Lecture 19

Fitting CAR and SAR Models

Colin Rundel

03/29/2017

Fitting areal models

CAR vs SAR

- Simultaneous Autogressve (SAR)

$$y(s_i) = \phi \sum_{j=1}^{n} W_{ij}\, y(s_j) + \epsilon$$

$$\boldsymbol{y} \sim \mathcal{N}(0,\ \sigma^2\, ((\boldsymbol{I} - \phi \boldsymbol{W})^{-1})((\boldsymbol{I} - \phi \boldsymbol{W})^{-1})^t)$$

- Conditional Autoregressive (CAR)

$$y(s_i)|\boldsymbol{y}_{-s_i} \sim \mathcal{N}\left(\phi \sum_{j=1}^{n} W_{ij}\, y(s_j),\ \sigma^2\right)$$

$$\boldsymbol{y} \sim \mathcal{N}(0,\ \sigma^2\, (\boldsymbol{I} - \phi \boldsymbol{W})^{-1})$$

Some specific generalizations

Generally speaking we will want to work with a scaled / normalized version of the weight matrix,

$$\frac{W_{ij}}{W_{i\cdot}}$$

When W is an adjacency matrix we can express this as

$$\boldsymbol{D}^{-1}\boldsymbol{W}$$

where $\boldsymbol{D} = \text{diag}(m_i)$ and $m_i = |N(s_i)|$.

We can also allow σ^2 to vary between locations, we can define this as $\boldsymbol{D}_\tau = \text{diag}(1/\sigma_i^2)$ and most often we use

$$\boldsymbol{D}_\tau = \text{diag}\left(\frac{1}{\sigma^2/|N(s_i)|}\right) = \boxed{\boldsymbol{D}/\sigma^2}$$

Revised CAR Model

- Conditional Model

$$y(s_i)|\boldsymbol{y}_{-s_i} \sim \mathcal{N}\left(\underline{X_{i\cdot}\beta} + \phi \sum_{j=1}^{n} \frac{W_{ij}}{D_{ii}}\,\big(y(s_j) - X_{j\cdot}\beta\big),\ \sigma^2 D_{ii}^{-1}\right)$$

- Joint Model

$$\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{X}\boldsymbol{\beta},\ \Sigma_{CAR})$$

$$\begin{aligned}
\Sigma_{CAR} &= (\boldsymbol{D}_{\sigma}(\boldsymbol{I} - \phi \boldsymbol{D}^{-1}\boldsymbol{W}))^{-1} \\
&= \left(1/\sigma^2 \boldsymbol{D}\,(\boldsymbol{I} - \phi \boldsymbol{D}^{-1}\boldsymbol{W})\right)^{-1} \\
&= \left(1/\sigma^2(\boldsymbol{D} - \phi \boldsymbol{W})\right)^{-1} \\
&= \sigma^2(\boldsymbol{D} - \phi \boldsymbol{W})^{-1}
\end{aligned}$$

Revised SAR Model

- Formula Model

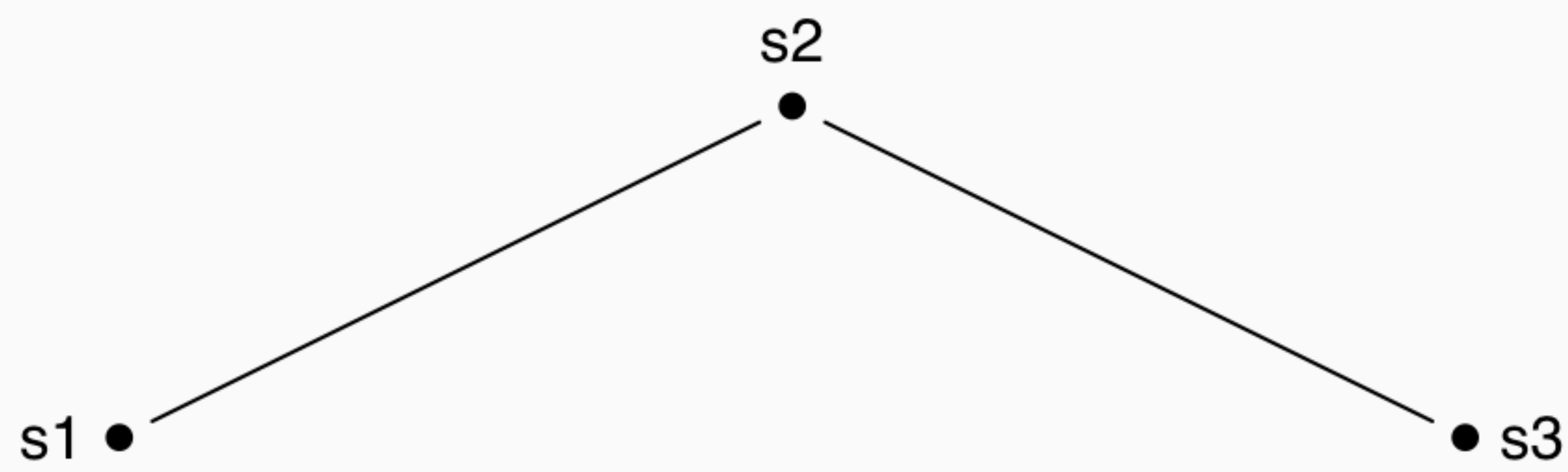
$$y(s_i) = X_{i\cdot}\beta + \phi \sum_{j=1}^{n} D_{jj}^{-1} W_{ij} \underbrace{\left(y(s_j) - X_{j\cdot}\beta\right)} + \epsilon_i$$

- Joint Model

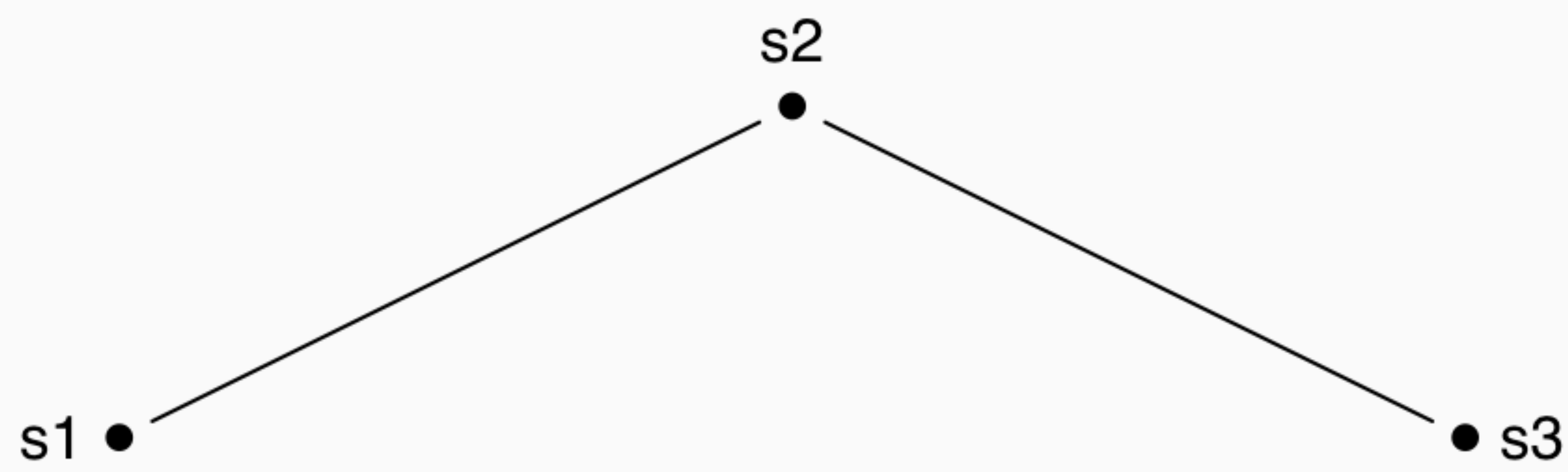
$$\begin{aligned}
\mathbf{y} &= X\boldsymbol{\beta} + \phi D^{-1}W\left(\mathbf{y} - X\boldsymbol{\beta}\right) + \boldsymbol{\epsilon} \\
\left(\mathbf{y} - X\boldsymbol{\beta}\right) &= \phi D^{-1}W\left(\mathbf{y} - X\boldsymbol{\beta}\right) + \boldsymbol{\epsilon} \\
\left(\mathbf{y} - X\boldsymbol{\beta}\right)(I - \phi D^{-1}W)^{-1} &= \boldsymbol{\epsilon} \\
\mathbf{y} &= X\boldsymbol{\beta} + \underbrace{(I - \phi D^{-1}W)}^{-1}\boldsymbol{\epsilon}
\end{aligned}$$

$$\mathbf{y} \sim \mathcal{N}\left(X\boldsymbol{\beta}, (I - \phi D^{-1}W)^{-1}\sigma^2 D^{-1}\left(\left(I - \phi D^{-1}W\right)^{-1}\right)^t\right)$$

Toy CAR Example



Toy CAR Example



$$W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{\Sigma} = \sigma^2 \left(D - \phi W\right)^{-1} = \sigma^2 \underbrace{\begin{pmatrix} 1 & -\phi & 0 \\ -\phi & 2 & -\phi \\ 0 & -\phi & 1 \end{pmatrix}}^{-1}$$

$x^t \Sigma x \geq 0$

When does Σ exist?

```
check_sigma = function(phi) {
  Sigma_inv = matrix(c(1,-phi,0,-phi,2,-phi,0,-phi,1), ncol=3, byrow=TRUE)
  solve(Sigma_inv)
}

check_sigma(phi=0)
##      [,1] [,2] [,3]
## [1,]    1  0.0    0
## [2,]    0  0.5    0
## [3,]    0  0.0    1

check_sigma(phi=0.5)
##           [,1]      [,2]      [,3]
## [1,] 1.1666667 0.3333333 0.1666667
## [2,] 0.3333333 0.6666667 0.3333333
## [3,] 0.1666667 0.3333333 1.1666667

check_sigma(phi=-0.6)
##          [,1]     [,2]     [,3]
## [1,]  1.28125 -0.46875  0.28125
## [2,] -0.46875  0.78125 -0.46875
## [3,]  0.28125 -0.46875  1.28125
```

```r
check_sigma(phi=1)
## Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactl

check_sigma(phi=-1)
## Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactl

check_sigma(phi=1.2)
##            [,1]      [,2]       [,3]
## [1,] -0.6363636 -1.363636 -1.6363636
## [2,] -1.3636364 -1.136364 -1.3636364
## [3,] -1.6363636 -1.363636 -0.6363636

check_sigma(phi=-1.2)
##            [,1]      [,2]       [,3]
## [1,] -0.6363636  1.363636 -1.6363636
## [2,]  1.3636364 -1.136364  1.3636364
## [3,] -1.6363636  1.363636 -0.6363636
```

$\boxed{-1 < \phi < 1}$

Conclusions

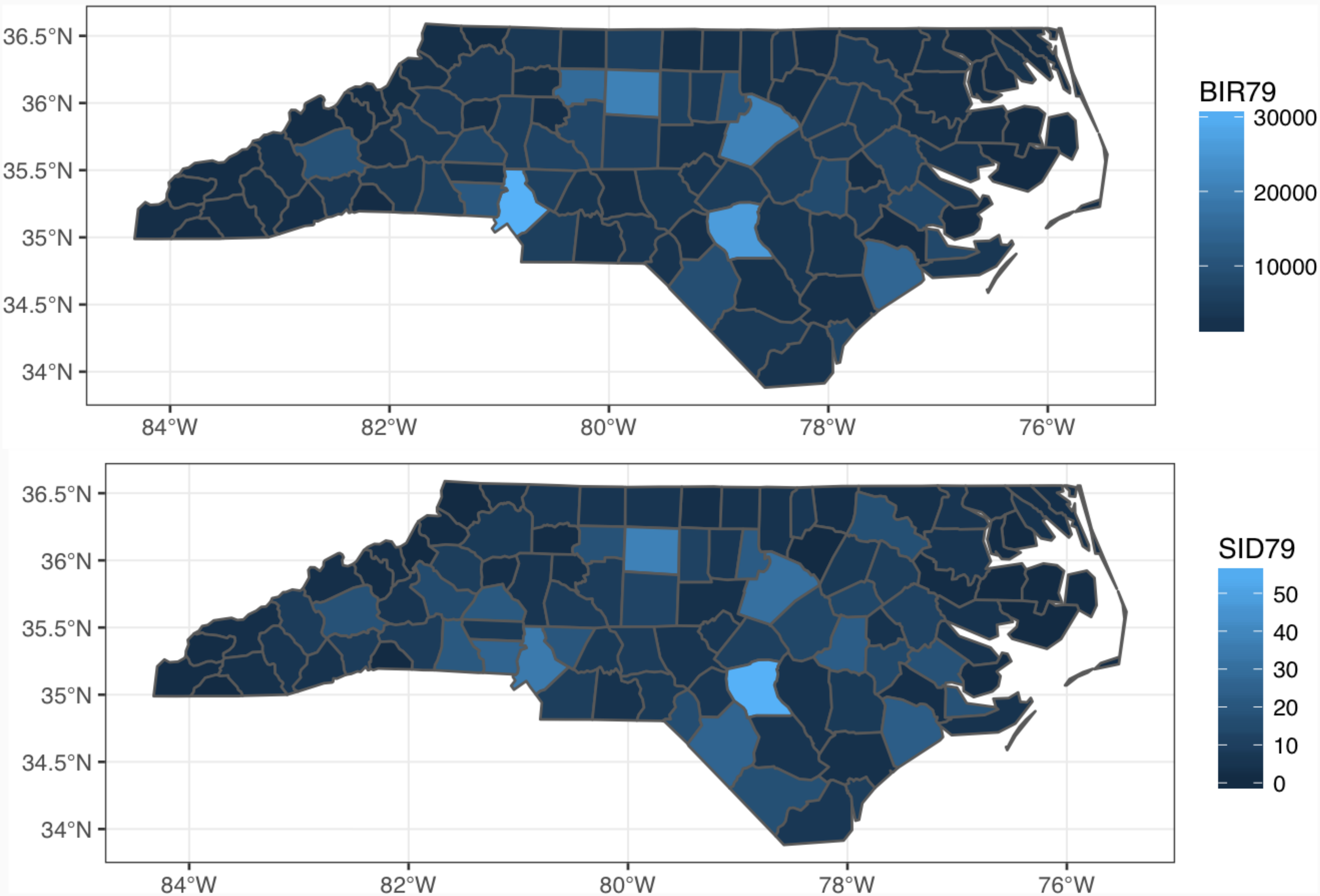
Generally speaking just like the AR(1) model for time series we require that $|\phi| < 1$ for the CAR model to be proper.

These results for ϕ also apply in the context where σ_i^2 is constant across locations (i.e. $\boldsymbol{\Sigma} = \left(\sigma^2 \left(\boldsymbol{I} - \phi \boldsymbol{D}^{-1} \boldsymbol{W}\right)\right)^{-1}$)

As a side note, the special case where $\phi = 1$ is known as an intrinsic autoregressive (IAR) model and they are popular as an *improper* prior for spatial random effects. An additional sum constraint is necessary for identifiability ($\sum i = 1^n y(s_i) = 0$).

$$y \sim N\left(0, \sigma^2 (D - \phi W)^{-1}\right)$$

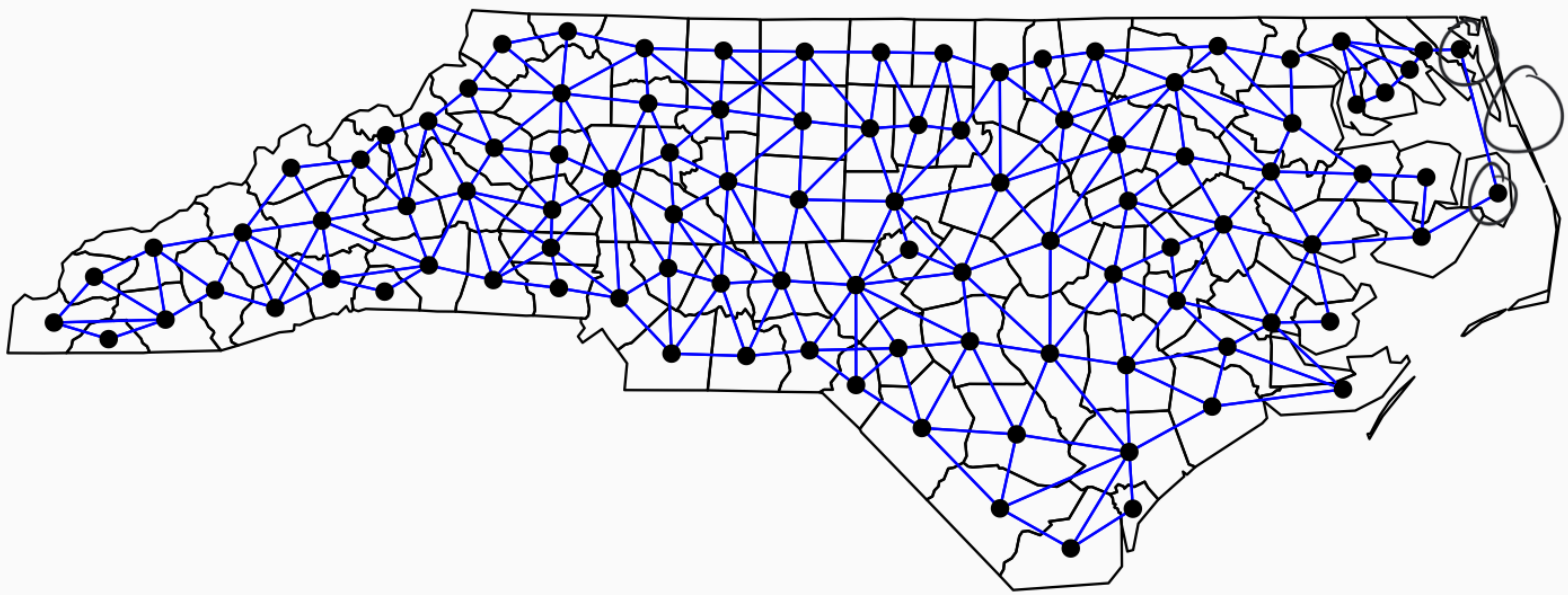
Example - NC SIDS



Using spautolm from spdep

```
library(spdep)

W = st_touches(nc, sparse=FALSE)
listW = mat2listw(W)

listW
## Characteristics of weights list object:
## Neighbour list object:
## Number of regions: 100
## Number of nonzero links: 490
## Percentage nonzero weights: 4.9
## Average number of links: 4.9
##
## Weights style: M
## Weights constants summary:
##     n    nn  S0  S1    S2
## M 100 10000 490 980 10696
```

```r
nc_coords = nc %>% st_centroid() %>% st_coordinates()

plot(st_geometry(nc))
plot(listW, nc_coords, add=TRUE, col="blue", pch=16)
```



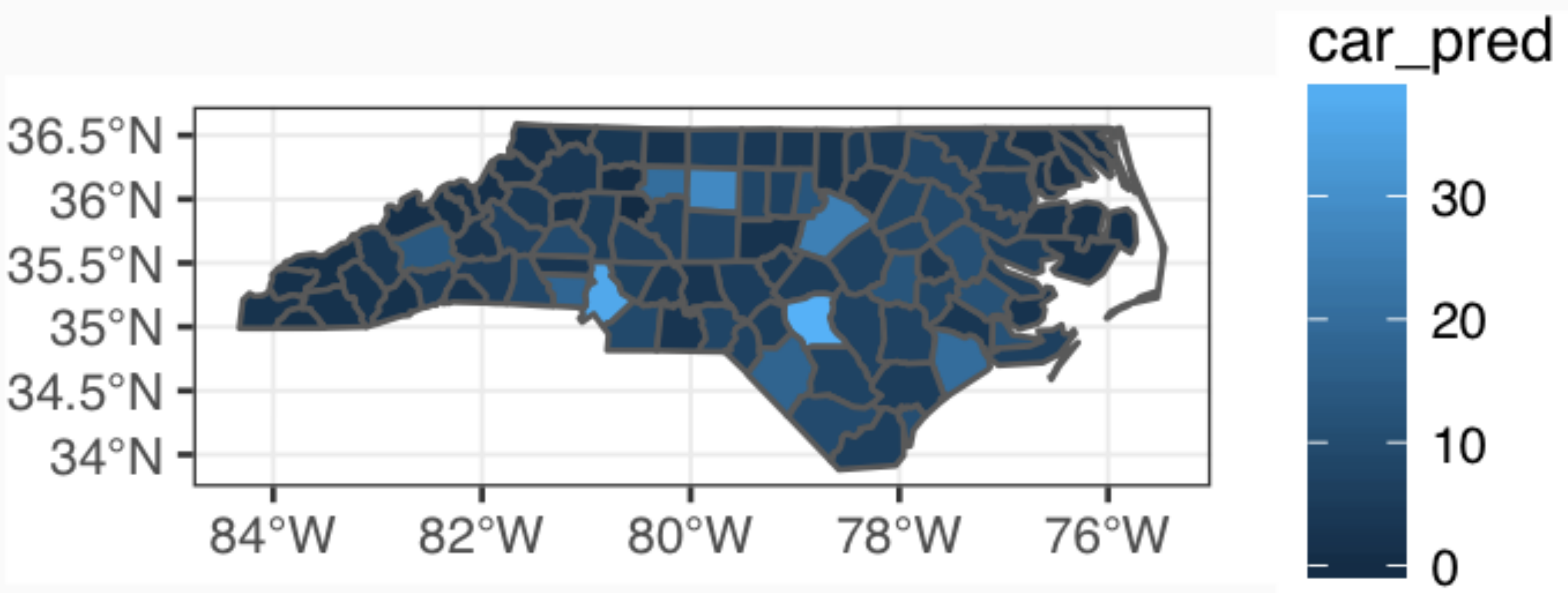
CAR Model

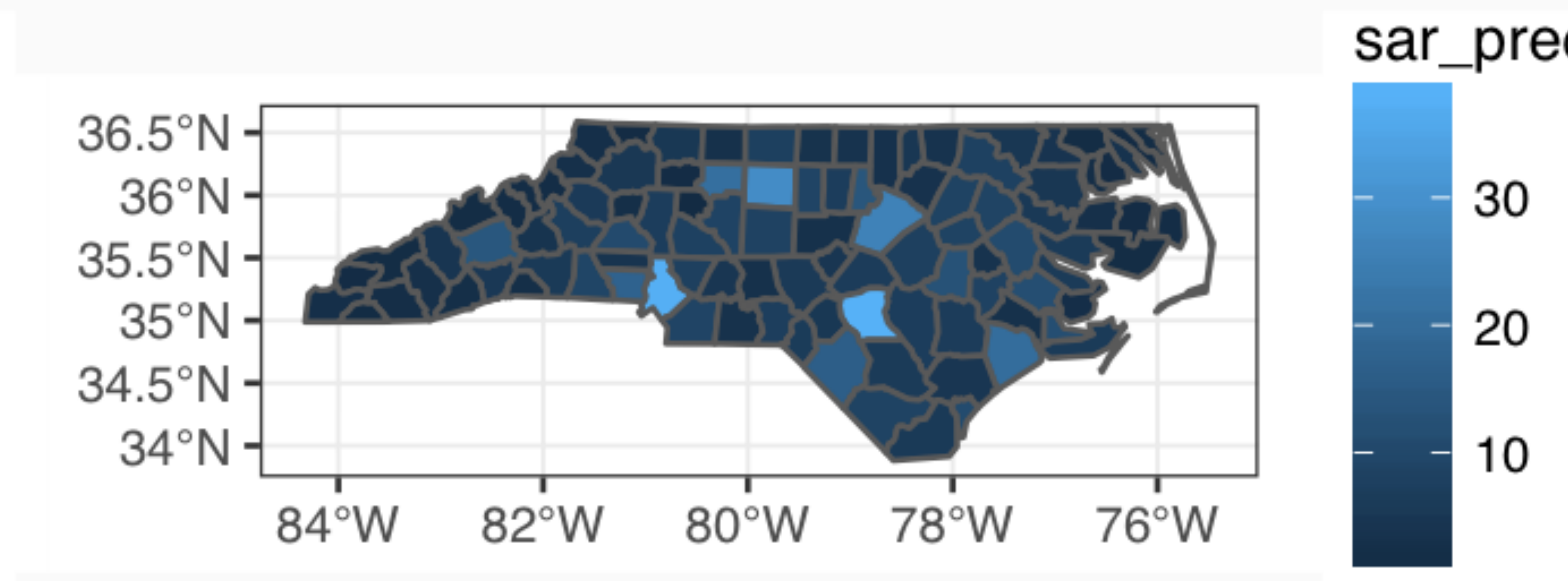
```
nc_car = spautolm(formula = SID74 ~ BIR74, data = nc,
                  listw = listW, family = "CAR")

summary(nc_car)
##
## Call:
## spautolm(formula = SID74 ~ BIR74, data = nc, listw = listW, family = "CAR")
##
## Residuals:
##       Min         1Q    Median        3Q       Max
## -10.38934  -1.58600  -0.52154   1.14729  13.54059
##
## Coefficients:
##                Estimate Std. Error z value Pr(>|z|)
## (Intercept) 1.06911902 0.67501301  1.5838   0.1132
## BIR74       0.00175249 0.00010107 17.3401   <2e-16
##
## Lambda: 0.13222 LR test value: 8.8654 p-value: 0.0029062
## Numerical Hessian standard error of lambda: 0.030094
##
## Log likelihood: -275.7655
## ML residual variance (sigma squared): 13.695, (sigma: 3.7007)
## Number of observations: 100
## Number of parameters estimated: 4
## AIC: 559.53
```

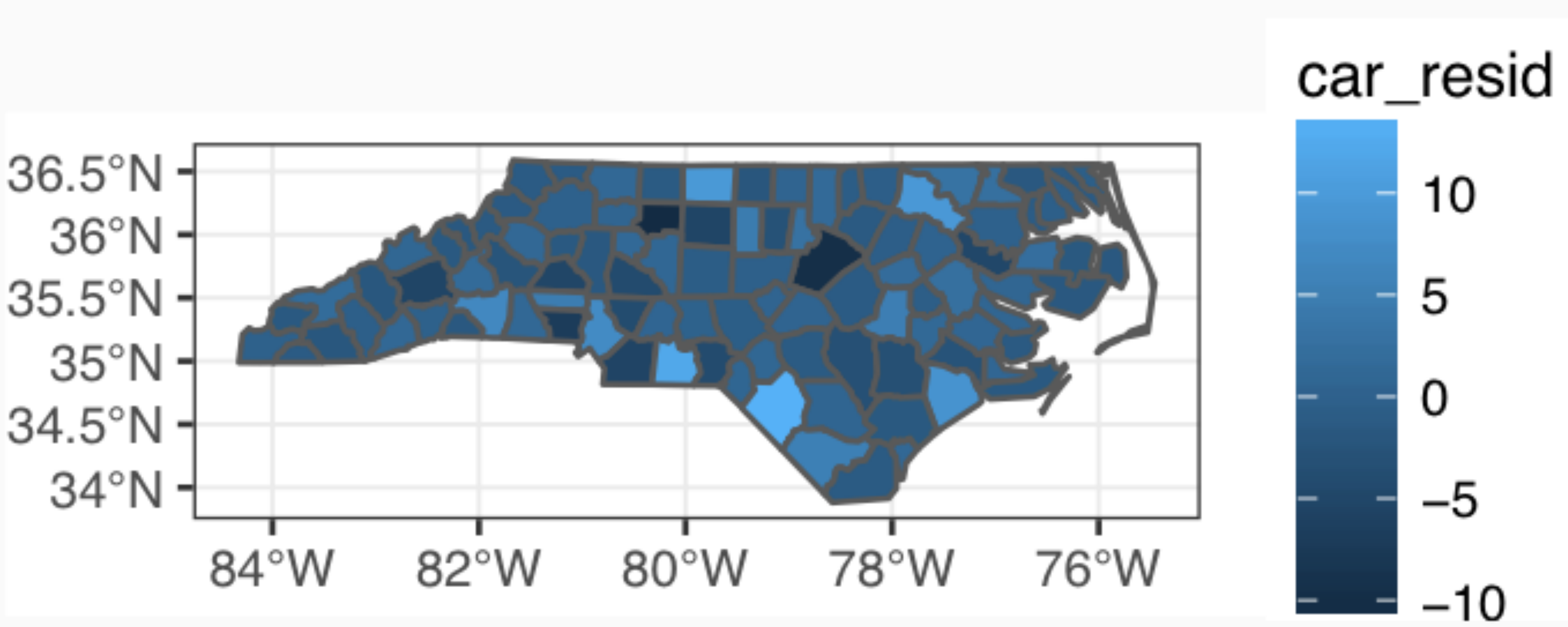
SAR Model

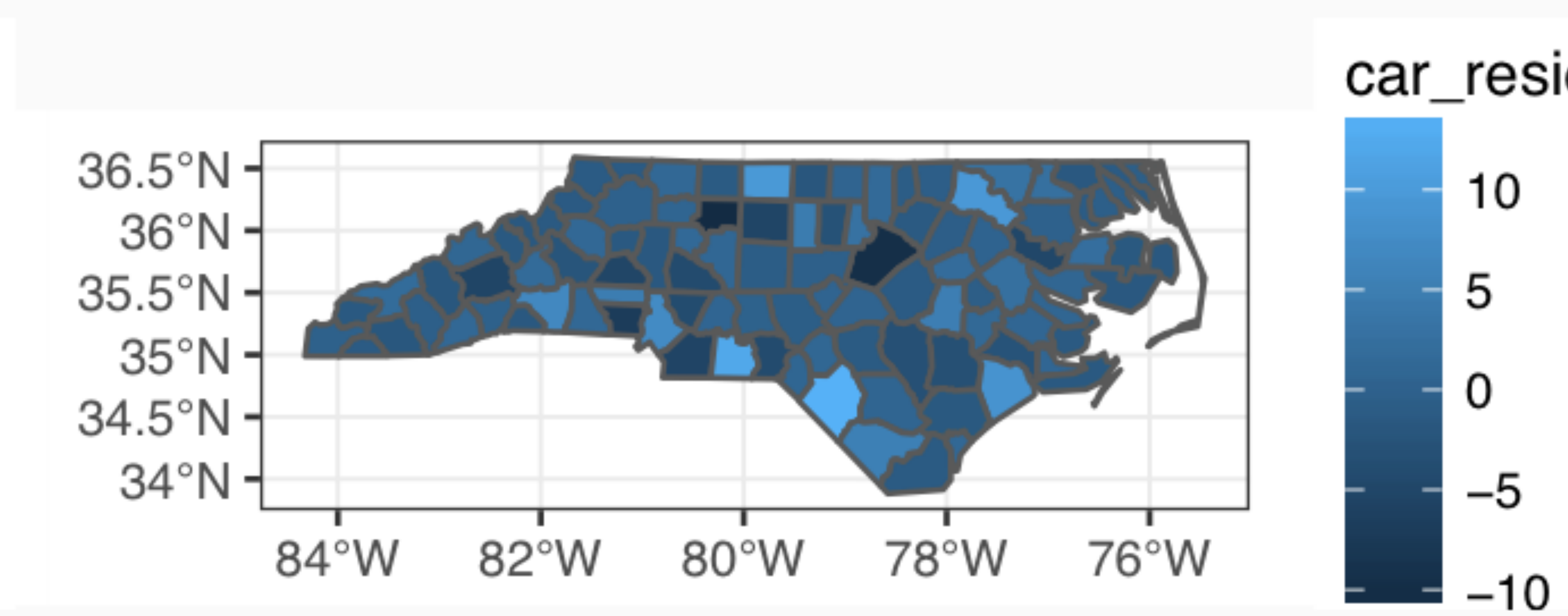
```
nc_sar = spautolm(formula = SID74 ~ BIR74, data = nc,
                  listw = listW, family = "SAR")

summary(nc_sar)
##
## Call:
## spautolm(formula = SID74 ~ BIR74, data = nc, listw = listW, family = "SAR")
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -10.94771  -1.72354  -0.56866   1.23273  14.70511
##
## Coefficients:
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept) 1.01971585 0.64910408   1.571   0.1162
## BIR74       0.00174741 0.00010105  17.292   <2e-16
##
## Lambda: 0.075265 LR test value: 8.4013 p-value: 0.0037495
## Numerical Hessian standard error of lambda: 0.024085
##
## Log likelihood: -275.9975
## ML residual variance (sigma squared): 14.158, (sigma: 3.7627)
## Number of observations: 100
## Number of parameters estimated: 4
## AIC: 560
```

Residuals

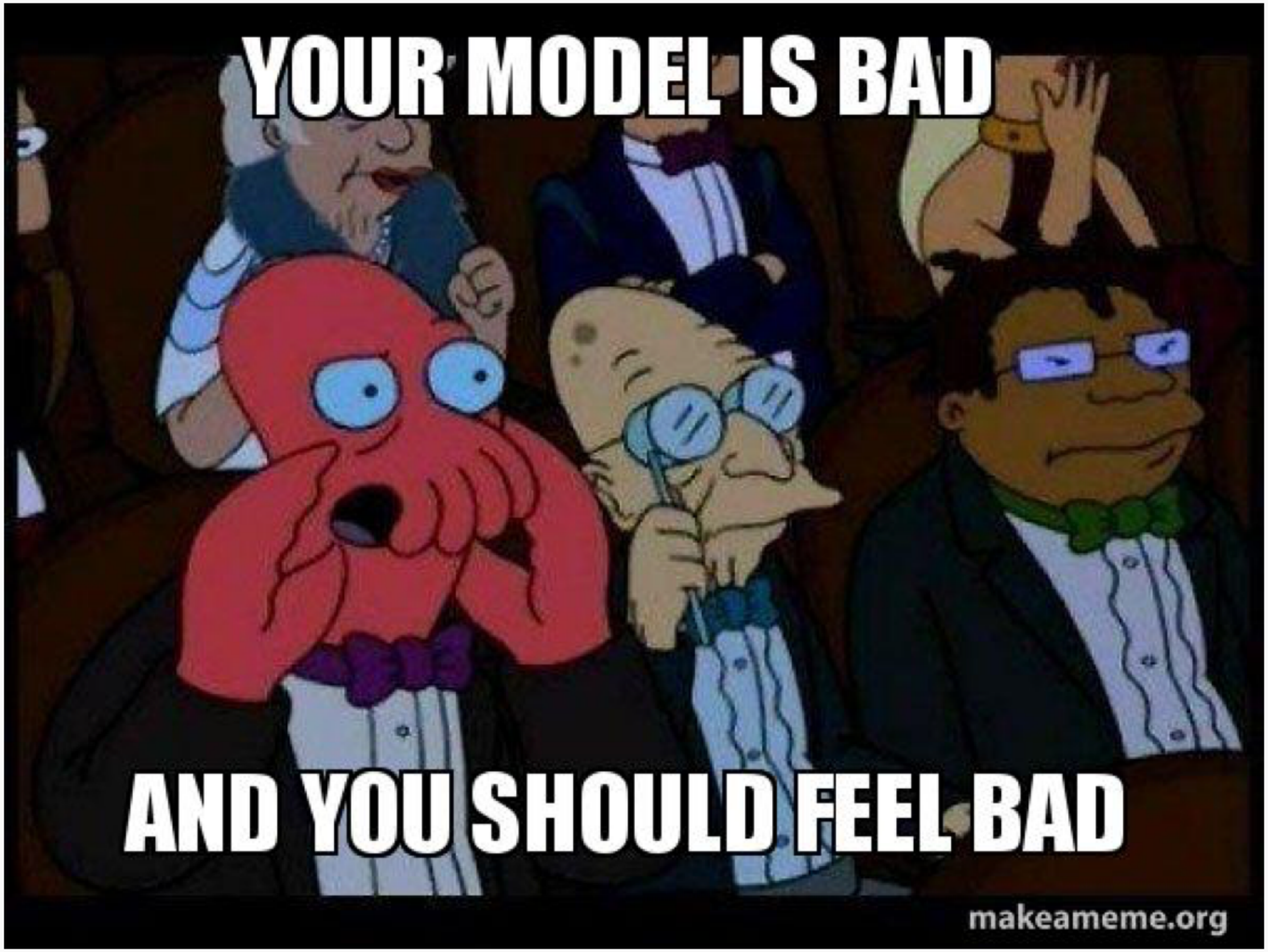

car_pred
36.5°N
36°N
35.5°N
35°N
34.5°N
34°N
84°W
82°W
80°W
78°W
76°W
30
20
10
0


sar_pred
36.5°N
36°N
35.5°N
35°N
34.5°N
34°N
84°W
82°W
80°W
78°W
76°W
30
20
10

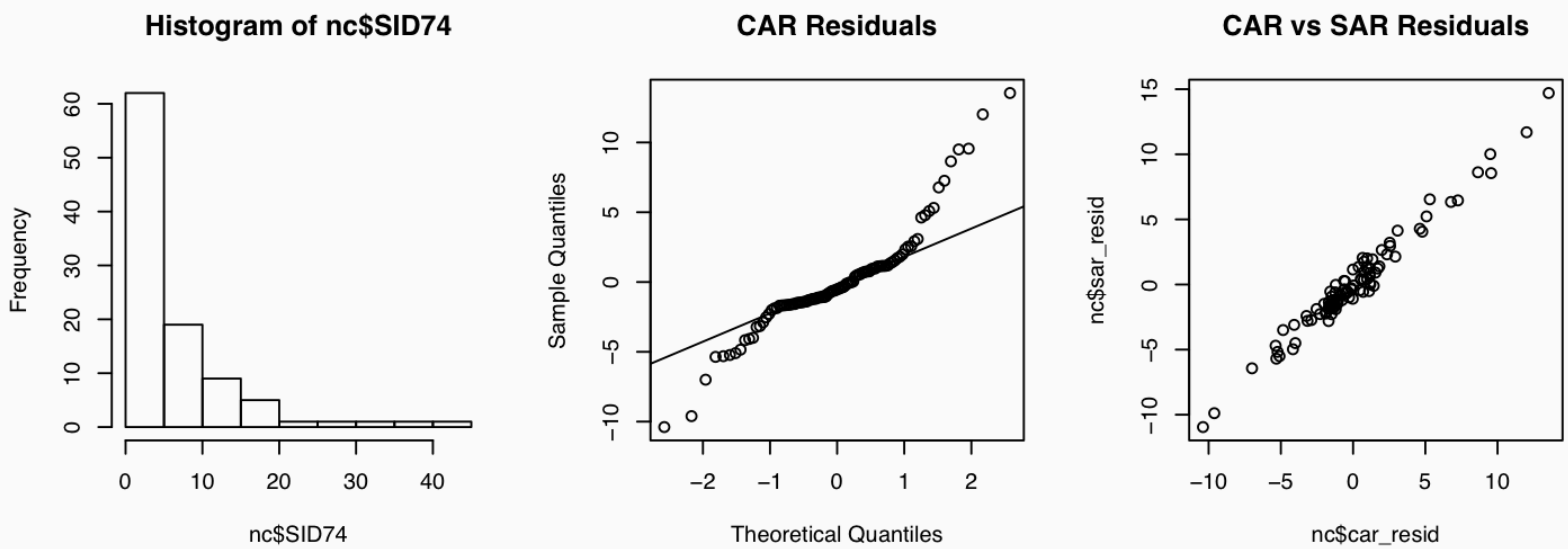

car_resid
36.5°N
36°N
35.5°N
35°N
34.5°N
34°N
84°W
82°W
80°W
78°W
76°W
10
5
0
−5
−10


car_resid
36.5°N
36°N
35.5°N
35°N
34.5°N
34°N
84°W
82°W
80°W
78°W
76°W
10
5
0
−5
−10

I agree ...



Why?



Histogram of nc$SID74
Frequency
nc$SID74
CAR Residuals
Sample Quantiles
Theoretical Quantiles
CAR vs SAR Residuals
nc$sar_resid
nc$car_resid

Jags CAR Model

```
## model{
##   y ~ dmnorm(beta0 + beta1*x, tau * (D - phi*W))
##   y_pred ~ dmnorm(beta0 + beta1*x, tau * (D - phi*W))
##
##   beta0 ~ dnorm(0, 1/100)
##   beta1 ~ dnorm(0, 1/100)
##
##   tau <- 1 / sigma2
##   sigma2 ~ dnorm(0, 1/100) T(0,)
##   phi ~ dunif(-0.99, 0.99)
## }
```

	MVN	Cond
Time	—	✓
Space	✓	~~JAGS~~

DAG $Y_t \longleftarrow Y_{t-1}$

```
y = nc$SID74
x = nc$BIR74

W = W * 1L
D = diag(rowSums(W))
```

$$y(s_1) \mid y(s_2)$$

$$y(s_2) \mid y(s_1)\, y(s_3)$$

...

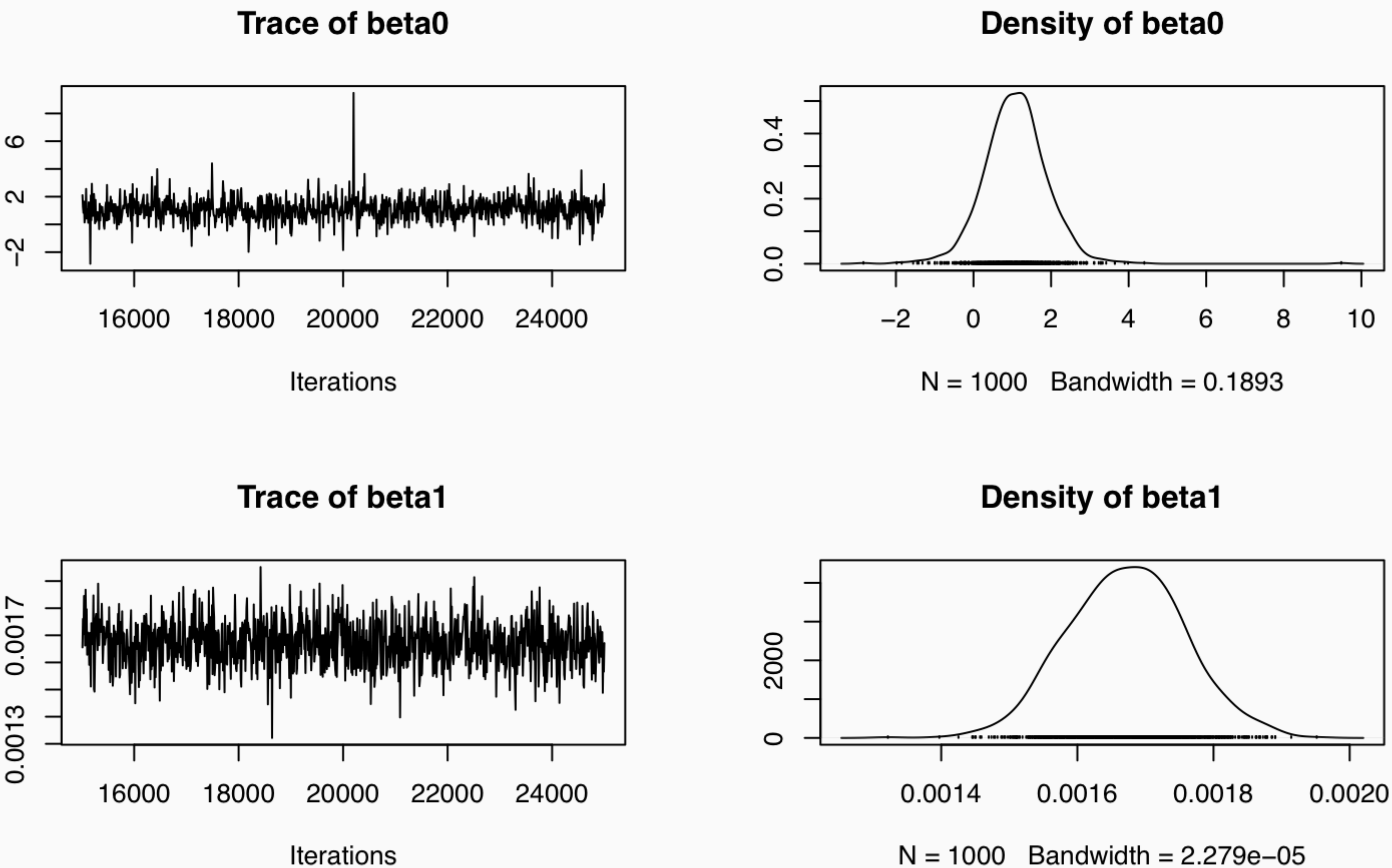
Jags CAR Model

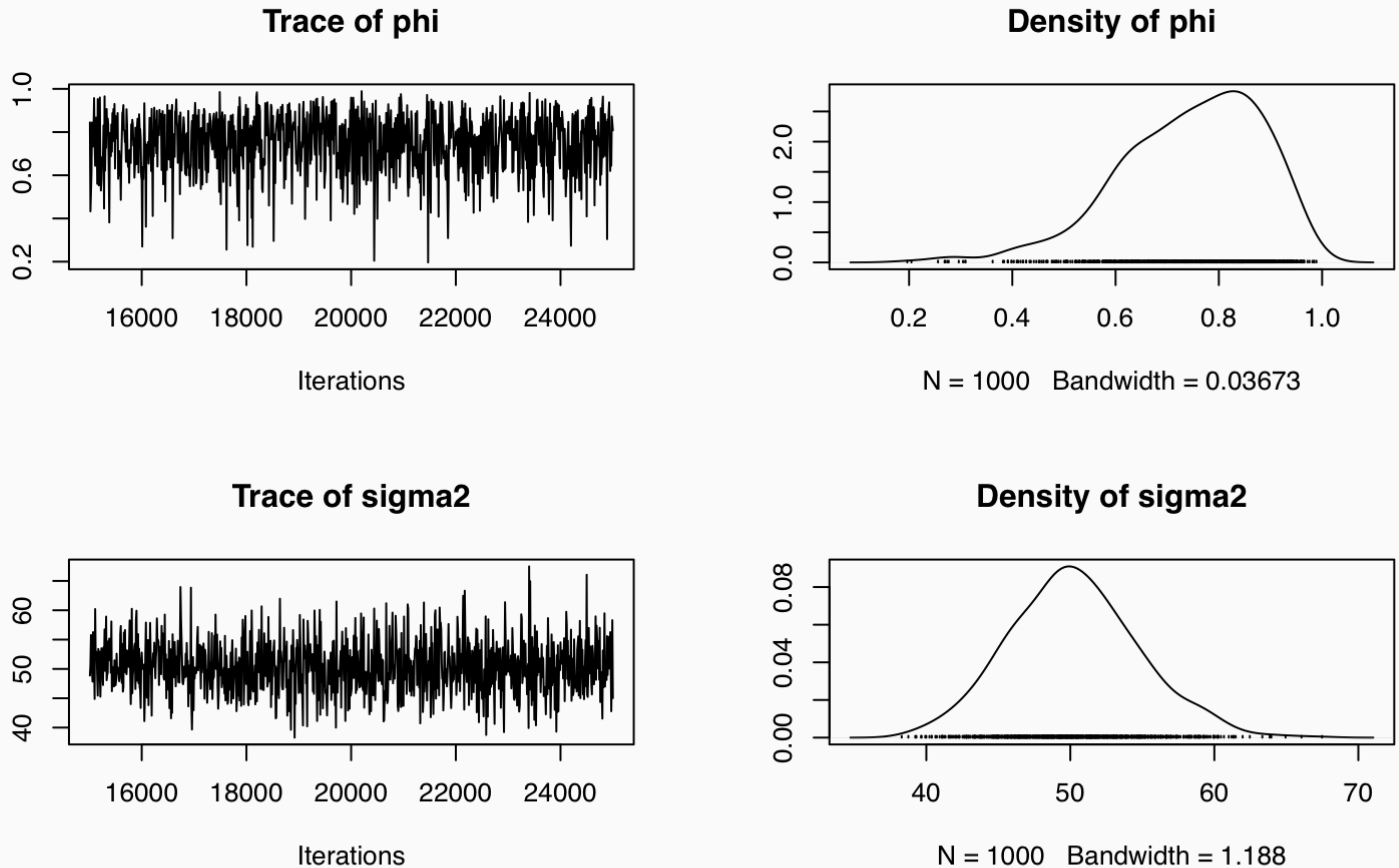
```
## model{
##   y ~ dmnorm(beta0 + beta1*x, tau * (D - phi*W))
##   y_pred ~ dmnorm(beta0 + beta1*x, tau * (D - phi*W))
##
##   beta0 ~ dnorm(0, 1/100)
##   beta1 ~ dnorm(0, 1/100)
##
##   tau <- 1 / sigma2
##   sigma2 ~ dnorm(0, 1/100) T(0,)
##   phi ~ dunif(-0.99, 0.99)
## }
```

```
y = nc$SID74
x = nc$BIR74

W = W * 1L
D = diag(rowSums(W))
```

Why don't we use the conditional definition for the *y*'s?

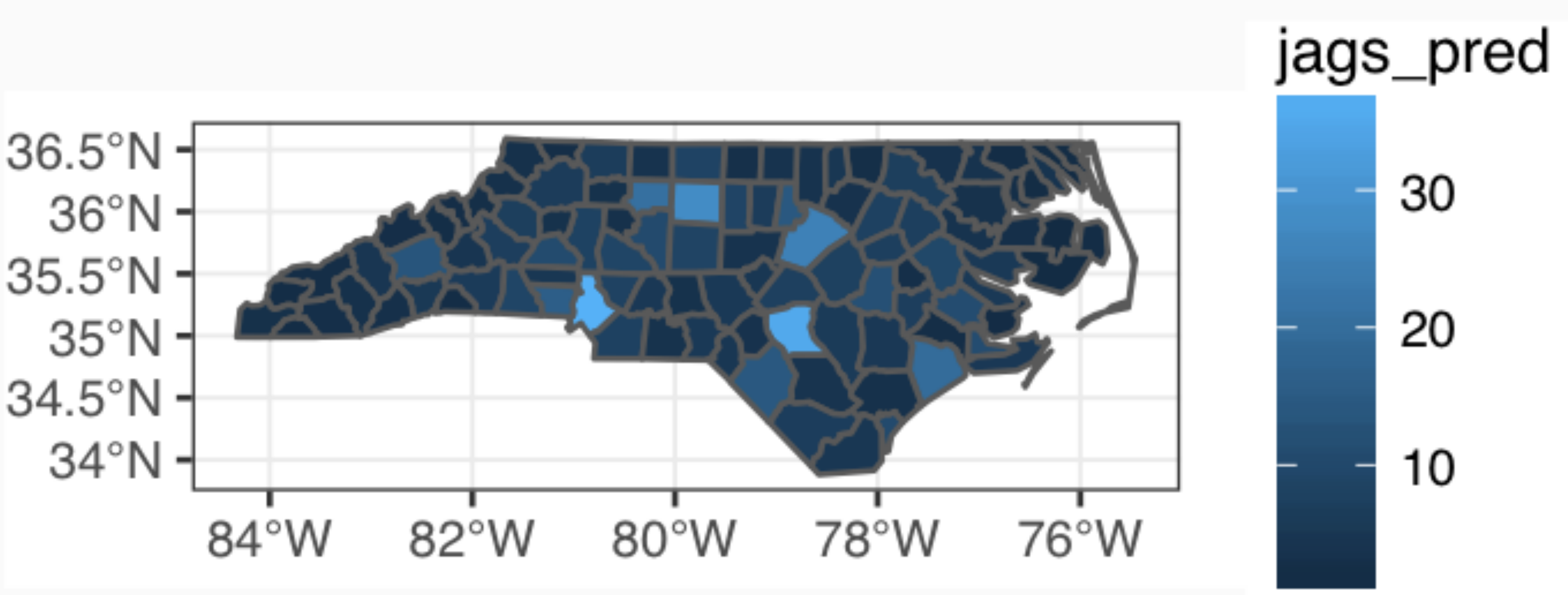
Model Results

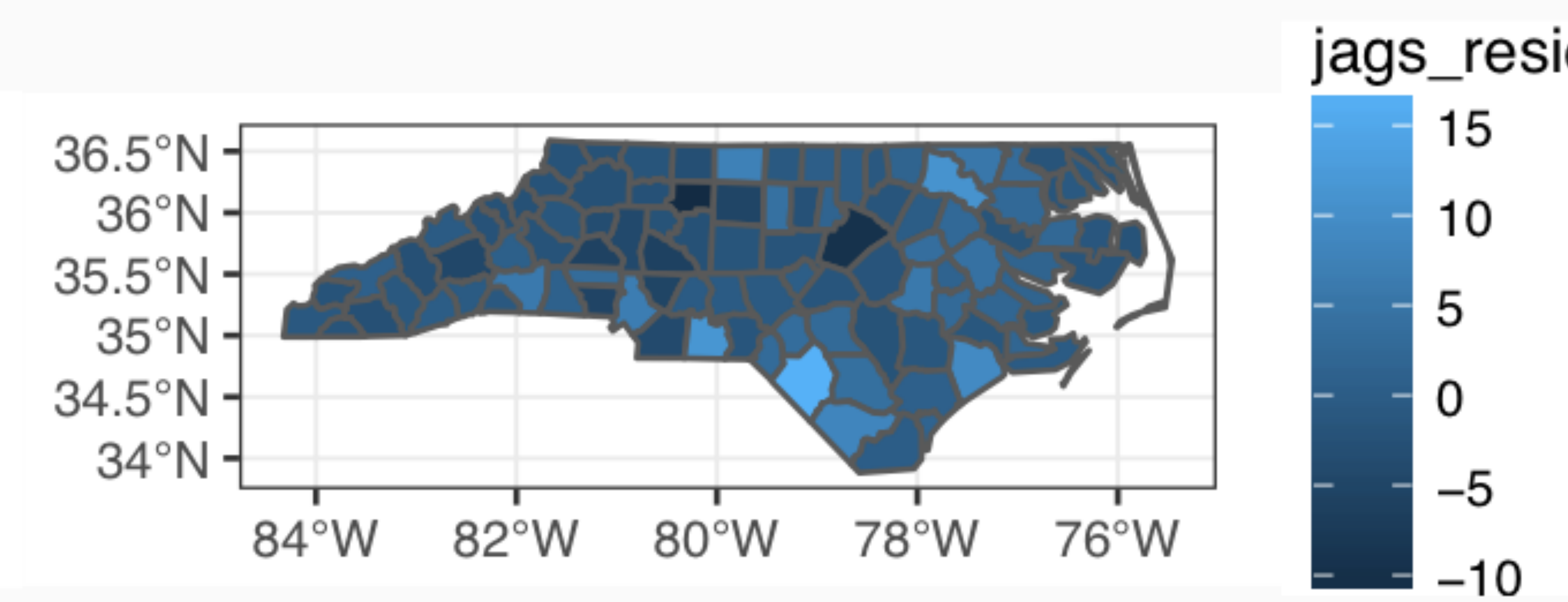


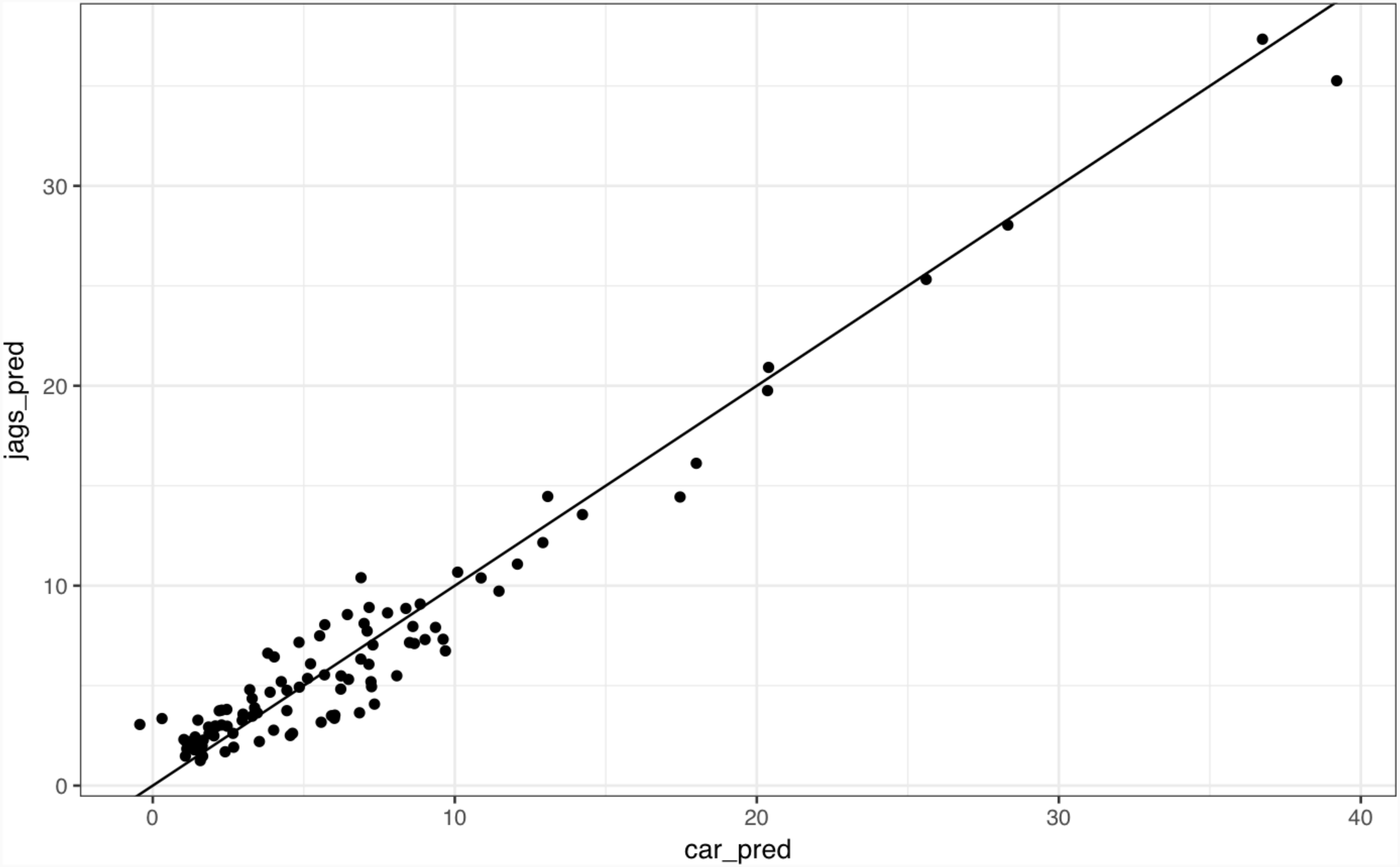

Trace of phi
Density of phi
Iterations
N = 1000 Bandwidth = 0.03673
Trace of sigma2
Density of sigma2
Iterations
N = 1000 Bandwidth = 1.188

Predictions

```
nc$jags_pred = y_pred$post_mean
nc$jags_resid = nc$SID74 - y_pred$post_mean

sqrt(mean(nc$jags_resid^2))
## [1] 3.987985
sqrt(mean(nc$car_resid^2))
## [1] 3.72107
sqrt(mean(nc$sar_resid^2))
## [1] 3.762664
```







jags_pred
car_pred
0
10
20
30
40

Brief Aside - SAR Precision Matrix

$$\Sigma_{SAR} = (I - \phi D^{-1} W)^{-1} \sigma^2 D^{-1} \left((I - \phi D^{-1} W)^{-1} \right)^t$$

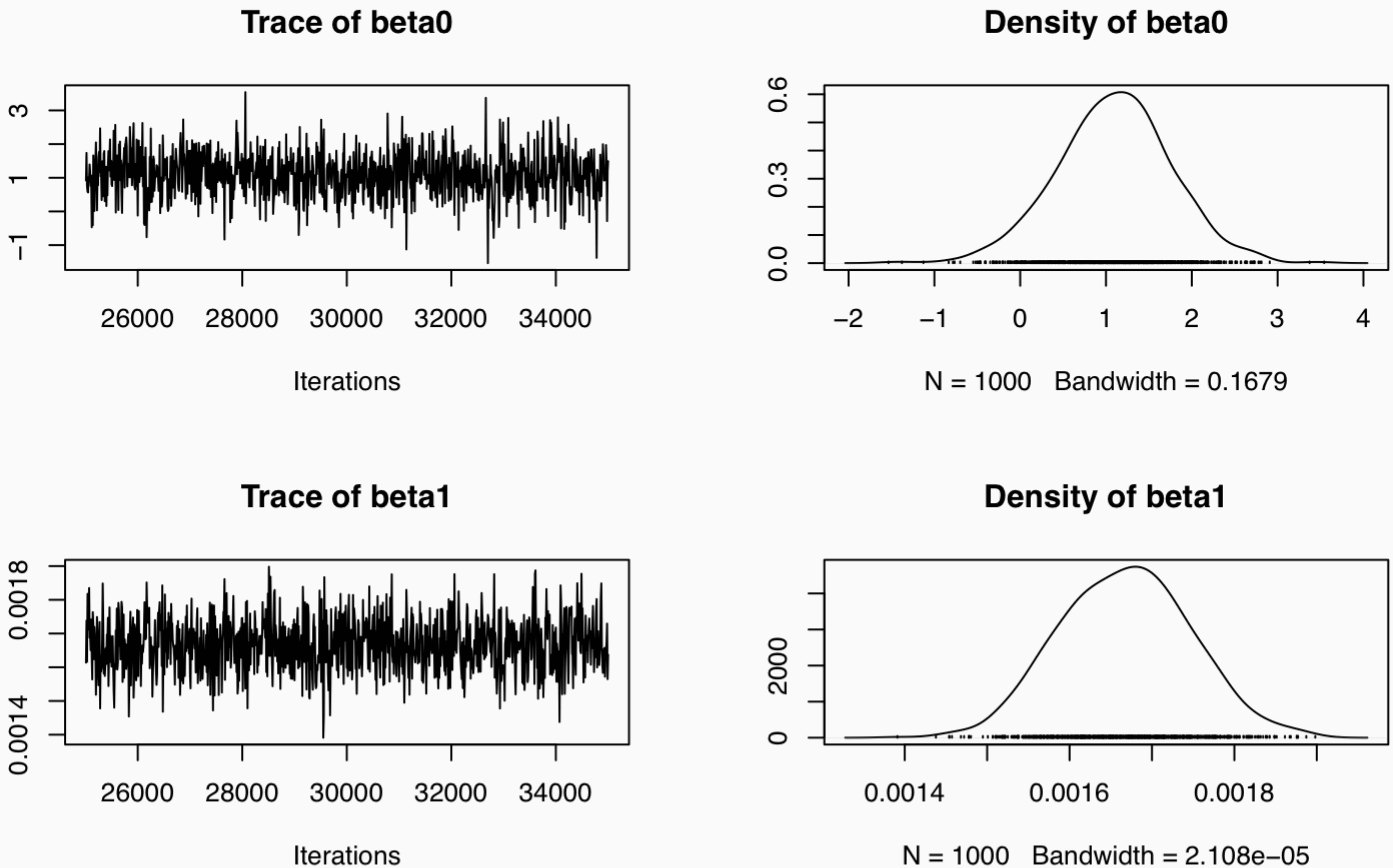
$$\begin{aligned}
\Sigma_{SAR}^{-1} &= \left((I - \phi D^{-1} W)^{-1} \sigma^2 D^{-1} \left((I - \phi D^{-1} W)^{-1} \right)^t \right)^{-1} \\
&= \left(\left((I - \phi D^{-1} W)^{-1} \right)^t \right)^{-1} \frac{1}{\sigma^2} D \left(I - \phi D^{-1} W \right) \\
&= \frac{1}{\sigma^2} (I - \phi D^{-1} W)^t D \left(I - \phi D^{-1} W \right)
\end{aligned}$$

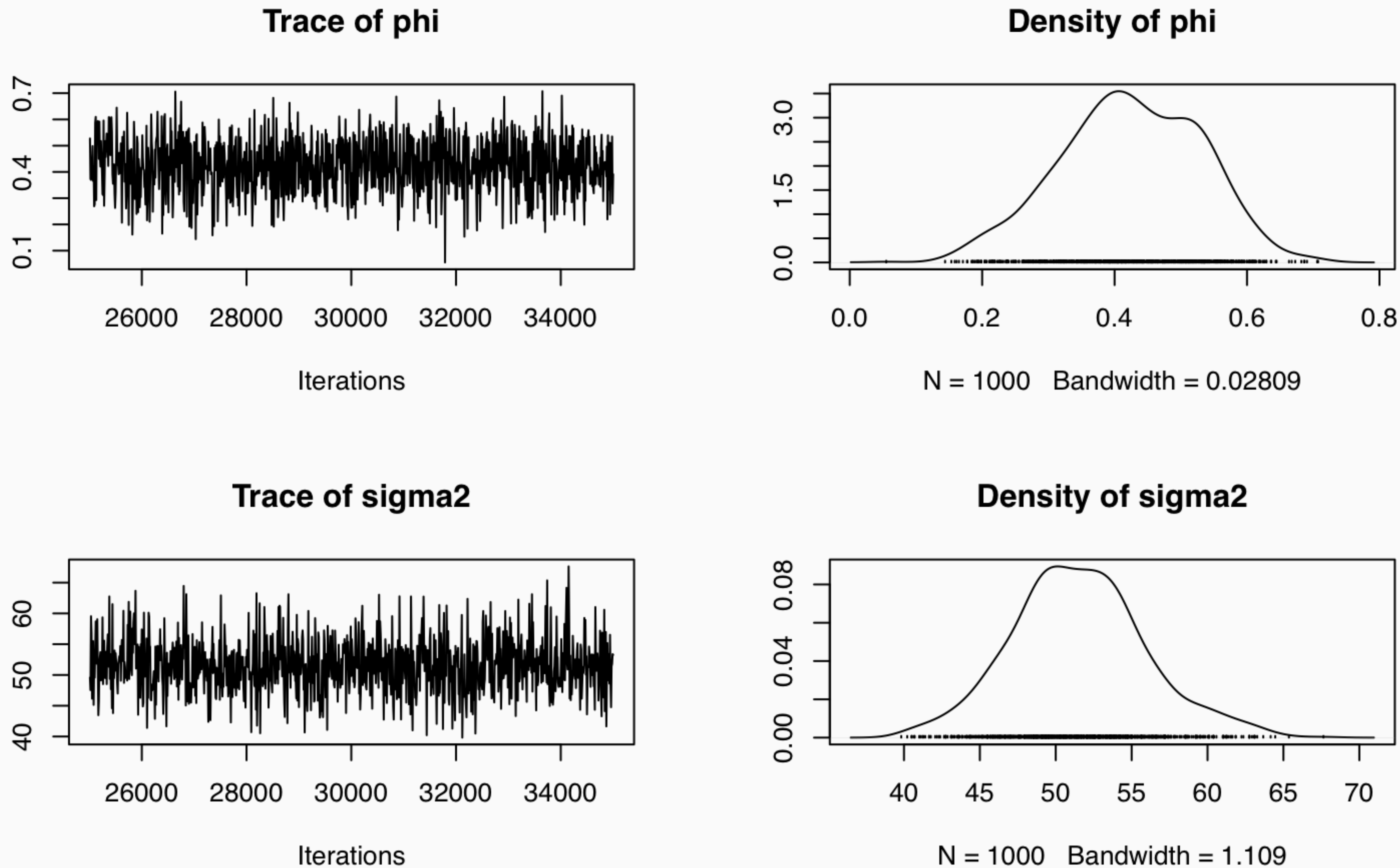
Jags SAR Model

```
## model{
##   y ~ dmnorm(beta0 + beta1*x, tau * (D - phi*W))
##   y_pred ~ dmnorm(beta0 + beta1*x, tau * (D - phi*W))
##
##   beta0 ~ dnorm(0, 1/100)
##   beta1 ~ dnorm(0, 1/100)
##
##   tau <- 1 / sigma2
##   sigma2 ~ dnorm(0, 1/100) T(0,)
##   phi ~ dunif(-0.99, 0.99)
## }
```

```
D_inv = diag(1/diag(D))
W_tilde = D_inv %*% W
I = diag(1, ncol=length(y), nrow=length(y))
```

Model Results




Trace of phi
0.1
0.4
0.7
26000
28000
30000
32000
34000
Iterations
Density of phi
0.0
1.5
3.0
0.0
0.2
0.4
0.6
0.8
N = 1000 Bandwidth = 0.02809
Trace of sigma2
40
50
60
26000
28000
30000
32000
34000
Iterations
Density of sigma2
0.00
0.04
0.08
40
45
50
55
60
65
70
N = 1000 Bandwidth = 1.109

Comparing Model Results