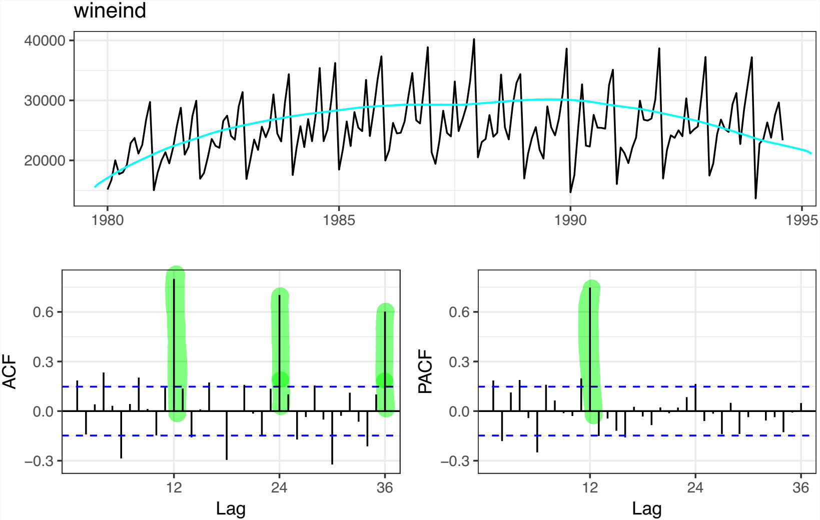
Lecture 10

Seasonal Arima

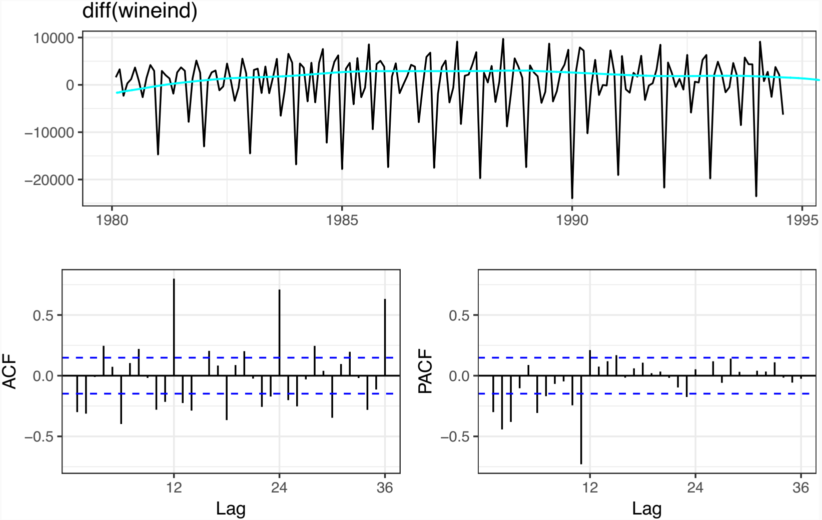
10/05/2018

Australian Wine Sales Example (Lecture 6)

Australian total wine sales by wine makers in bottles <= 1 litre. Jan 1980 – Aug 1994.



Differencing



We can extend the existing Arima model to handle these higher order lags (without having to include all of the intervening lags).

Seasonal ARIMA $(p, d, q) \times (P, D, Q)_s$:

$$\Phi_P(L^s)\, \phi_p(L)\, \Delta_s^D\, \Delta^d\, y_t = \delta + \Theta_Q(L^s)\, \theta_q(L)\, w_t$$

Seasonal Arima

We can extend the existing Arima model to handle these higher order lags (without having to include all of the intervening lags).

Seasonal ARIMA $(p, d, q) \times (P, D, Q)_s$:

$$\Phi_P(L^s)\,\phi_p(L)\,\Delta_s^D\,\Delta^d\,y_t = \delta + \Theta_Q(L^s)\,\theta_q(L)\,w_t$$

where

$$\begin{aligned}
\phi_p(L) &= 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p \\
\theta_q(L) &= 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_p L^q \\
\Delta^d &= (1 - L)^d
\end{aligned}$$

$$\begin{aligned}
\Phi_P(L^s) &= 1 - \Phi_1 L^s - \Phi_2 L^{2s} - \ldots - \Phi_P L^{Ps} \\
\Theta_Q(L^s) &= 1 + \Theta_1 L^{s} + \Theta_2 L^{2s} + \ldots + \theta_p L^{Qs} \\
\Delta_s^D &= (1 - L^s)^D
\end{aligned}$$

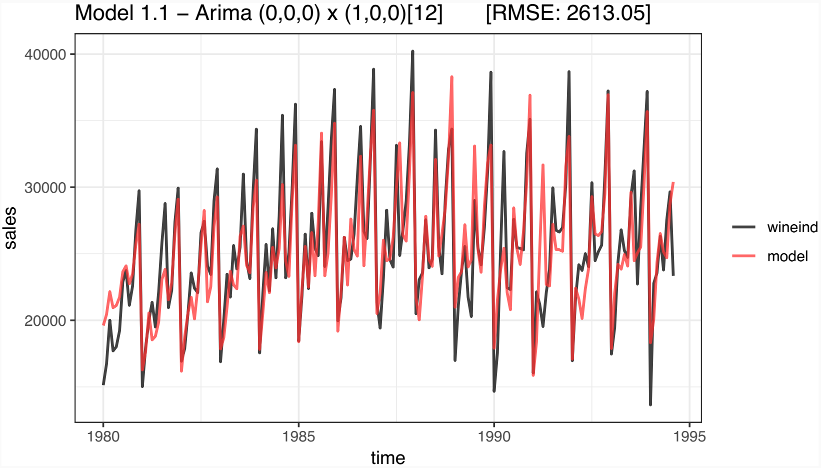
Seasonal Arima for wineind - AR

Lets consider an $\text{ARIMA}(0,0,0) \times (1,0,0)_{12}$:

$$(1 - \Phi_1 L^{12})\, y_t = \delta + w_t$$
$$y_t = \Phi_1 y_{t-12} + \delta + w_t$$

```
(m1.1 = forecast::Arima(wineind, seasonal=list(order=c(1,0,0), period=12)))
## Series: wineind
## ARIMA(0,0,0)(1,0,0)[12] with non-zero mean
##
## Coefficients:
##         sar1      mean
##       0.8780  24489.24
## s.e.  0.0314   1154.48
##
## sigma^2 estimated as 6906536:  log likelihood=-1643.39
## AIC=3292.78   AICc=3292.92   BIC=3302.29
```

Fitted model



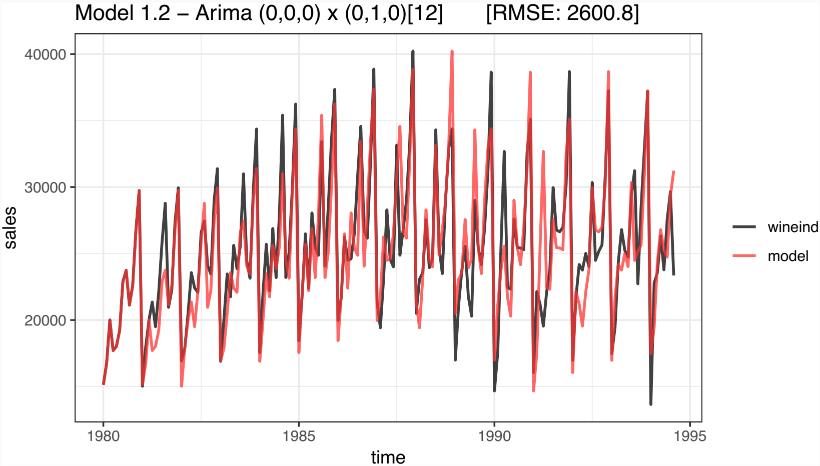
Seasonal Arima for `wineind` - Diff

Lets consider an ARIMA$(0,0,0) \times (0,1,0)_{12}$:

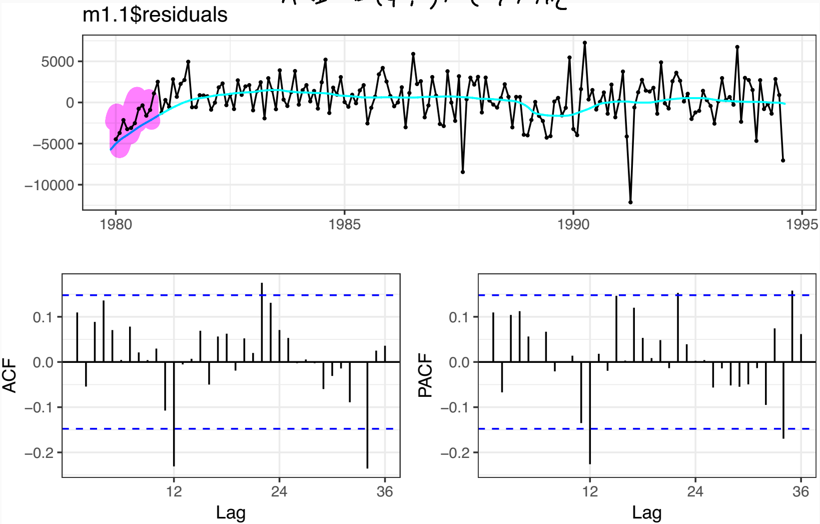
$$
\begin{aligned}
(1 - L^{12})\, y_t &= \delta + w_t \\
y_t &= y_{t-12} + \delta + w_t
\end{aligned}
$$

```
(m1.2 = forecast::Arima(wineind, seasonal=list(order=c(0,1,0), period=12)))
## Series: wineind
## ARIMA(0,0,0)(0,1,0)[12]
##
## sigma^2 estimated as 7259076:  log likelihood=-1528.12
## AIC=3058.24   AICc=3058.27   BIC=3061.34
```

Fitted model

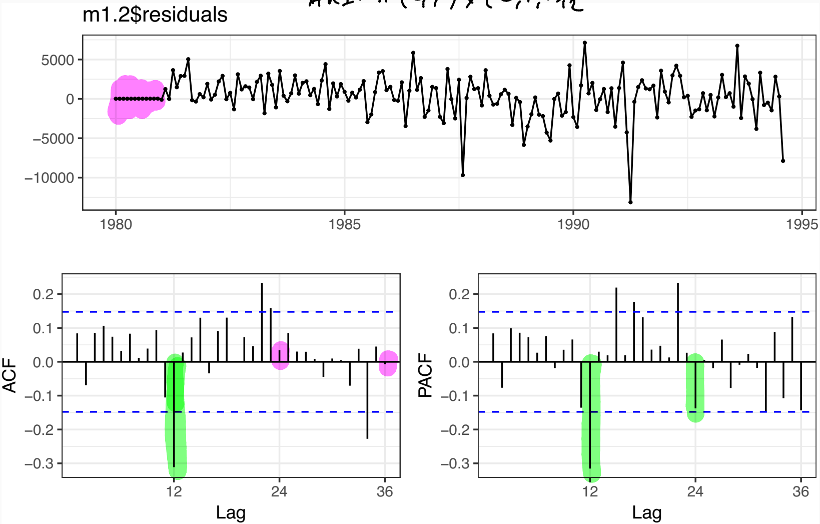


ARIMA (0,0,0) x (1,0,0)$_{12}$



Residuals - Model 1.2

ARIMA (0,0,0) x (0,1,0)$_{12}$



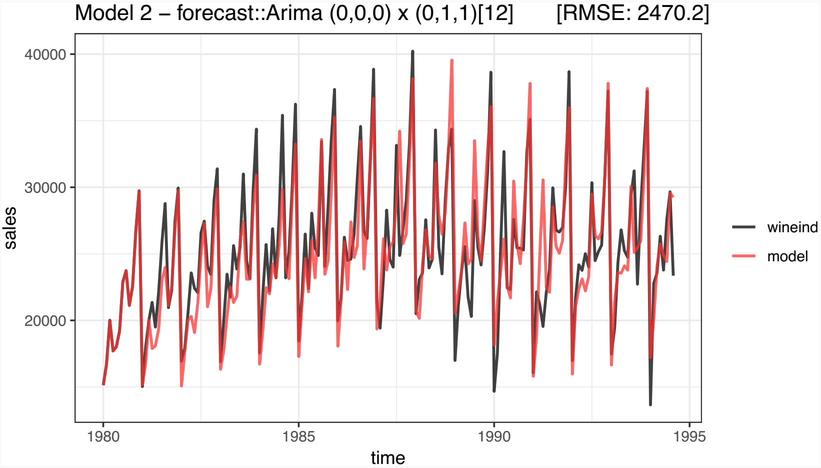
Model 2

ARIMA$(0,0,0) \times (0,1,1)_{12}$:

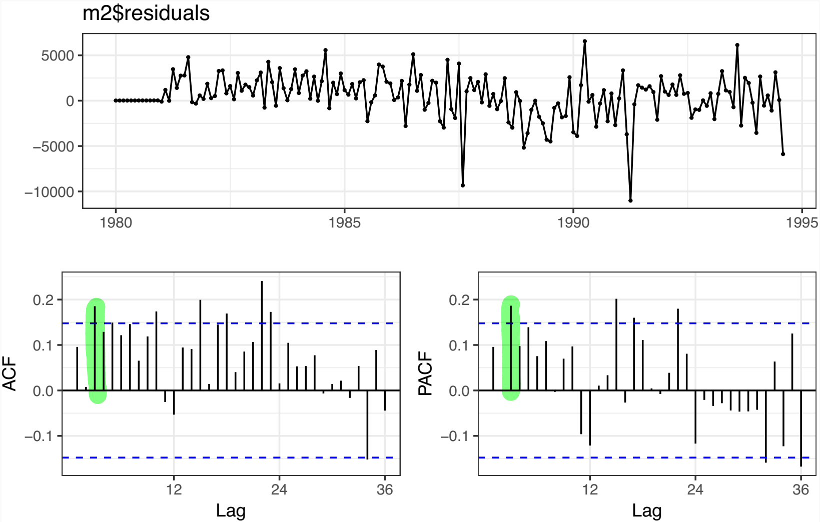
$$\begin{aligned}(1 - L^{12})y_t &= \delta + (1 + \Theta_1 L^{12})w_t \\ y_t - y_{t-12} &= \delta + w_t + \Theta_1 w_{t-12} \\ y_t &= \delta + y_{t-12} + w_t + \Theta_1 w_{t-12}\end{aligned}$$

```
(m2 = forecast::Arima(wineind, order=c(0,0,0),
                      seasonal=list(order=c(0,1,1), period=12)))
## Series: wineind
## ARIMA(0,0,0)(0,1,1)[12]
##
## Coefficients:
##          sma1
##       -0.3246
## s.e.   0.0807
##
## sigma^2 estimated as 6588531:  log likelihood=-1520.34
## AIC=3044.68   AICc=3044.76   BIC=3050.88
```

Fitted model



Residuals

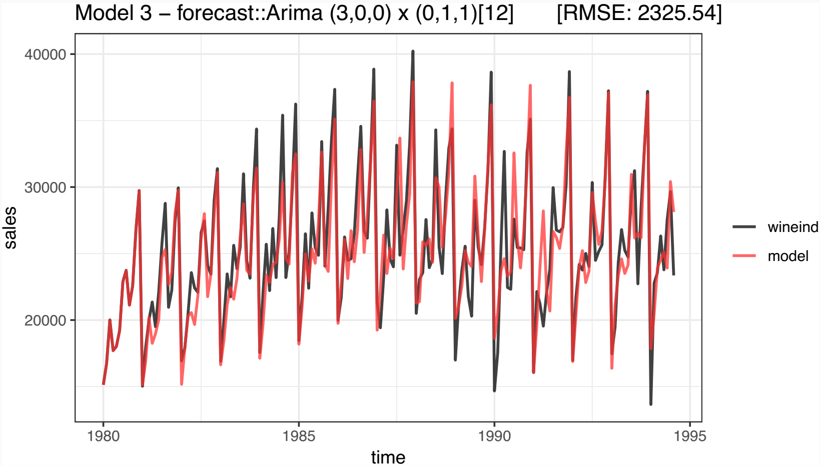

m2$residuals
5000
0
−5000
−10000
1980
1985
1990
1995
ACF
0.2
0.1
0.0
−0.1
12
24
36
Lag
PACF
0.2
0.1
0.0
−0.1
12
24
36
Lag

ARIMA$(3,0,0) \times (0,1,1)_{12}$

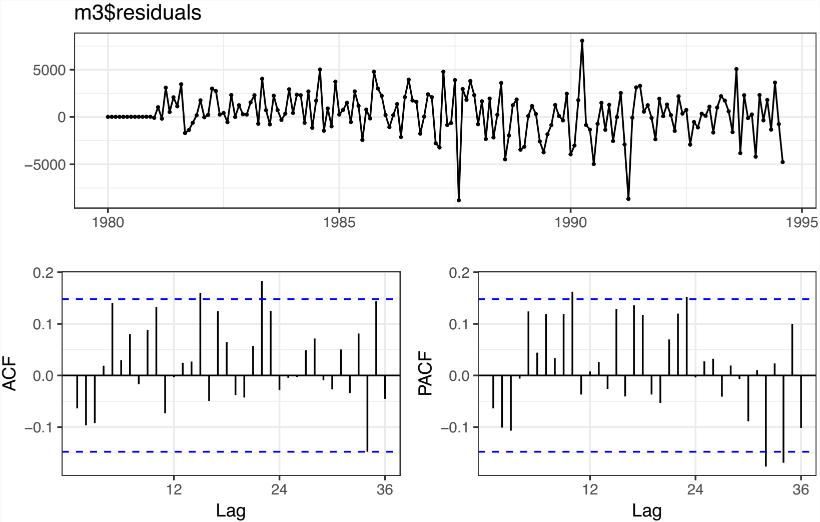
$$(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3)\,(1 - L^{12}) y_t = \delta + (1 + \Theta_1 L) w_t$$
$$(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3)\,(y_t - y_{t-12}) = \delta + w_t + w_{t-12}$$
$$y_t = \delta + \sum_{i=1}^{3} \phi_i y_{t-1} + y_{t-12} - \sum_{i=1}^{3} \phi_i y_{t-12-i} + w_t + w_{t-12}$$

```
(m3 = forecast::Arima(wineind, order=c(3,0,0),
                      seasonal=list(order=c(0,1,1), period=12)))
## Series: wineind
## ARIMA(3,0,0)(0,1,1)[12]
##
## Coefficients:
##          ar1     ar2     ar3     sma1
##       0.1402  0.0806  0.3040  -0.5790
## s.e.  0.0755  0.0813  0.0823   0.1023
##
## sigma^2 estimated as 5948935:  log likelihood=-1512.38
## AIC=3034.77   AICc=3035.15   BIC=3050.27
```

Fitted model



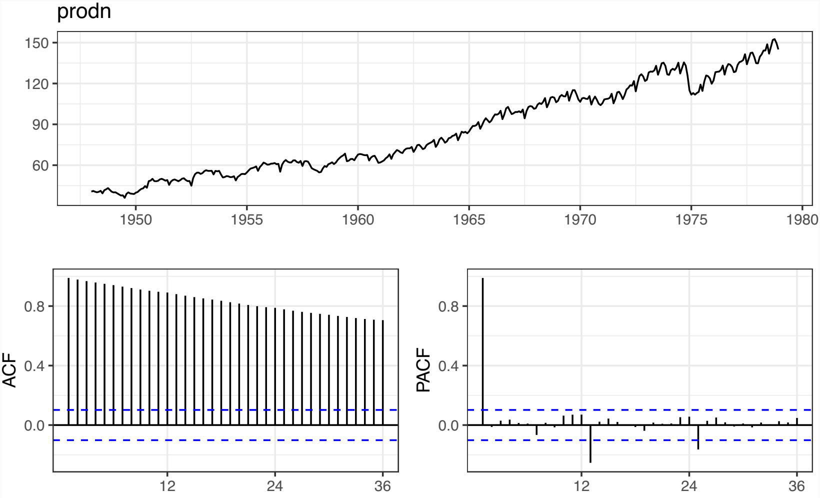
Model - Residuals



prodn from the astsa package

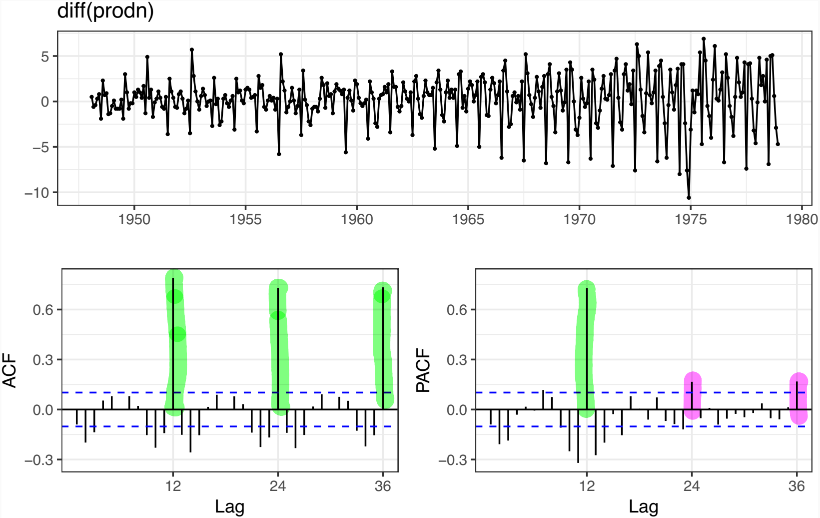
Monthly Federal Reserve Board Production Index (1948-1978)

```
data(prodn, package="astsa"); forecast::ggtsdisplay(prodn, points = FALSE)
```



Differencing

Based on the ACF it seems like standard differencing may be required



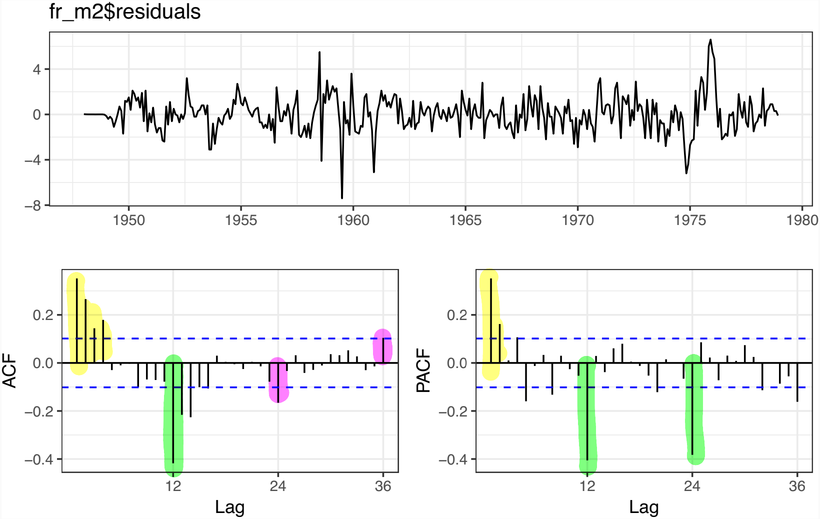
Differencing + Seasonal Differencing

Additional seasonal differencing also seems warranted

```
(fr_m1 = forecast::Arima(prodn, order = c(0,1,0),
            seasonal = list(order=c(0,0,0), period=12)))
## Series: prodn
## ARIMA(0,1,0)
##
## sigma^2 estimated as 7.147:  log likelihood=-891.26
## AIC=1784.51   AICc=1784.52   BIC=1788.43
```

```
(fr_m2 = forecast::Arima(prodn, order = c(0,1,0),
            seasonal = list(order=c(0,1,0), period=12)))
## Series: prodn
## ARIMA(0,1,0)(0,1,0)[12]
##
## sigma^2 estimated as 2.52:  log likelihood=-675.29
## AIC=1352.58   AICc=1352.59   BIC=1356.46
```

Residuals



fr_m2$residuals
4
0
−4
−8
1950
1955
1960
1965
1970
1975
1980
ACF
0.2
0.0
−0.2
−0.4
12
24
36
Lag
PACF
0.2
0.0
−0.2
−0.4
12
24
36
Lag

Adding Seasonal MA

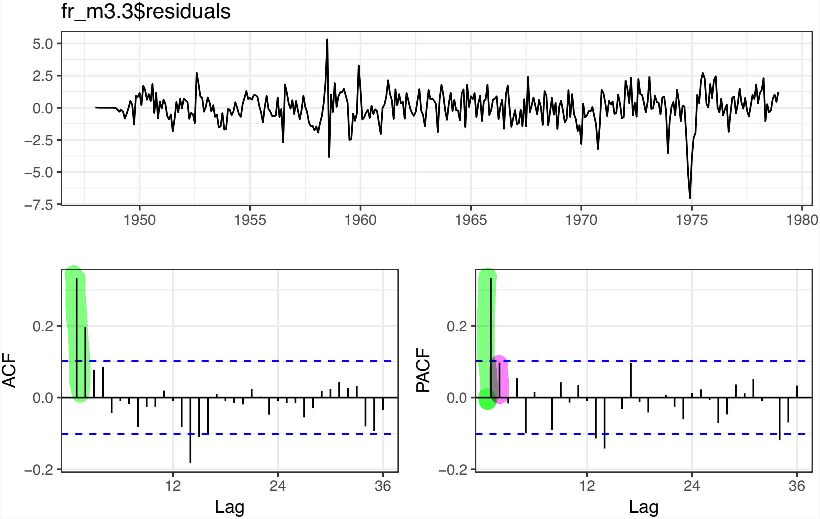
```r
(fr_m3.1 = forecast::Arima(prodn, order = c(0,1,0),
            seasonal = list(order=c(0,1,1), period=12)))
## Series: prodn
## ARIMA(0,1,0)(0,1,1)[12]
##
## Coefficients:
##          sma1
##       -0.7151
## s.e.   0.0317
##
## sigma^2 estimated as 1.616:  log likelihood=-599.29
## AIC=1202.57   AICc=1202.61   BIC=1210.34
```

```r
(fr_m3.2 = forecast::Arima(prodn, order = c(0,1,0),
            seasonal = list(order=c(0,1,2), period=12)))
## Series: prodn
## ARIMA(0,1,0)(0,1,2)[12]
##
## Coefficients:
##          sma1    sma2
##       -0.7624  0.0520
## s.e.   0.0689  0.0666
##
## sigma^2 estimated as 1.615:  log likelihood=-598.98
## AIC=1203.96   AICc=1204.02   BIC=1215.61
```

Adding Seasonal MA (cont.)

```
(fr_m3.3 = forecast::Arima(prodn, order = c(0,1,0),
            seasonal = list(order=c(0,1,3), period=12)))
## Series: prodn
## ARIMA(0,1,0)(0,1,3)[12]
##
## Coefficients:
##          sma1     sma2    sma3
##       -0.7853  -0.1205  0.2624
## s.e.   0.0529   0.0644  0.0529
##
## sigma^2 estimated as 1.506:  log likelihood=-587.58
## AIC=1183.15   AICc=1183.27   BIC=1198.69
```

Residuals - Model 3.3

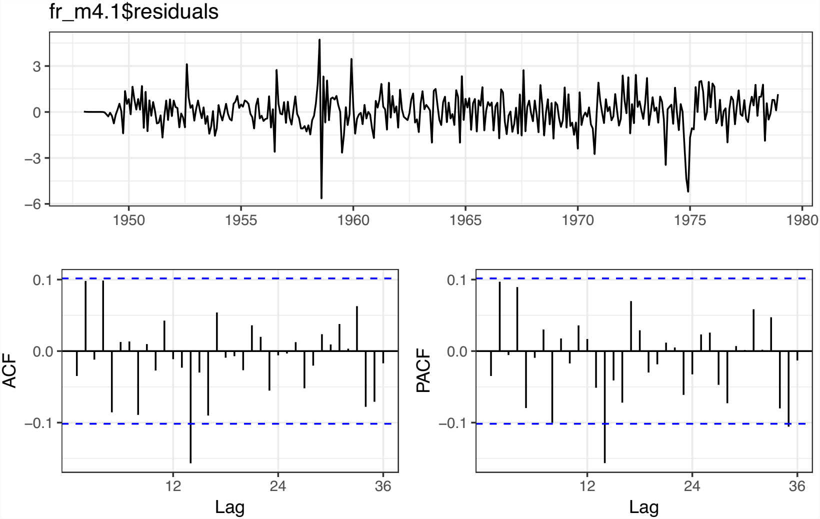


Adding AR

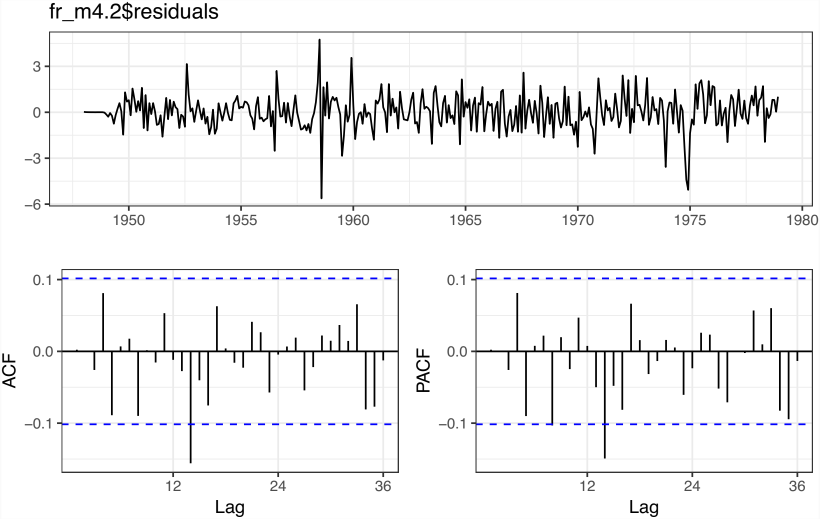
```r
(fr_m4.1 = forecast::Arima(prodn, order = c(1,1,0),
            seasonal = list(order=c(0,1,3), period=12)))
## Series: prodn
## ARIMA(1,1,0)(0,1,3)[12]
##
## Coefficients:
##          ar1     sma1     sma2    sma3
##       0.3393  -0.7619  -0.1222  0.2756
## s.e.  0.0500   0.0527   0.0646  0.0525
##
## sigma^2 estimated as 1.341:  log likelihood=-565.98
## AIC=1141.95   AICc=1142.12   BIC=1161.37
```

```r
(fr_m4.2 = forecast::Arima(prodn, order = c(2,1,0),
            seasonal = list(order=c(0,1,3), period=12)))
## Series: prodn
## ARIMA(2,1,0)(0,1,3)[12]
##
## Coefficients:
##          ar1     ar2     sma1     sma2    sma3
##       0.3038  0.1077  -0.7393  -0.1445  0.2815
## s.e.  0.0526  0.0538   0.0539   0.0653  0.0526
##
## sigma^2 estimated as 1.331:  log likelihood=-563.98
## AIC=1139.97   AICc=1140.2   BIC=1163.26
```

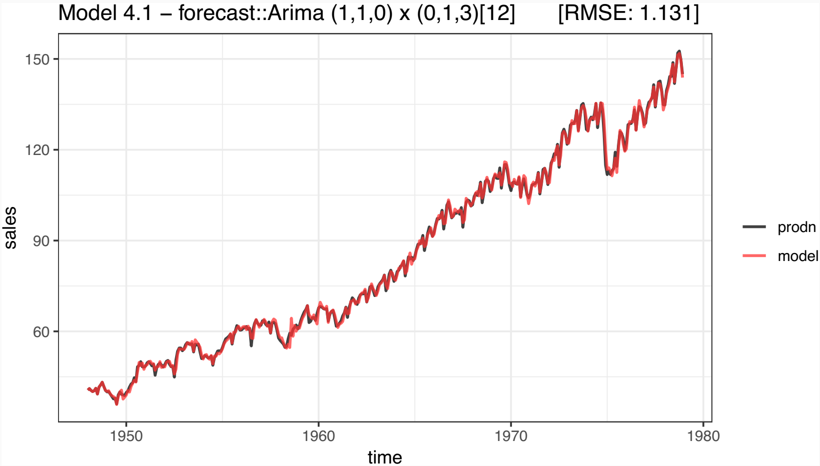
Residuals - Model 4.1



Residuals - Model 4.2

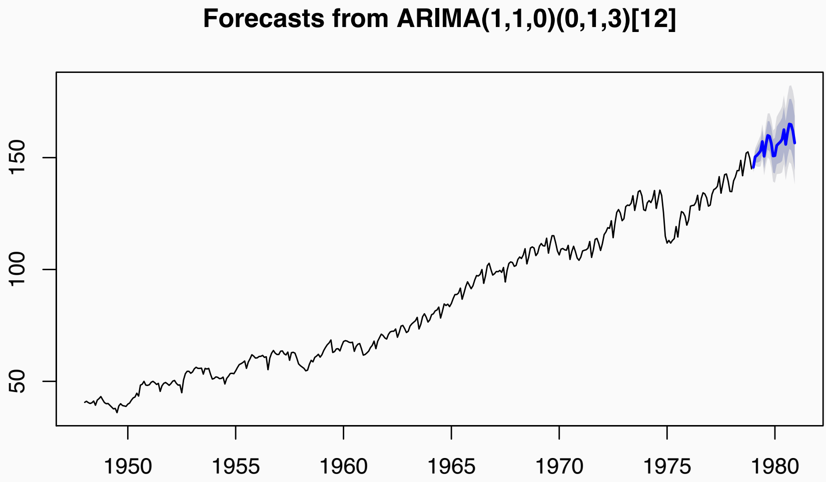


Model Fit



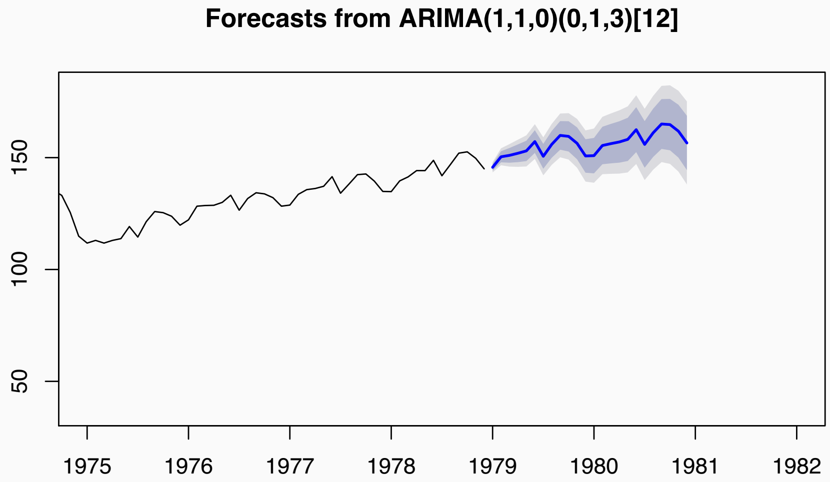
Model 4.1 – forecast::Arima (1,1,0) x (0,1,3)[12]   [RMSE: 1.131]
sales
150
120
90
60
1950
1960
1970
1980
time
prodn
model

```r
forecast::forecast(fr_m4.1) %>% plot()
```



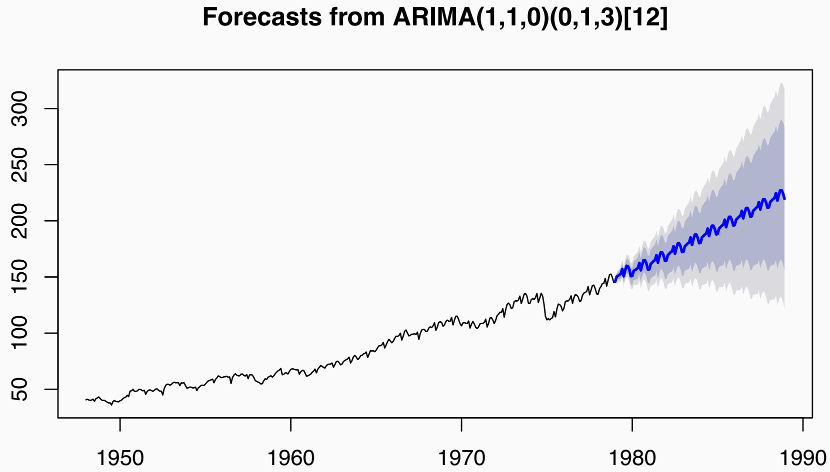
Model Forecast (cont.)

```
forecast::forecast(fr_m4.1) %>% plot(xlim=c(1975,1982))
```



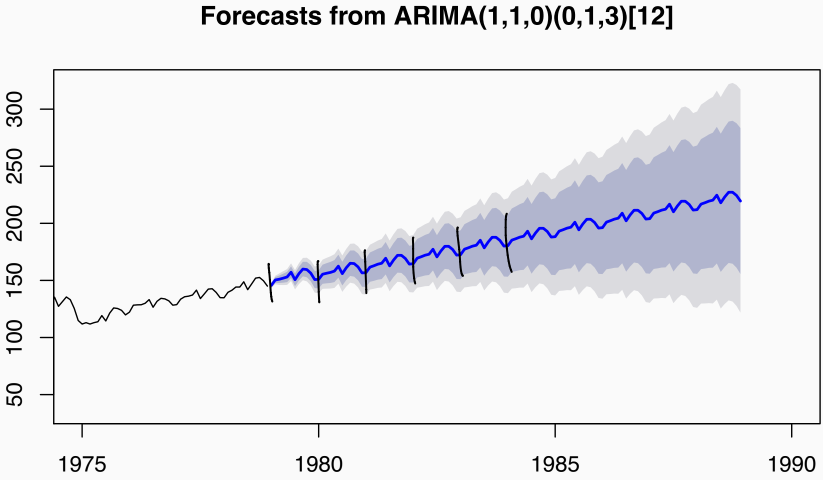
Model Forecast (cont.)

```
forecast::forecast(fr_m4.1, 120) %>% plot()
```



Model Forecast (cont.)

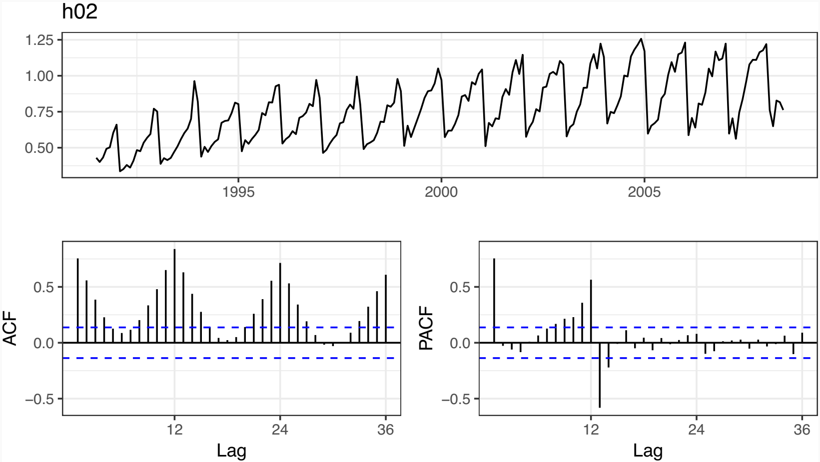
```
forecast::forecast(fr_m4.1, 120) %>% plot(xlim=c(1975,1990))
```



Exercise - Cortecosteroid Drug Sales

Monthly cortecosteroid drug sales in Australia from 1992 to 2008.

```
data(h02, package="fpp")
forecast::ggtsdisplay(h02,points=FALSE)
```



Forecasting

Forecasting ARMA

Forecasts for stationary models necessarily revert to mean

- Remember, $E(y_t) \neq \delta$ but rather $\delta/(1 - \sum_{i=1}^{p} \phi_i)$.
- Differenced models revert to trend (usually a line)
- Why? AR gradually damp out, MA terms disappear

Like any other model, accuracy decreases as we extrapolate and the prediction interval expands rapidly

Take a fitted ARMA(1,1) process where we know δ, ϕ, and θ then

$$y_t = \delta + \phi\, y_{t-1} + w_t + \theta w_{t-1}$$

$$E(w_t \mid y_1 \dots y_t\ \theta, \phi, \delta, \sigma_w^2)$$

$$E(y_t \mid \dots) = \delta + \phi y_{t-1} + E(w_t \mid \dots) + \theta E(w_{t-1} \mid \dots)$$

$$y_t = \delta + \phi y_{t-1} + E(w_t \mid \dots) + \theta E(w_{t-1} \mid \dots)$$

$$E(w_t \mid \dots) = y_t - (\delta + \phi y_{t-1} + \theta E(w_{t-1} \mid \dots))$$

$$= y_t - (\delta + \phi y_{t-1} + \theta w_{t-1})$$

$$E(w_{t+1} \mid y_1 \dots y_t\ \theta \phi \sigma_w \delta) = 0$$

$$E(y_{t+h} \mid y_t \dots y_1, \theta, \phi, \sigma_w^2) = ?$$

Take a fitted ARMA(1,1) process where we know δ, ϕ, and θ then

$$E(y_{t+1} \mid y_t \cdots y_1, \theta, \phi, \sigma_w^2) = E(\delta + \phi y_t + \theta w_t + w_{t+1} \mid \cdots)$$

$$= \delta + \phi y_t + \theta E(w_t \mid \cdots) + E(w_{t+1} \mid \cdots)$$

$$= \delta + \phi y_t + \theta E(w_t \mid \cdots)$$

$$E(y_{t+2} \mid \cdots) = E(\delta + \phi y_{t+1} + \theta w_{t+1} + w_{t+2} \mid \cdots)$$

$$= \delta + \phi E(y_{t+1} \mid \cdots) + 0 + 0$$

$$= \delta + \phi(\delta + \phi y_t + \theta E(w_t \mid \cdots))$$

$$= \delta + \phi\delta + \phi^2 y_t + \phi\theta E(w_t \mid \cdots)$$

Take a fitted ARMA(1,1) process where we know δ, ϕ, and θ then

$$E(y_{t+h} \mid \dots) = \delta \sum_{i=0}^{h-1} \phi^i + \phi^h y_t + \phi^{h-1} \theta E(w_t \mid \dots)$$

$$\lim_{h \to \infty} E(y_{t+h} \mid \dots) = \delta \sum_{i=0}^{\infty} \phi^i = \delta \left(\frac{1}{1-\phi} \right)$$

$$= \frac{\delta}{1-\phi}$$

Process is stationary

$$\phi_3(L)\,\Delta^1 y_t = \theta_1(L)\,v_t + \delta$$

$$(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3)(1 - L)\, y_t = (1 + \theta L) v_t + \delta$$

$$(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3)(y_t - y_{t-1}) = v_t + \theta v_{t-1} + \delta$$

$$y_t - (1 + \phi_1) y_{t-1} + (\phi_1 - \phi_2) y_{t-2} + (\phi_2 - \phi_3) y_{t-3} + \phi_4 y_{t-4} = v_t + \theta v_{t-1} + \delta$$

$$y_t = (1 + \phi_1) y_{t-1} - (\phi_1 - \phi_2) y_{t-2} - (\phi_2 - \phi_3) y_{t-3} - \phi_4 y_{t-4} + v_{t-1} + v_{t-1}\delta$$

$$E(y_{t+1} \mid y_1 \ldots y_t\, \underline{\theta}) =$$

$$(1 + \phi_1) y_t - (\phi_1 - \phi_2) y_{t-1} - (\phi_2 - \phi_3) y_{t-2} - \phi_4 y_{t-3} + \theta E(v_{t}\mid\ldots) + \delta$$