{F}\,\boldsymbol{\xi}_{t-1} \\
&= \boldsymbol{\delta} + \mathbf{w}_t + \mathbf{F}\left(\boldsymbol{\delta} + \mathbf{w}_{t-1}\right) + \mathbf{F}^2\left(\boldsymbol{\delta} + \mathbf{w}_{t-2}\right) + \cdots \\
&\qquad\qquad + \mathbf{F}^{t-1}\left(\boldsymbol{\delta} + \mathbf{w}_1\right) + \mathbf{F}^t\left(\boldsymbol{\delta} + \mathbf{w}_0\right) \\
&= (\sum_{i=0}^{t} F^i)\boldsymbol{\delta} + \sum_{i=0}^{t} F^i\, w_{t-i}
\end{aligned}$$

and therefore we need $\lim_{t\to\infty} F^t \to 0$.

Proof sketch (cont.)

We can find the eigen decomposition such that $\mathbf{F} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ where the columns of $\mathbf{Q}$ are the eigenvectors of $\mathbf{F}$ and $\mathbf{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues.

A useful property of the eigen decomposition is that

$$\mathbf{F}^i = \mathbf{Q}\mathbf{\Lambda}^i\mathbf{Q}^{-1}$$

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A useful property of the eigen decomposition is that

$$\mathbf{F}^i = \mathbf{Q}\boldsymbol{\Lambda}^i\mathbf{Q}^{-1}$$

Using this property we can rewrite our equation from the previous slide as

$$\begin{aligned}\boldsymbol{\xi}_t &= (\sum_{i=0}^{t} F^i)\boldsymbol{\delta} + \sum_{i=0}^{t} F^i\, w_{t-i} \\ &= (\sum_{i=0}^{t} \mathbf{Q}\boldsymbol{\Lambda}^i\mathbf{Q}^{-1})\boldsymbol{\delta} + \sum_{i=0}^{t} \mathbf{Q}\boldsymbol{\Lambda}^i\mathbf{Q}^{-1}\, w_{t-i}\end{aligned}$$

Proof sketch (cont.)

$$\mathbf{\Lambda}^i = \begin{bmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^i \end{bmatrix}$$

Therefore,

$$\lim_{t\to\infty} F^t \to 0$$

when

$$\lim_{t\to\infty} \Lambda^t \to 0$$

which requires that

$$|\lambda_i| < 1 \quad \text{for all } i$$

Proof sketch (cont.)

Eigenvalues are defined such that for $\boldsymbol{\lambda}$,

$$\det(\mathbf{F} - \boldsymbol{\lambda}\,\mathbf{I}) = 0$$

based on our definition of $\mathbf{F}$ our eigenvalues will therefore be the roots of

$$\lambda^p - \phi_1\,\lambda^{p-1} - \phi_2\,\lambda^{p-2} - \cdots - \phi_{p_1}\,\lambda^1 - \phi_p = 0$$

Proof sketch (cont.)

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$$\lambda^p - \phi_1\,\lambda^{p-1} - \phi_2\,\lambda^{p-2} - \cdots - \phi_{p_1}\,\lambda^1 - \phi_p = 0$$

which if we multiply by $1/\lambda^p$ where $L = 1/\lambda$ gives

$$1 - \phi_1\,L - \phi_2\,L^2 - \cdots - \phi_{p_1}\,L^{p-1} - \phi_p\,L^p = 0$$

Properties of $AR(2)$

For a stationary $AR(2)$ process where w_t has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

$$E(y_t) = E(y_{t-h})$$

(1) $$E(y_t) = E(\delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t)$$

$$E(y_t) = \delta + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2}) + 0$$

$$(1-\phi_1-\phi_2) E(y_t) = \delta$$

$$E(y_t) = \frac{\delta}{1-\phi_1-\phi_2}$$

Properties of $AR(2)$

For a *stationary* $AR(2)$ process where w_t has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

$$\tilde{y}_t = y_t - E(y_t)$$

(2)
$$\gamma(h) = Cov(y_t, y_{t-h})$$
$$= E\left((y_t - E(y_t))(y_{t-h} - E(y_{t-h})) \right)$$
$$= E(\tilde{y}_t \, \tilde{y}_{t-h}) = E\left(\phi_1 \tilde{y}_{t-1} \tilde{y}_{t-h} + \phi_2 \tilde{y}_{t-2} \tilde{y}_{t-h} + w_t \tilde{y}_{t-h} \right)$$
$$= \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \sigma_w^2 \mathbb{1}_{h=0}$$

$$\boxed{\tilde{y}_t = \phi_1 \tilde{y}_{t-1} + \phi_2 \tilde{y}_{t-2} + w_t}$$

For a *stationary* $AR(2)$ process where w_t has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \sigma_w^2 \mathbb{1}_{h=0}$$

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma_w^2$$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1) = \frac{1}{1-\phi_2} \phi_1 \gamma(0)$$

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0) = \frac{\phi_1^2 + \phi_1 (1-\phi_2)}{1-\phi_2} \gamma(0)$$

$$\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)$$

Properties of $AR(p)$

For a *stationary* $AR(p)$ process where w_t has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

$$E(Y_t) = \frac{\delta}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

$$\begin{aligned} Var(y_t) = \gamma(0) &= \phi_1 \gamma(1) + \phi_2 \gamma(2) + ... + \phi_p \gamma(p) + \sigma_w^2 \\ \gamma(h) &= \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + ... + \phi_p \gamma(h-p) \end{aligned}$$

$$\rho(h) = \phi_1 \, \rho(h-1) + \phi_2 \, \rho(h-2) + ... + \phi_p \, \rho(h-p)$$

Moving Average (MA) Processes

MA(1)

A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$MA(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

Properties:

① $E(y_t) = E(\delta) + E(w_t) + \theta E(w_{t-1}) = \delta$

② $Var(y_t) = \gamma(0) = Var(\delta + w_t + \theta w_{t-1})$

$= Var(w_t) + Var(\theta w_{t-1})$

$= \sigma_w^2 + \theta^2 \sigma_w^2 = (1+\theta^2)\sigma_w^2$

A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$MA(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

$$\tilde{y}_t = y_t - \delta = w_t + \theta w_{t-1}$$

Properties:

③ $\gamma(h) = E(\tilde{y}_t \, \tilde{y}_{t-h})$

$= E\left[(w_t + \theta w_{t-1})(w_{t-h} + \theta w_{t-h-1})\right]$

$= E(w_t w_{t-h}) + \theta E(w_t w_{t-h-1})$

$+ \theta E(w_{t-1} w_{t-h}) + \theta^2 E(w_{t-1} w_{t-h-1})$

$$= \begin{cases} (1+\theta^2)\sigma_w^2 & \text{if } h = 0 \\ \theta \sigma_w^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

MA(1)

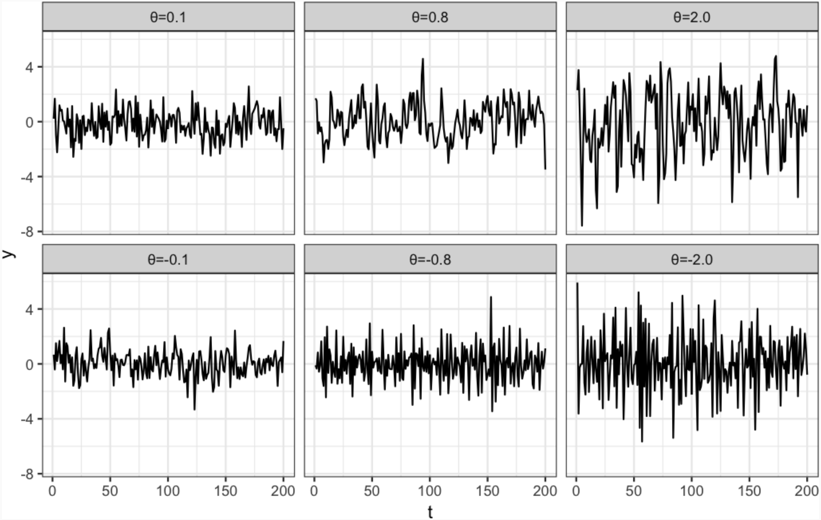
A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$MA(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

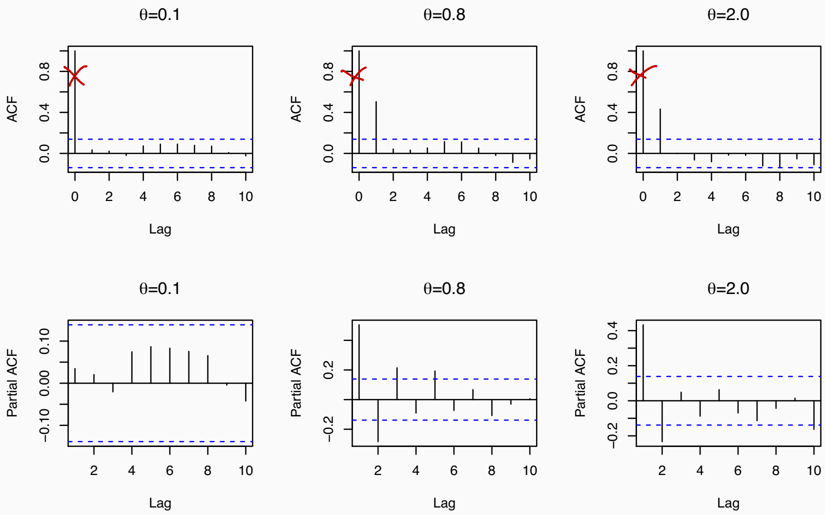
Properties:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & \text{if } h = 0 \\ \frac{\theta}{1+\theta^2} & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Time series



ACF



MA(q)

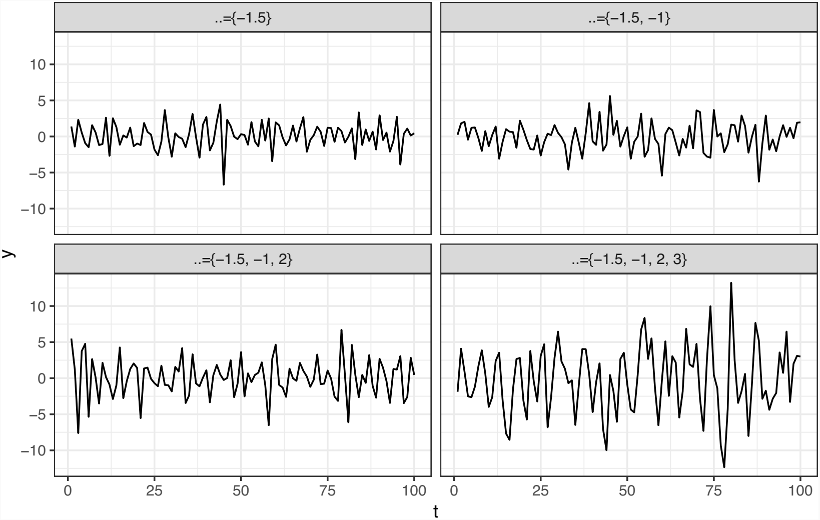
$$MA(q): \qquad y_t = \delta + w_t + \theta_1 \, w_{t-1} + \theta_2 \, w_{t-2} + \cdots + \theta_q \, w_{t-q}$$

Properties:

$$E(y_t) = \delta$$

$$\gamma(0) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \, \sigma_w^2$$
$$\gamma(h) = \begin{cases} -\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q+k}\theta_q & \text{if } |k| \in \{1, \dots, q\} \\ 0 & \text{otherwise} \end{cases}$$

Example series



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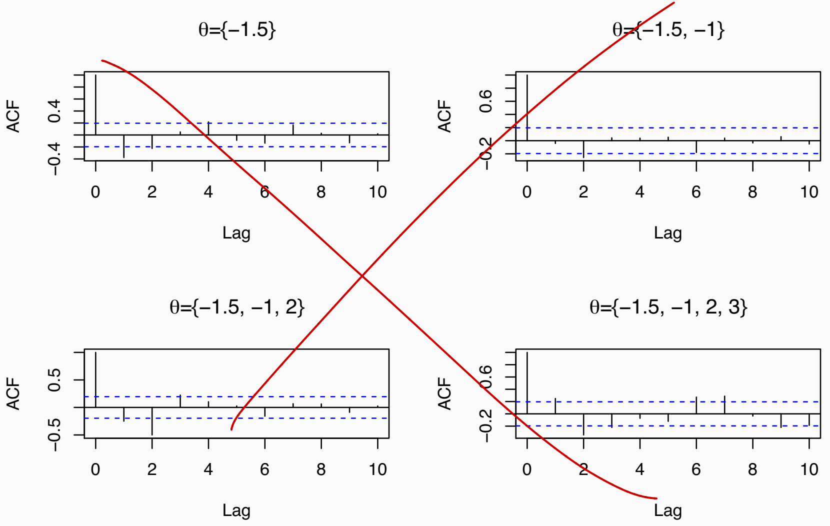


ARMA Model

ARMA Model

An ARMA model is a composite of AR and MA processes,

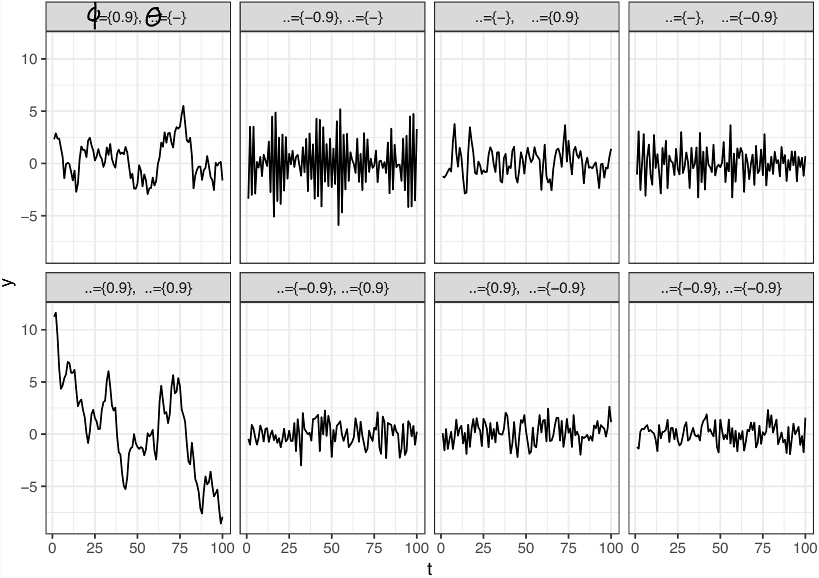
$ARMA(p, q)$:

$$y_t = \delta + \phi_1 \, y_{t-1} + \cdots \phi_p \, y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t_q}$$

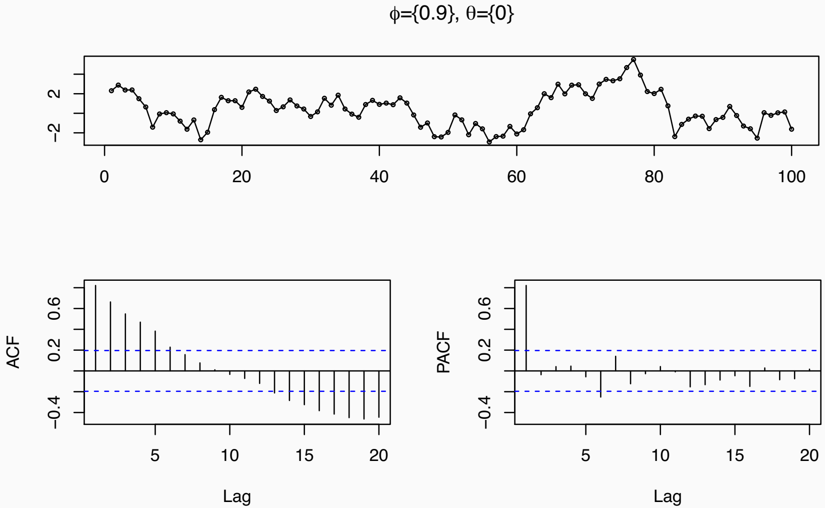
$$\phi_p(L) y_t = \delta + \theta_q(L) w_t$$

Since all MA processes are stationary, we only need to examine the AR aspect to determine stationarity (roots of $\phi_p(L)$ lie outside the complex unit circle).

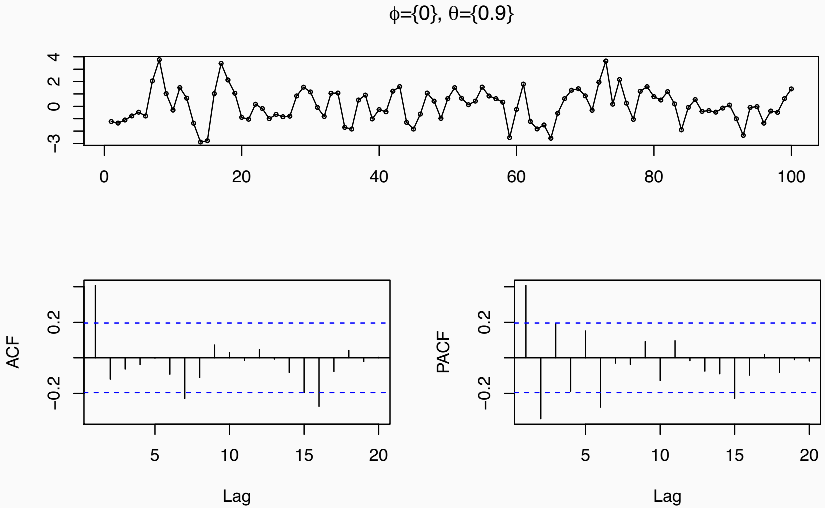
Time series



$\phi = 0.9, \theta = 0$



$\phi = 0, \theta = 0.9$



$\phi = 0.9, \theta = 0.9$