Lecture 6

Discrete Time Series

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Stationary Processes

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Weak Stationary

Strict stationary is too strong for most applications, so instead we often opt for *weak stationary* which requires the following,

1. The process has finite variance

$$E(y_t^2) < \infty \text{ for all } t$$

2. The mean of the process in constant

$$E(y_t) = \mu \text{ for all } t$$

3. The second moment only depends on the lag

$$Cov(y_t, y_s) = Cov(y_{t+k}, y_{s+k}) \text{ for all } t, s, k$$

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When we say stationary in class we almost always mean this version of *weakly stationary*.

Autocorrelation

For a stationary time series, where $E(y_t) = \mu$ and $\text{Var}(y_t) = \sigma^2)$ for all t, we define the autocorrelation at lag k as

$$\begin{aligned}
\rho_k &= Cor(y_t,\ y_{t+k}) \\
&= \frac{Cov(y_t, y_{t+k})}{\sqrt{Var(y_t)Var(y_{t+k})}} \\
&= \frac{E\left((y_t - \mu)(y_{t+k} - \mu)\right)}{\sigma^2}
\end{aligned}$$

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this is also sometimes written in terms of the autocovariance function (γ_k) as

$$\begin{aligned} \gamma_k &= \gamma(t, t+k) = Cov(y_t, y_{t+k}) \\ \rho_k &= \frac{\gamma(t, t+k)}{\sqrt{\gamma(t,t)\gamma(t+k, t+k)}} = \frac{\gamma(k)}{\gamma(0)} \end{aligned}$$

Covariance Structure

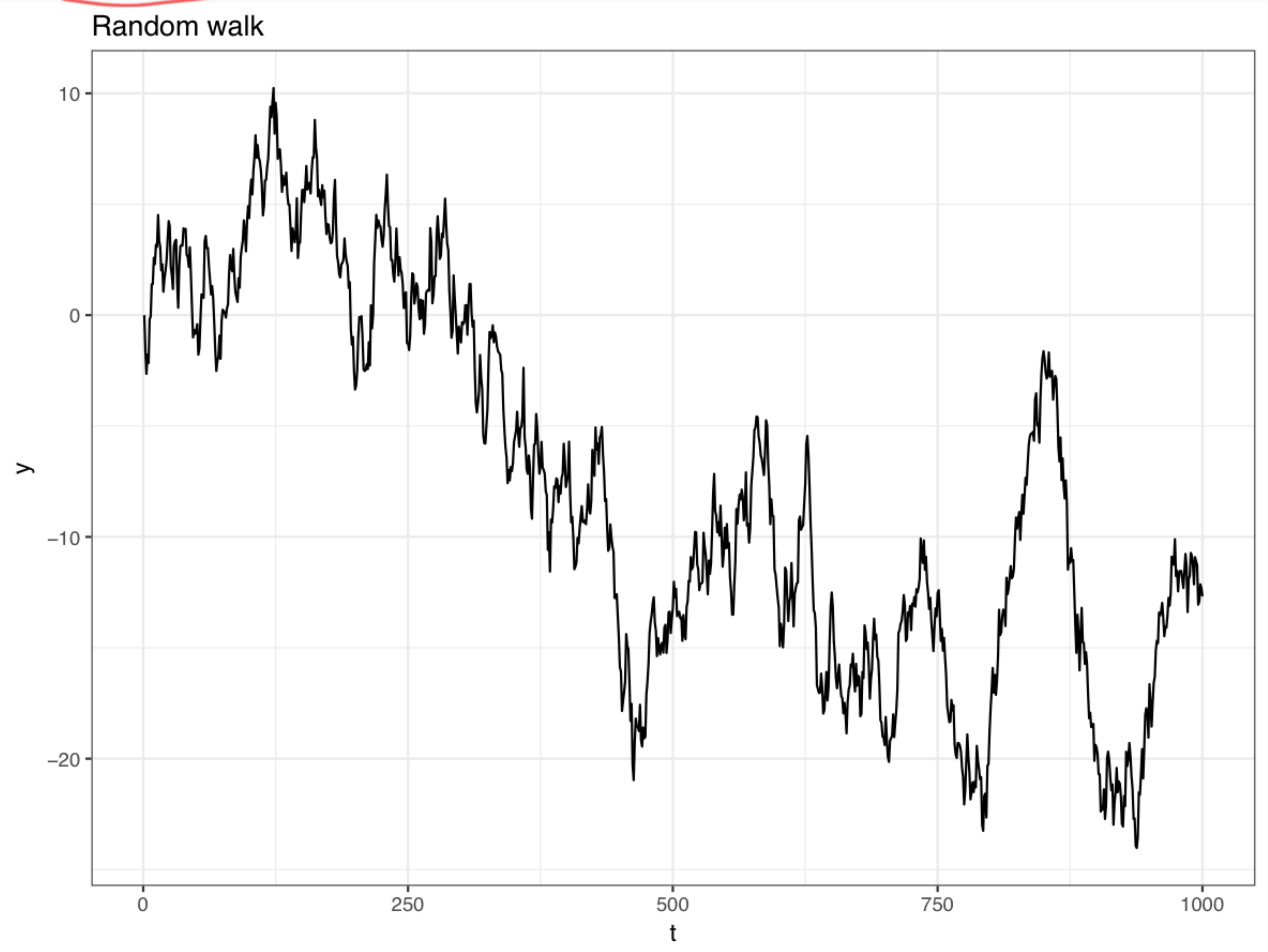
Based on our definition of a (weakly) stationary process, it implies a covariance of the following structure,

$$\begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots & \gamma(n) \\ \gamma(1) & \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(n-1) \\ \gamma(2) & \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(n-2) \\ \gamma(3) & \gamma(2) & \gamma(1) & \gamma(0) & \cdots & \gamma(n-3) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(n) & \gamma(n-1) & \gamma(n-2) & \gamma(n-3) & \cdots & \gamma(0) \end{pmatrix}$$

Example - Random walk

Let $y_t = y_{t-1} + w_t$ with $y_0 = 0$ and $w_t \sim \mathcal{N}(0, 1)$. Is y_t stationary?

not



$$Y_t = Y_{t-1} + w_t$$

$$Y_0 = 0$$

$$Y_1 = w_1$$

$$Y_2 = w_2 + w_1$$

$$Y_3 = w_3 + w_2 + w_1$$

$$\vdots$$

$$\boxed{Y_t = \sum_{i=1}^{t} w_i}$$

$$E(Y_t) = E\left(\sum w_i\right)$$

$$= \sum E(w_i)$$

$$= \sum 0 = 0$$

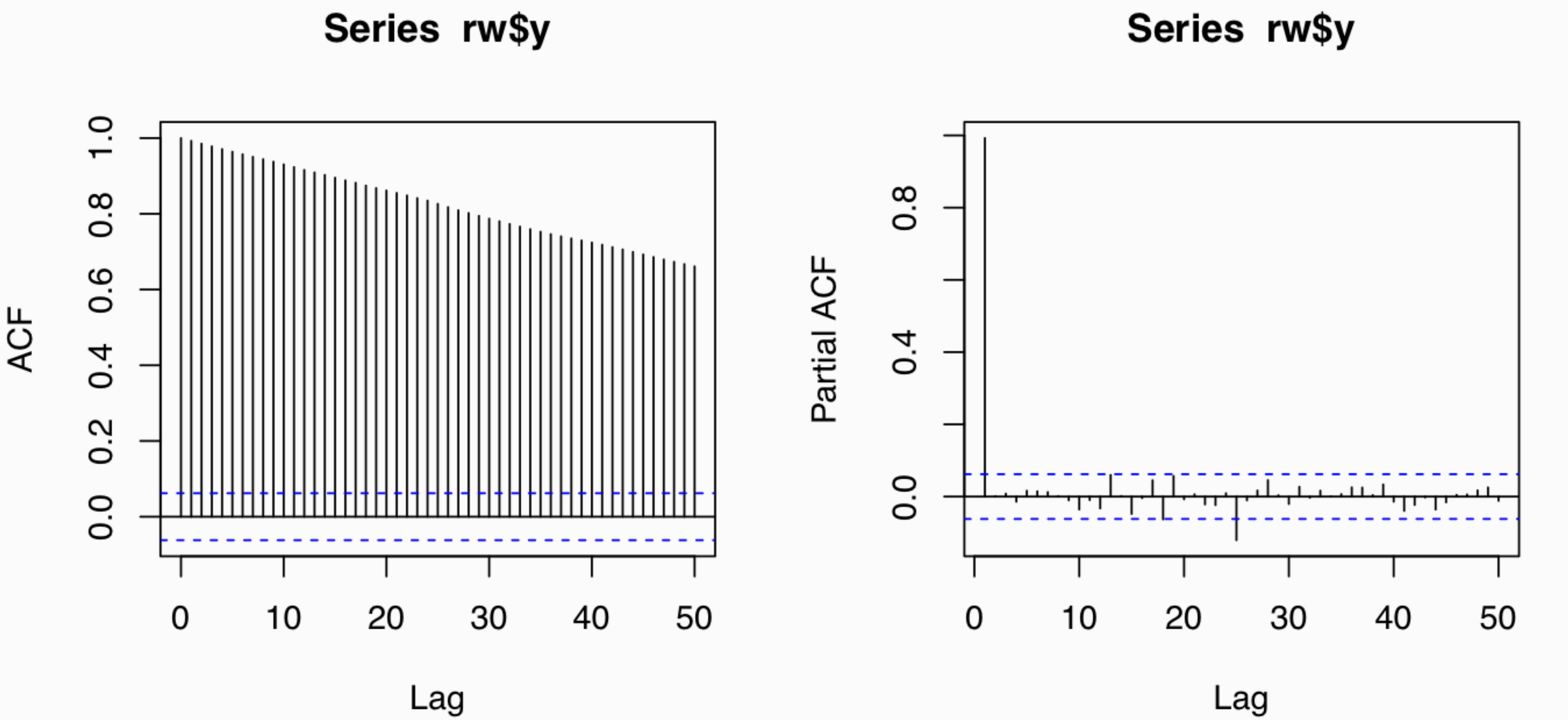
$$Cov(Y_t, Y_{t+h})$$

$$= Cov(w_1 + w_2 + \dots + w_t,\ w_1 + w_2 + w_3 \dots + w_t + \underbrace{w_{t+1} + \dots + w_{t+h}})$$

$$= \sum_{i=1}^{t} Cov(w_i, w_i)$$

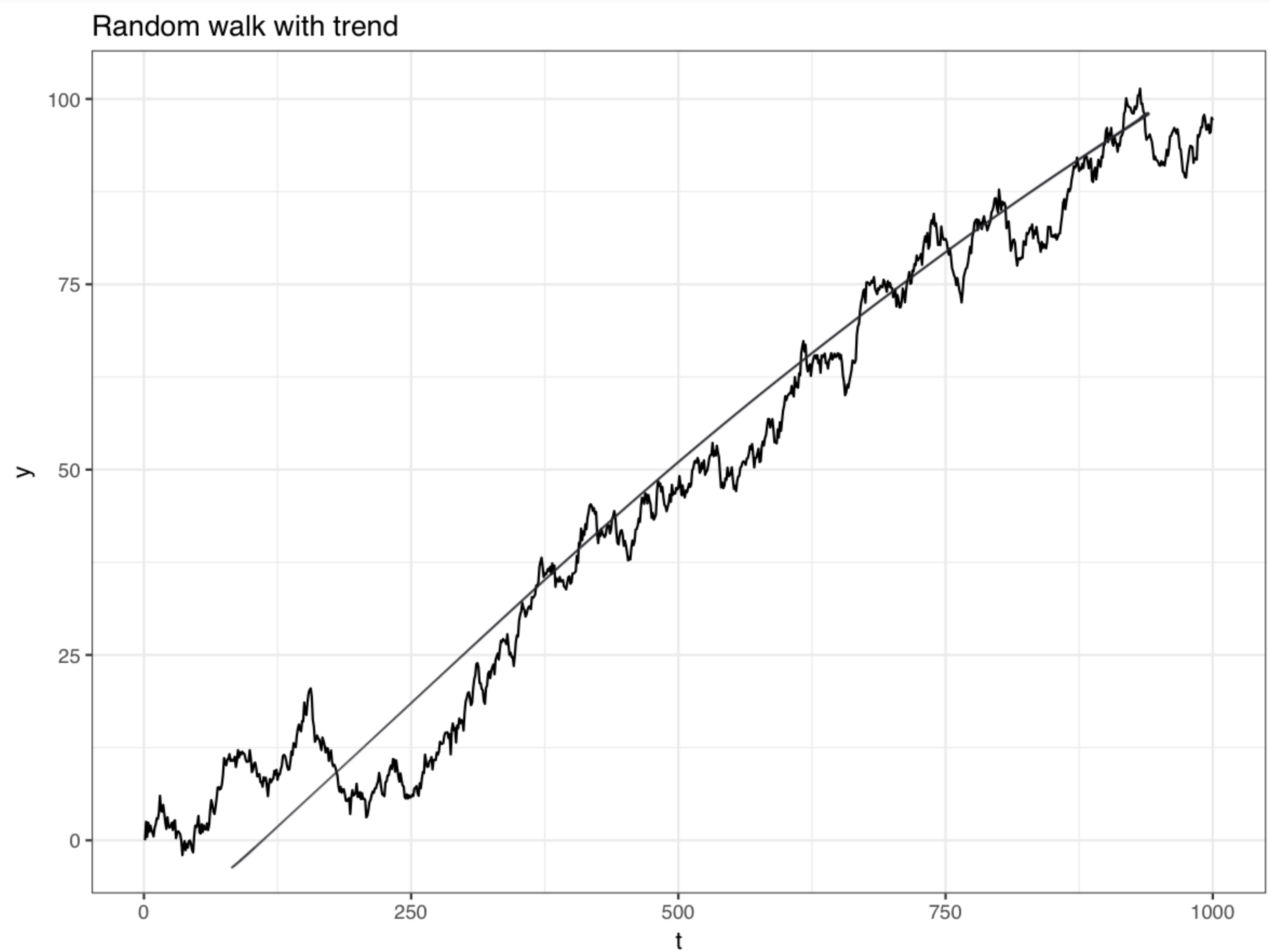
$$= \sum_{i=1}^{t} \sigma^2 = \underline{t}\ \sigma^2$$

ACF + PACF


Series rw$y
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
Series rw$y
Partial ACF
Lag
0.8
0.4
0.0
0
10
20
30
40
50

Example - Random walk with drift

Let $y_t = \delta + y_{t-1} + w_t$ with $y_0 = 0$ and $w_t \sim \mathcal{N}(0, 1)$. Is y_t stationary?



$$Y_0 = 0$$

$$Y_1 = \delta + u_1$$

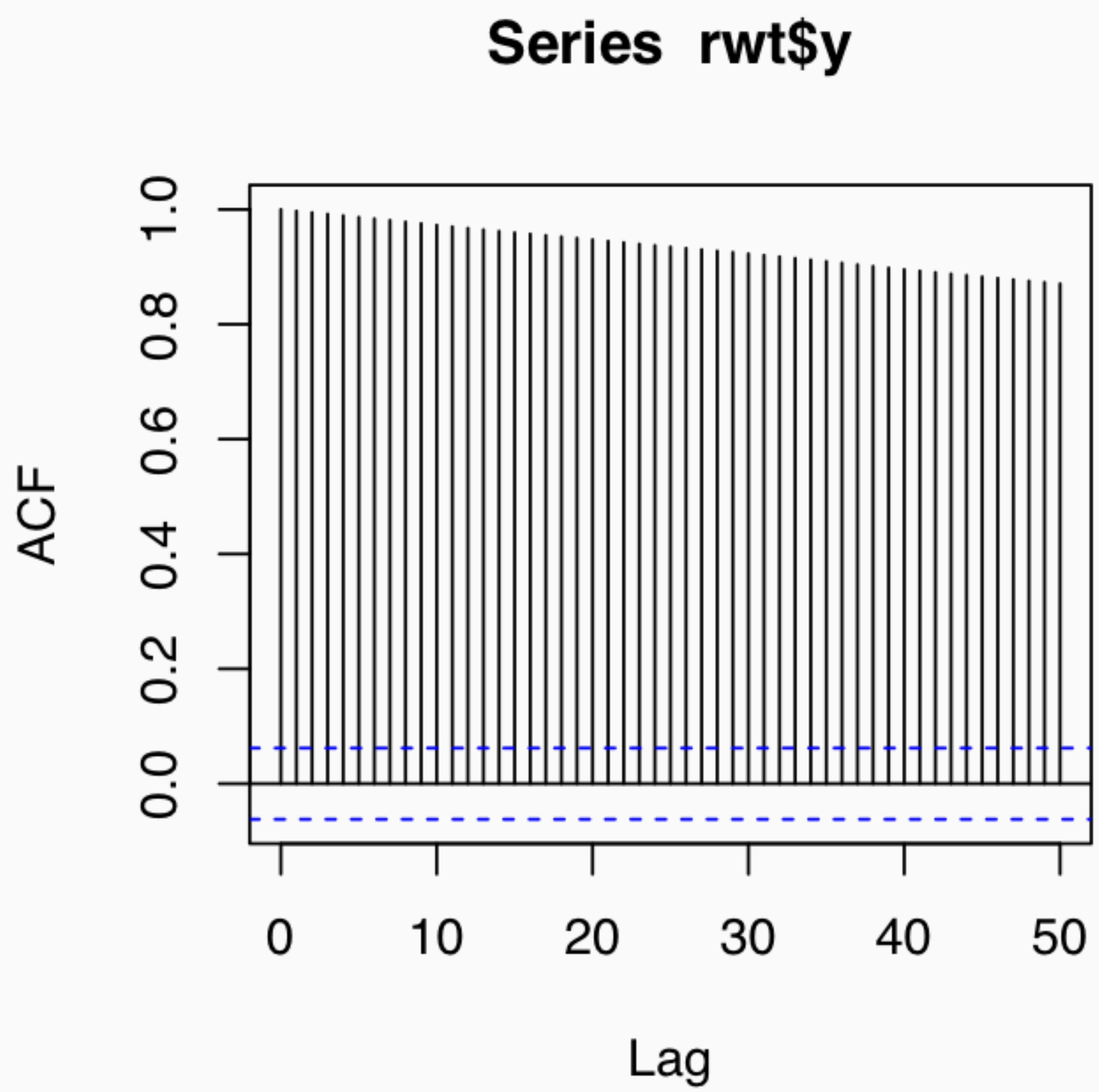
$$Y_2 = \delta + \delta + u_1 + u_2$$

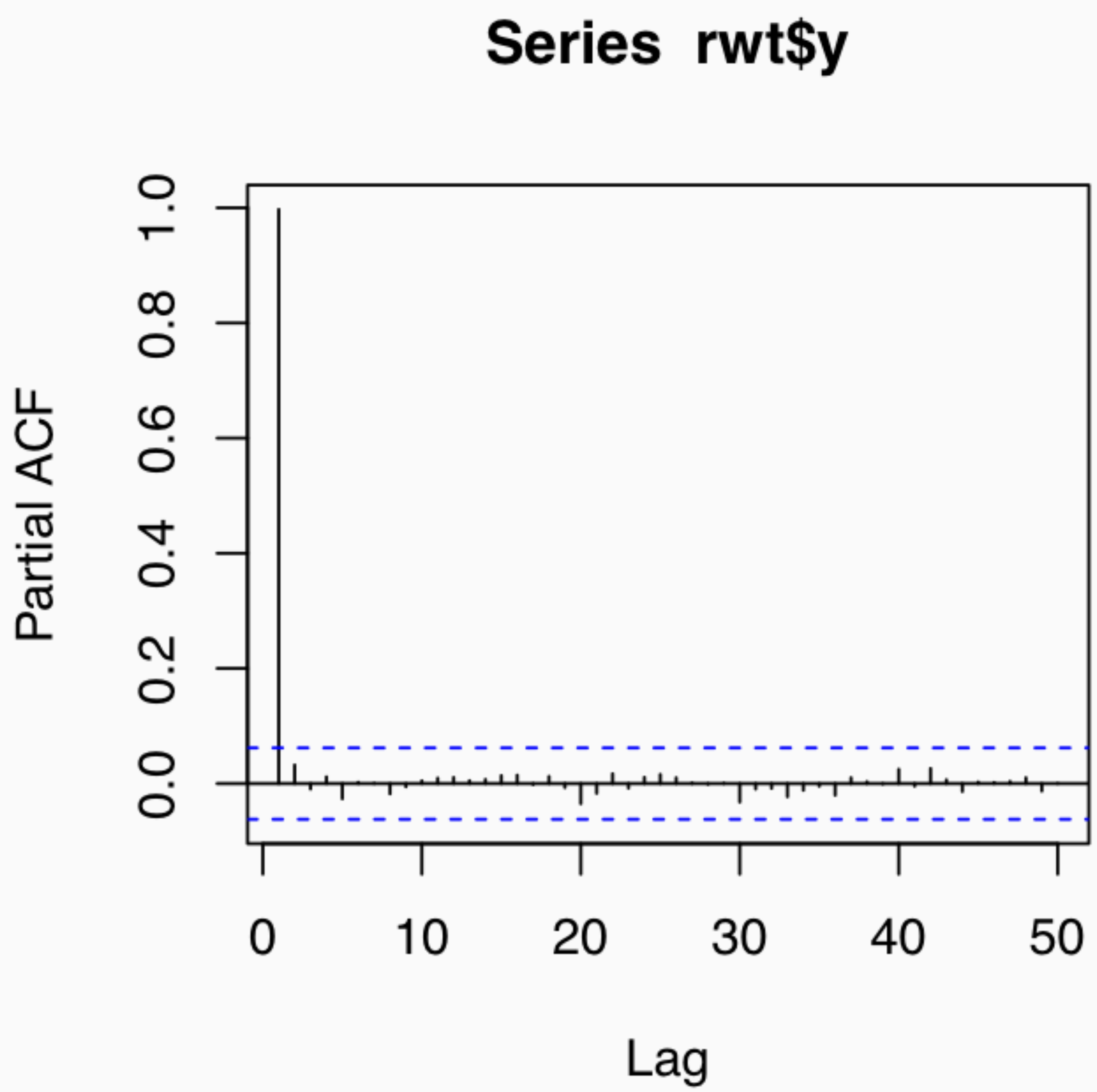
$$\vdots$$

$$Y_t = \delta t + \sum_{i=1}^{t} u_i$$

$$E(Y_t) = \delta t$$

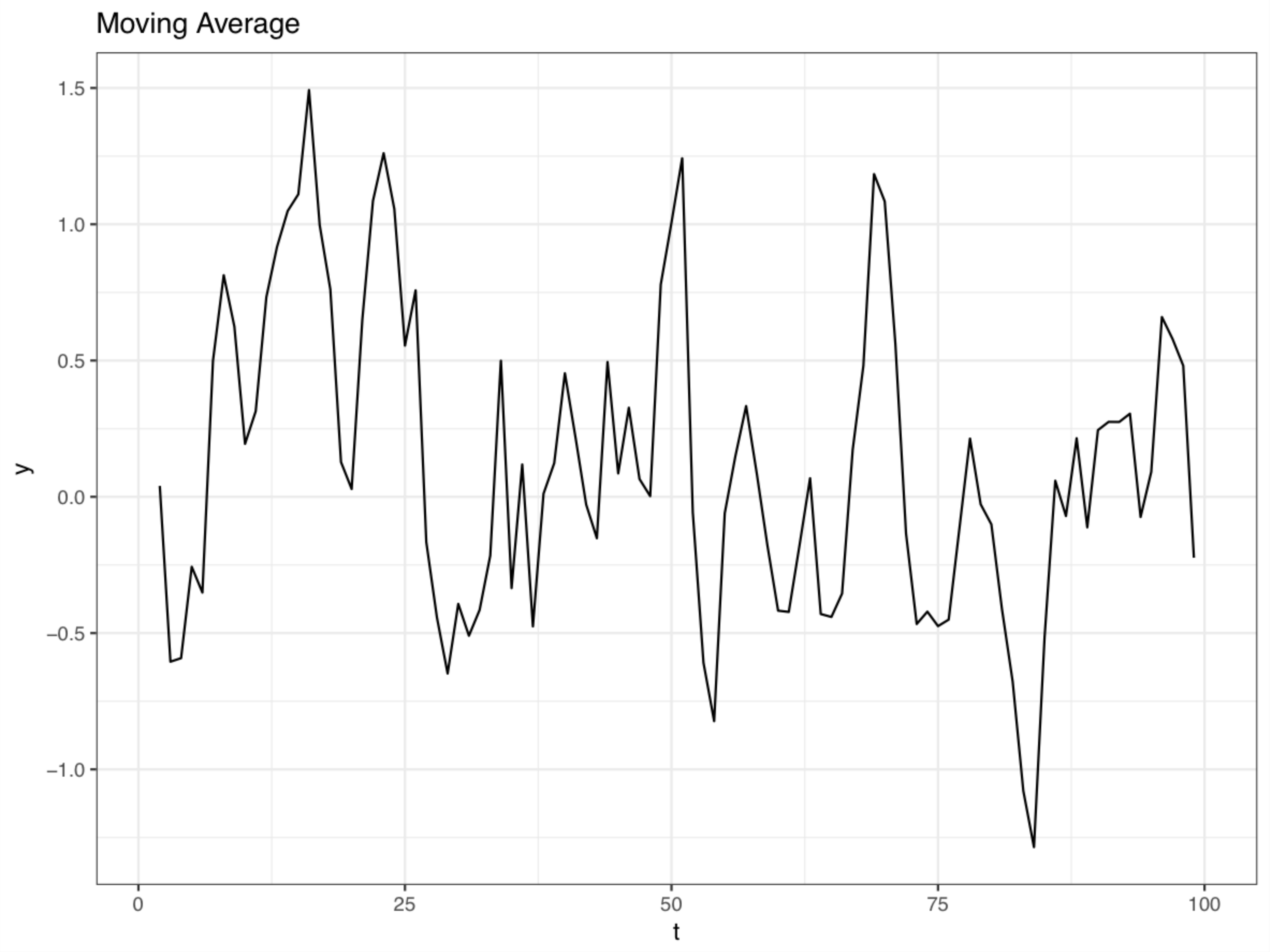
ACF + PACF


Series rwt$y
ACF
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
Lag


Series rwt$y
Partial ACF
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
Lag

Example - Moving Average

Let $w_t \sim \mathcal{N}(0,1)$ and $y_t = \frac{1}{3}\left(w_{t-1} + w_t + w_{t+1}\right)$, is y_t stationary?



$$E(Y_t) = E\left(\frac{1}{3}\left(v_{t-1} + w_t + w_{t+1}\right)\right)$$

$$= \frac{1}{3}E(v_{t-1}) + \frac{1}{3}E(v_t) + \frac{1}{3}E(v_{t+1})$$

$$= 0$$

$$\text{Cov}(Y_t, Y_{t+h}) =$$

<u>$h = 0$</u>

$$\text{Cov}(Y_t, Y_t) = \text{Var}(Y_t) = \text{Var}\left(\frac{1}{3}\left(v_{t-1} + v_t + v_{t+1}\right)\right)$$

$$= \frac{1}{9}(1+1+1) = 3/9$$

<u>$h = \pm 1$</u>

$$\text{Cov}(Y_t, Y_{t+1}) = \text{Cov}\left(\frac{1}{3}\left(v_{t-1} + v_t + w_{t+1}\right), \frac{1}{3}\left(w_t + v_{t+1} + v_{t+2}\right)\right)$$

$$= \frac{1}{9}\left(\text{Cov}(w_t, w_t) + \text{Cov}(w_{t+1}, v_{t+1})\right)$$

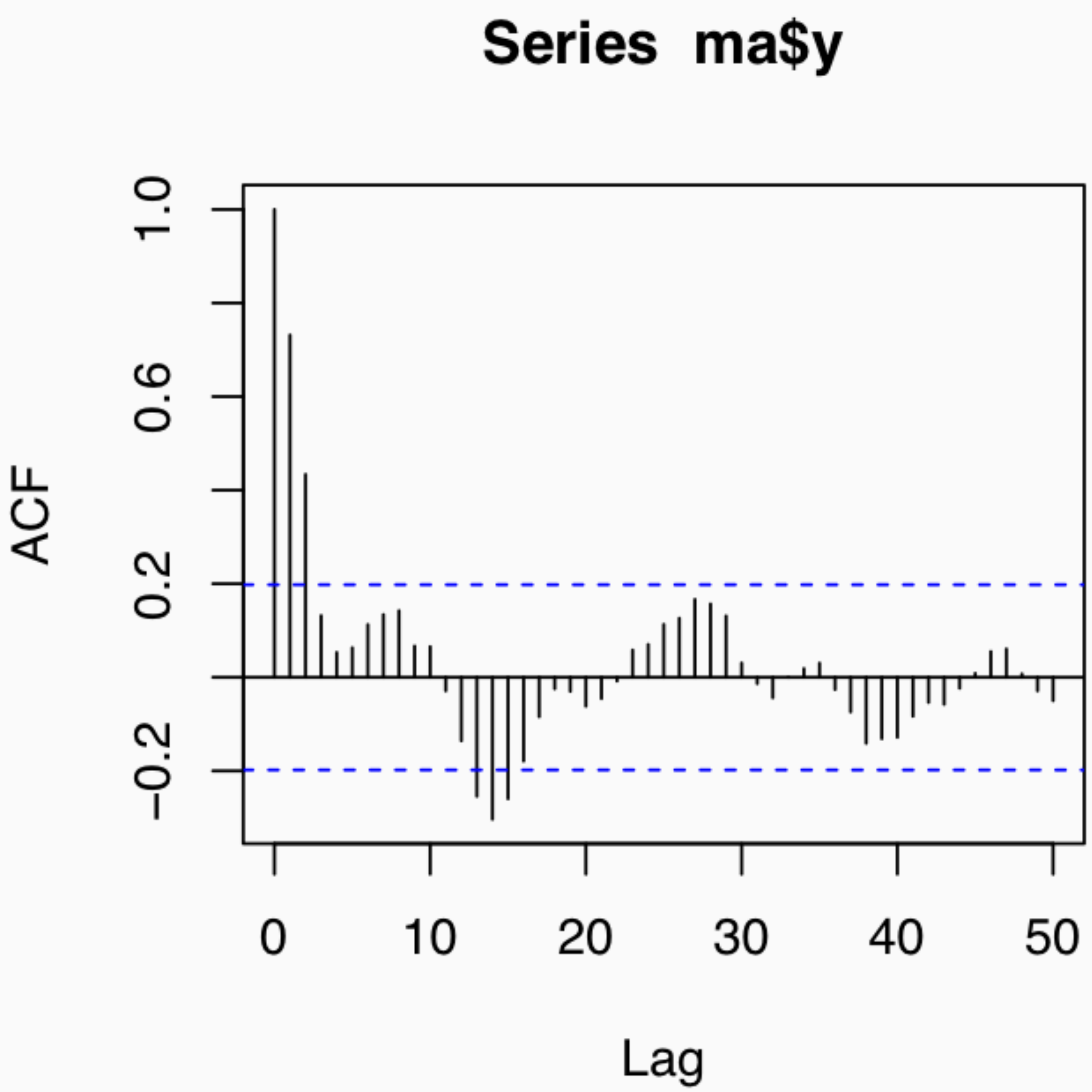
$$= 2/9$$

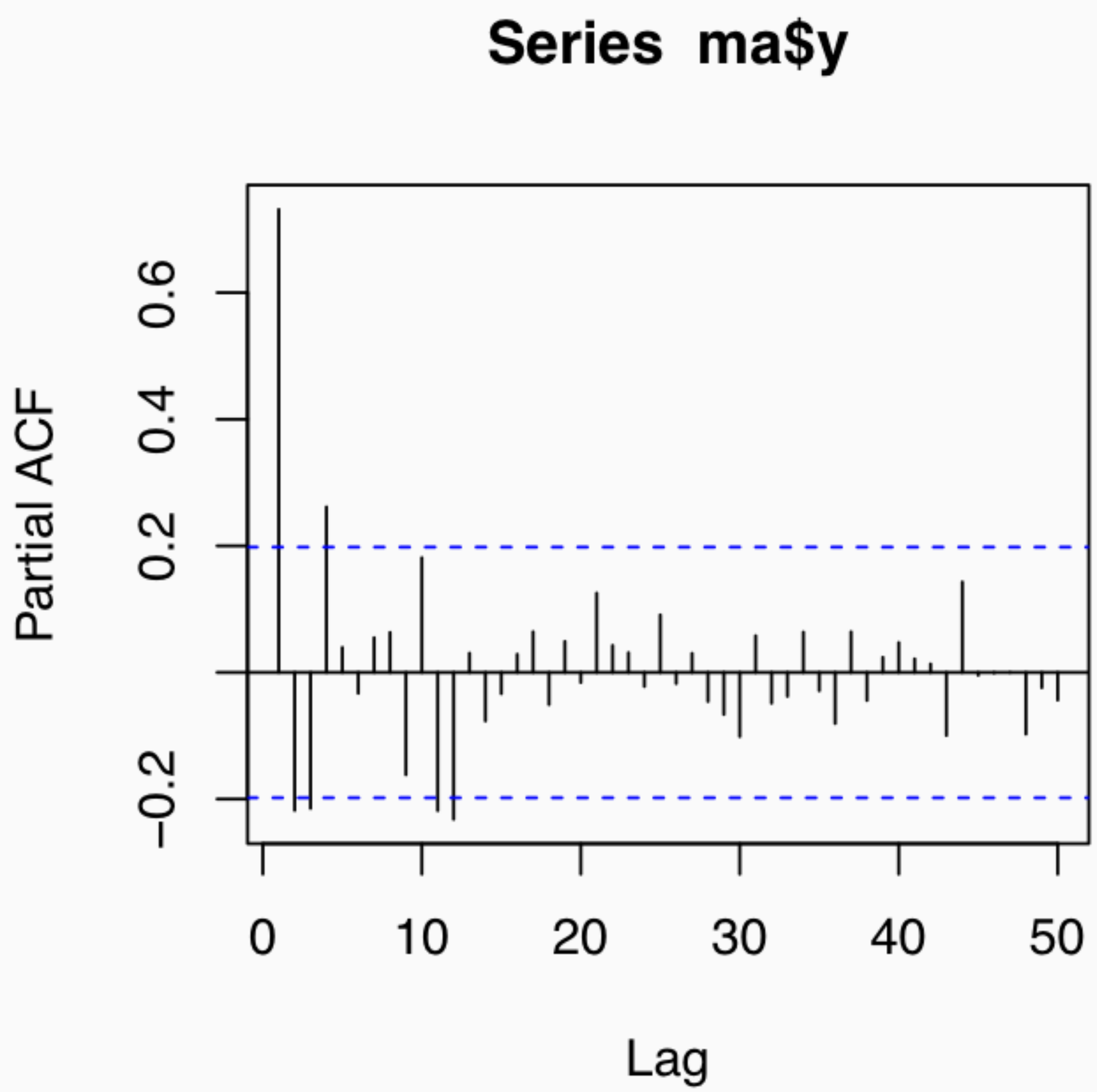
$h = \pm 2$ $\quad Cov(Y_t, Y_{t+h}) = 1/9$

$|h| > 2$ $\quad Cov(Y_t, Y_{t+h}) = 0$

$$\gamma_h = \begin{cases} 3/9 & \text{if } h = 0 \\ 2/9 & \text{if } h = \pm 1 \\ 1/9 & \text{if } h = \pm 2 \\ 0 & \text{if } |h| > 2 \end{cases}$$

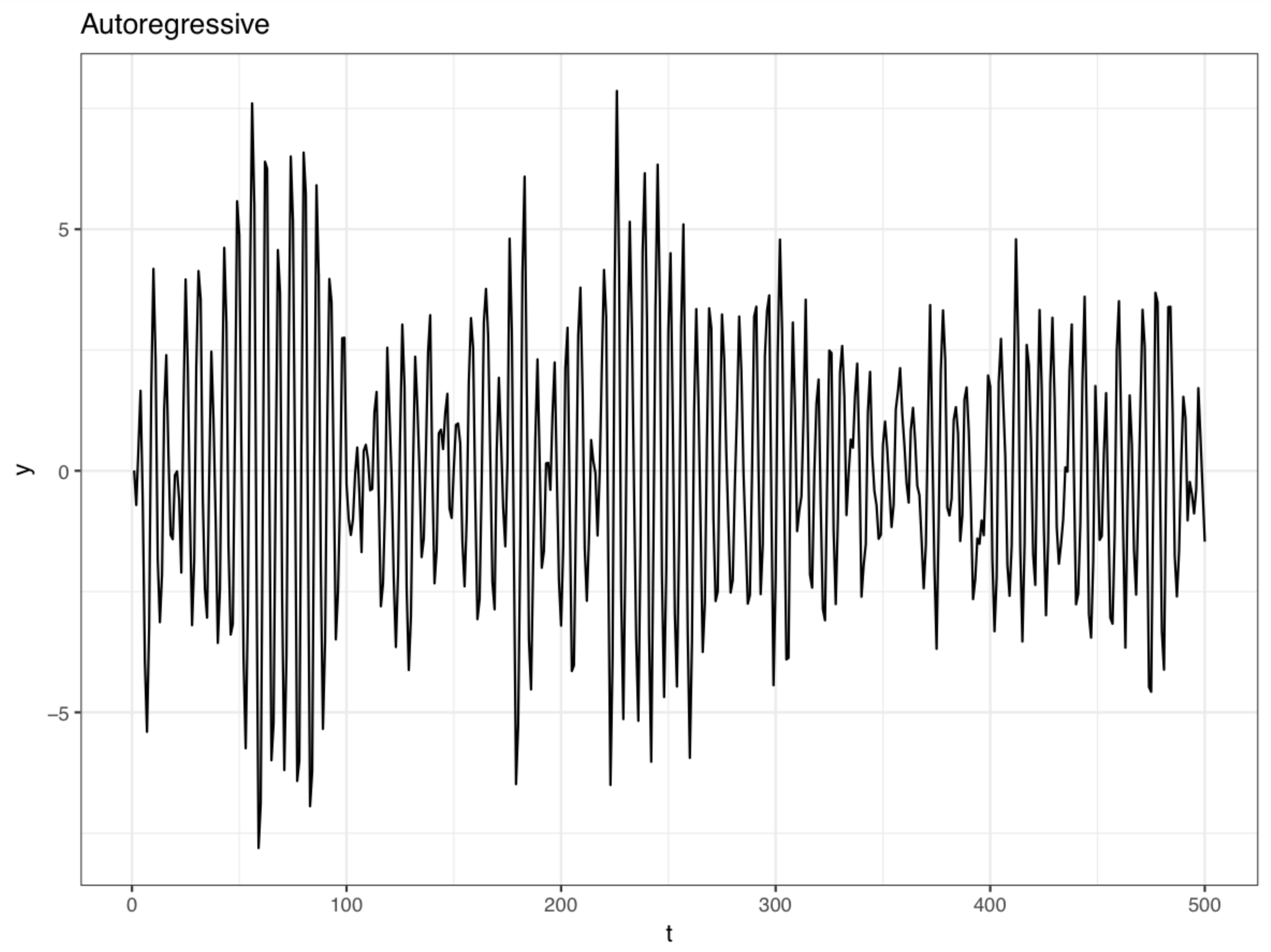
ACF + PACF


Series ma$y
ACF
1.0
0.6
0.2
−0.2
0
10
20
30
40
50
Lag

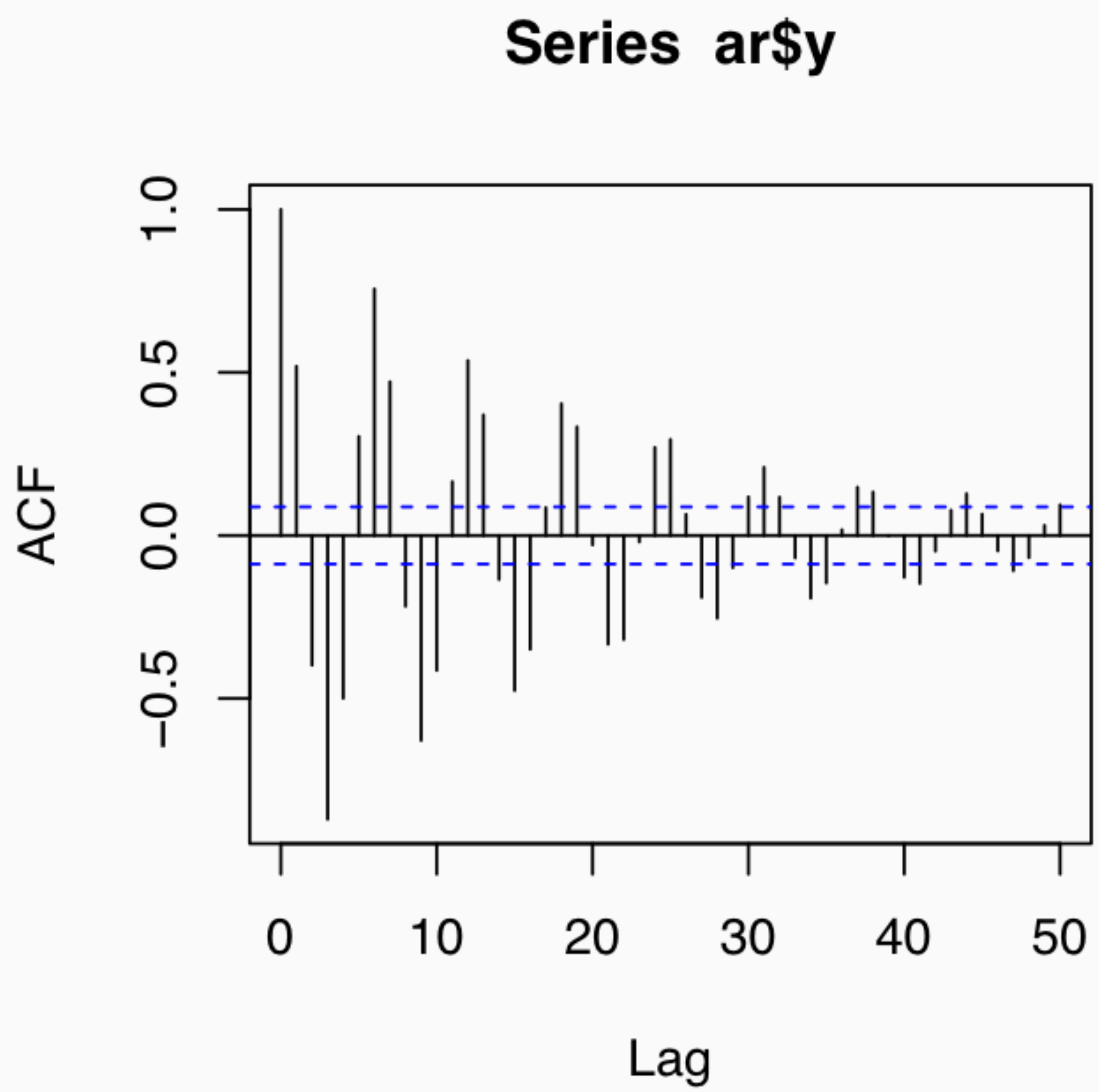

Series ma$y
Partial ACF
0.6
0.4
0.2
−0.2
0
10
20
30
40
50
Lag

Autoregression

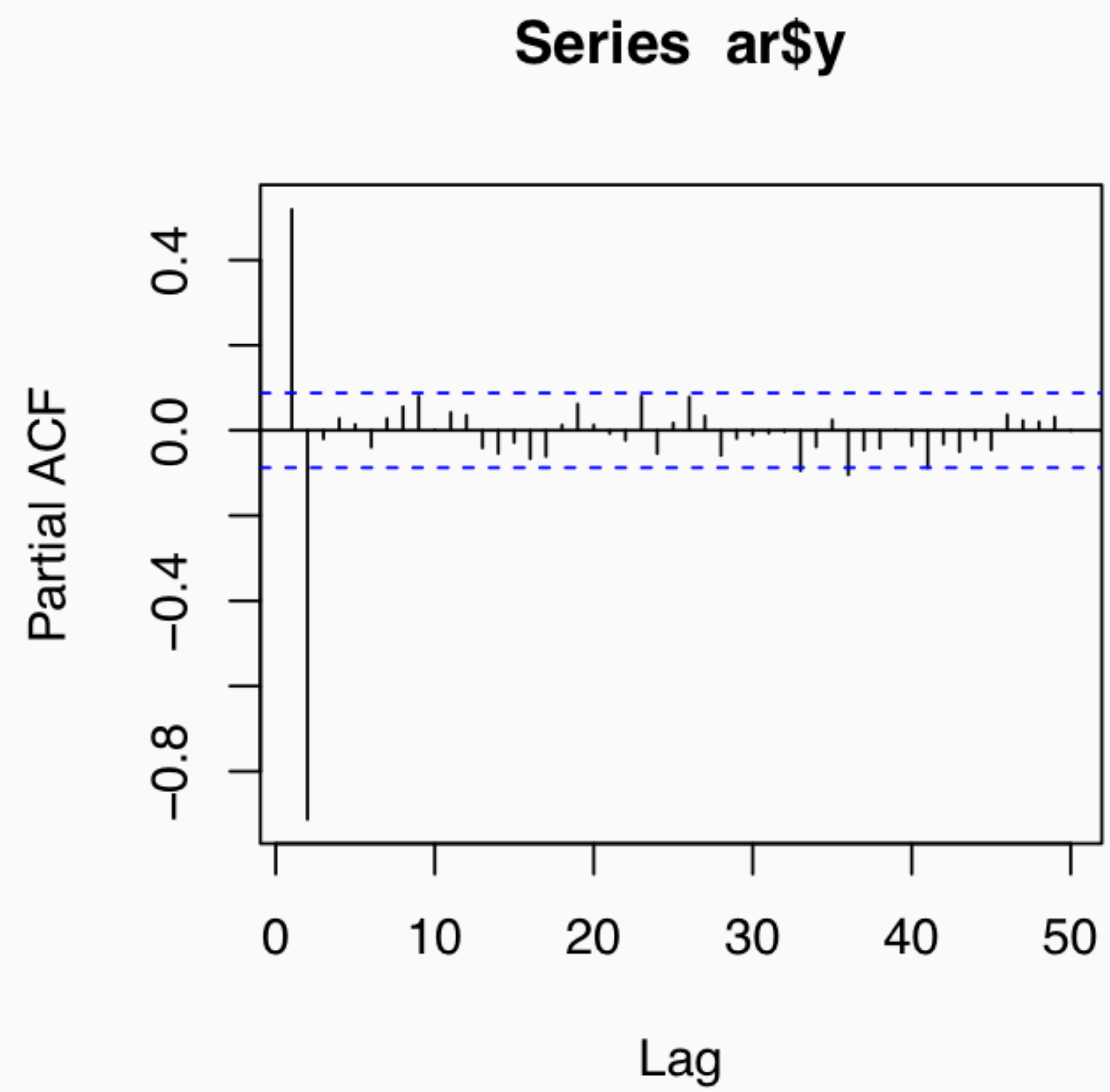
Let $w_t \sim \mathcal{N}(0,1)$ and $y_t = y_{t-1} - 0.9y_{t-2} + w_t$ with $y_t = 0$ for $t < 1$, is y_t stationary?



ACF + PACF



Series ar$y
ACF
1.0
0.5
0.0
−0.5
0
10
20
30
40
50
Lag



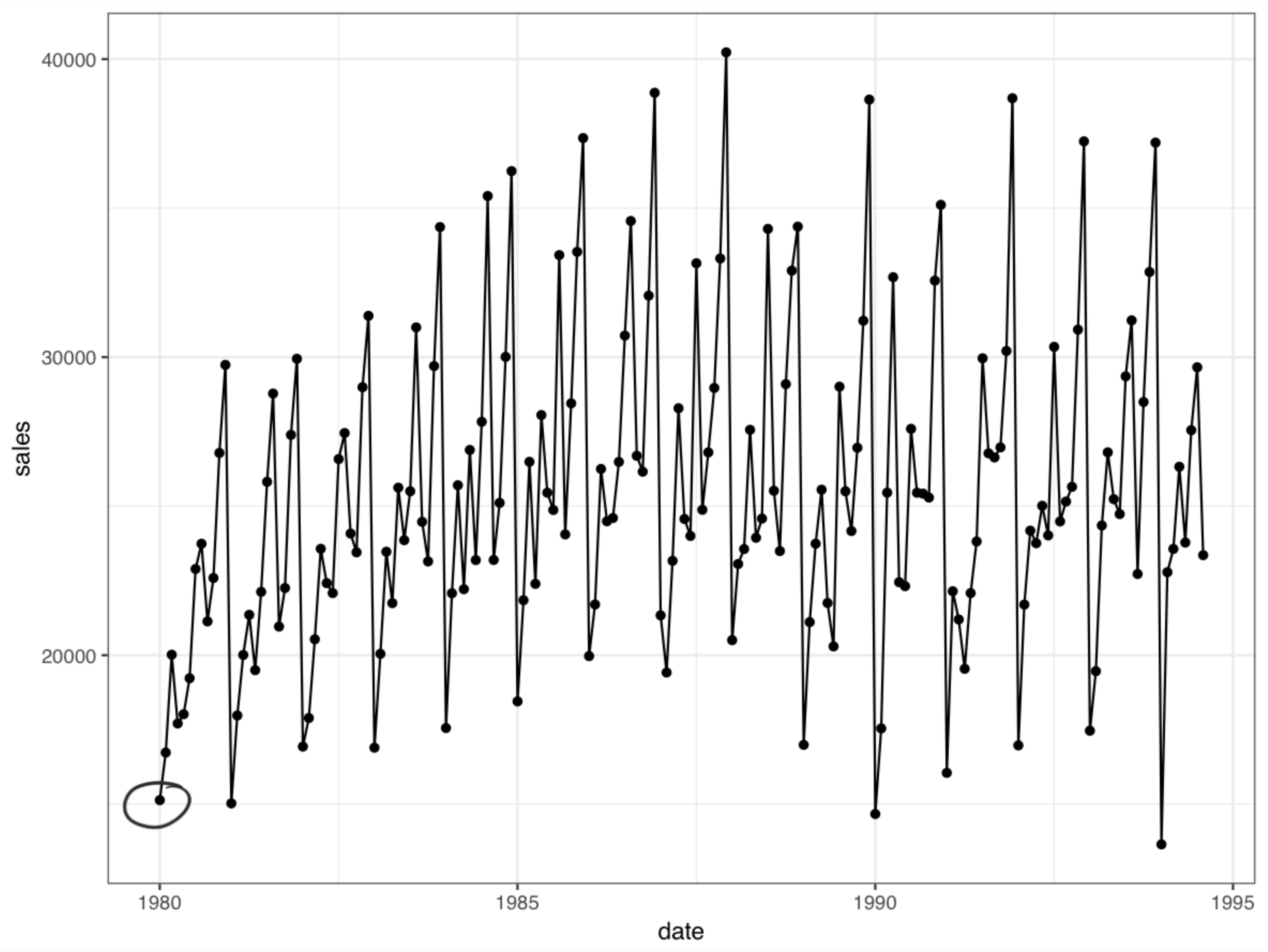
Series ar$y
Partial ACF
0.4
0.0
−0.4
−0.8
0
10
20
30
40
50
Lag

Example - Australian Wine Sales

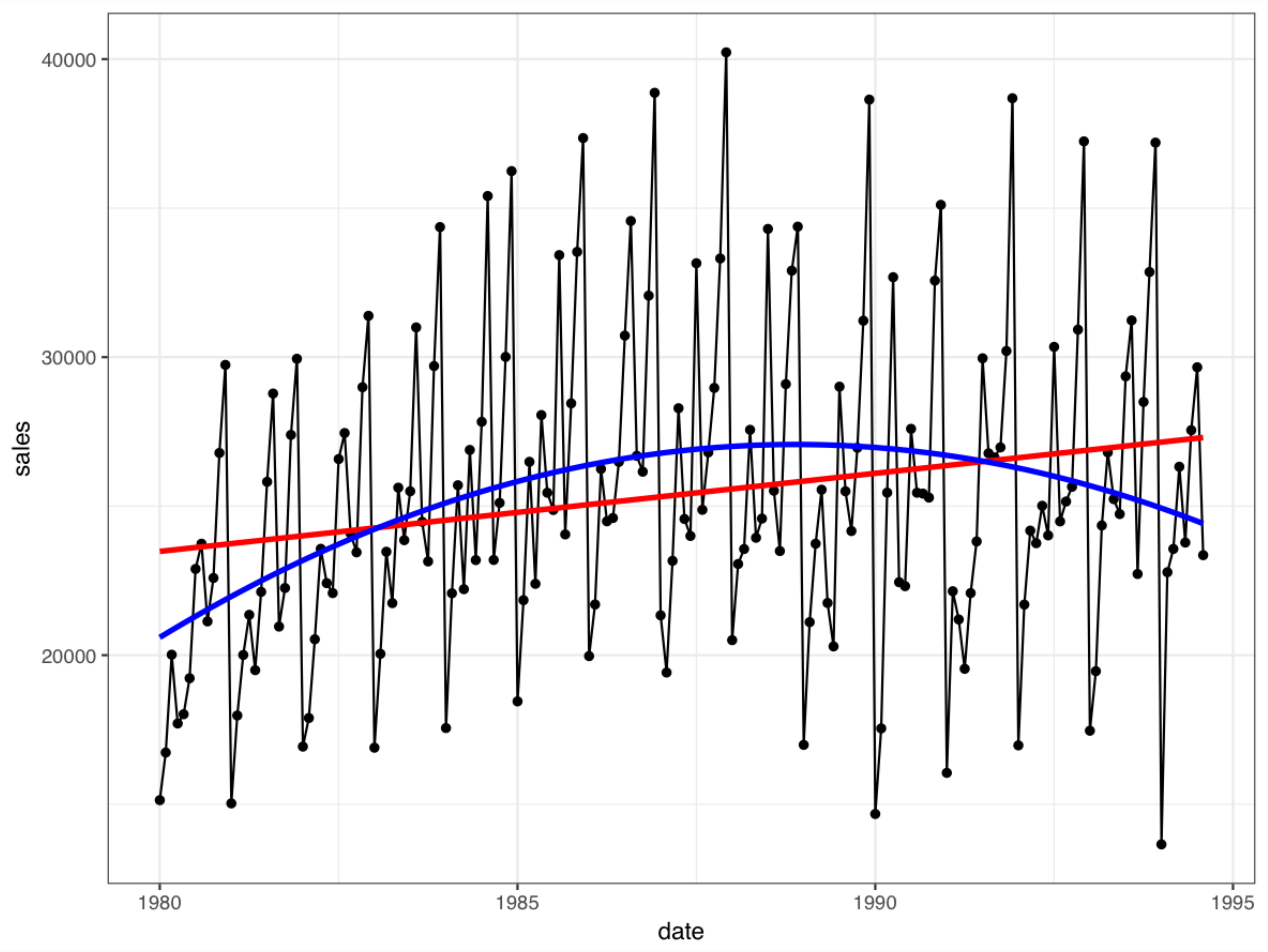
Australian total wine sales by wine makers in bottles <= 1 litre. Jan 1980 – Aug 1994.

```r
load(url("http://www.stat.duke.edu/~cr173/Sta444_Sp17/data/aus_wine.Rdata"))
aus_wine
## # A tibble: 176 × 2
##         date sales
##        <dbl> <dbl>
## 1  1980.000 15136
## 2  1980.083 16733
## 3  1980.167 20016
## 4  1980.250 17708
## 5  1980.333 18019
## 6  1980.417 19227
## 7  1980.500 22893
## 8  1980.583 23739
## 9  1980.667 21133
## 10 1980.750 22591
## # ... with 166 more rows
```

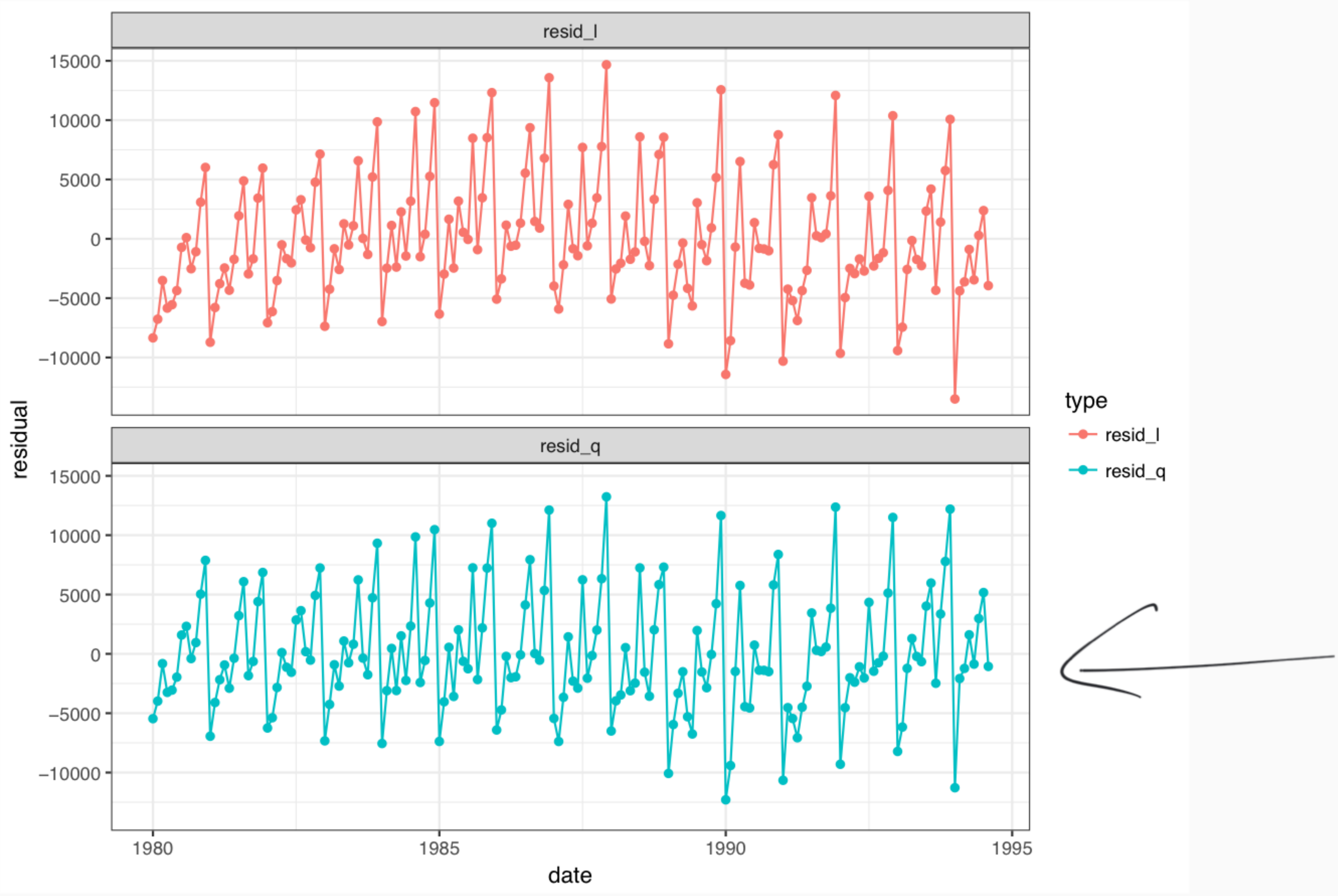
Time series



Basic Model Fit

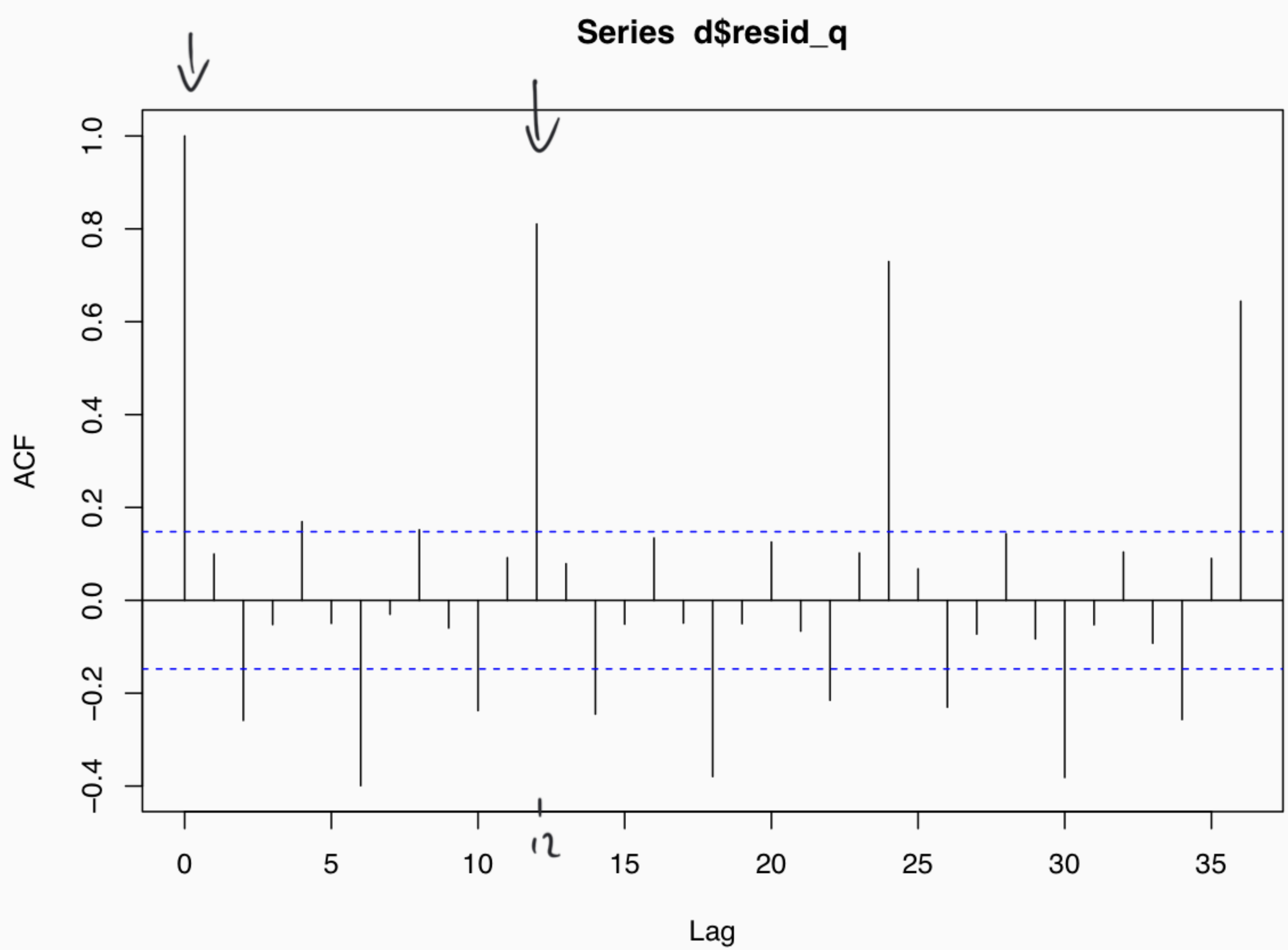


Residuals

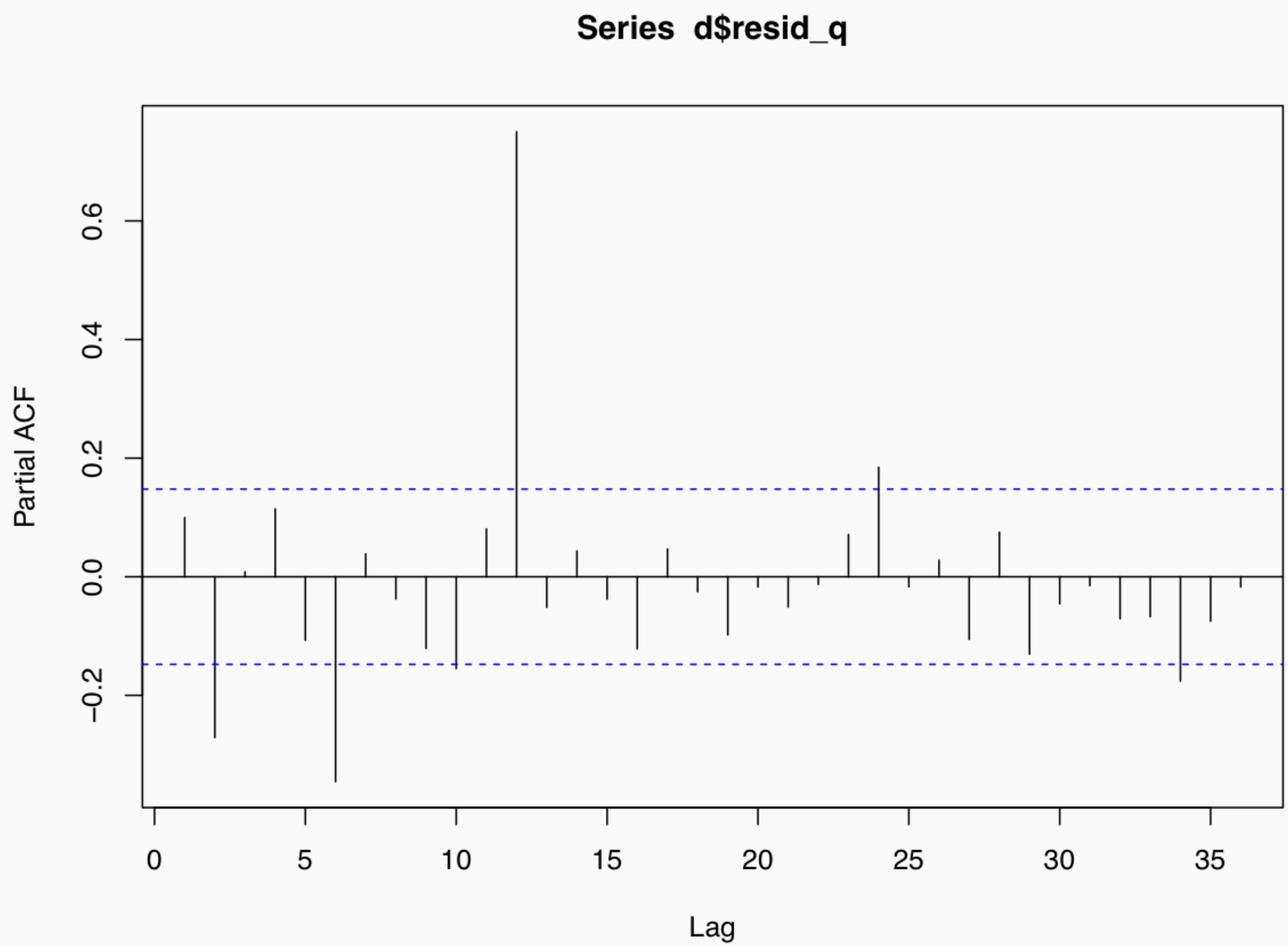


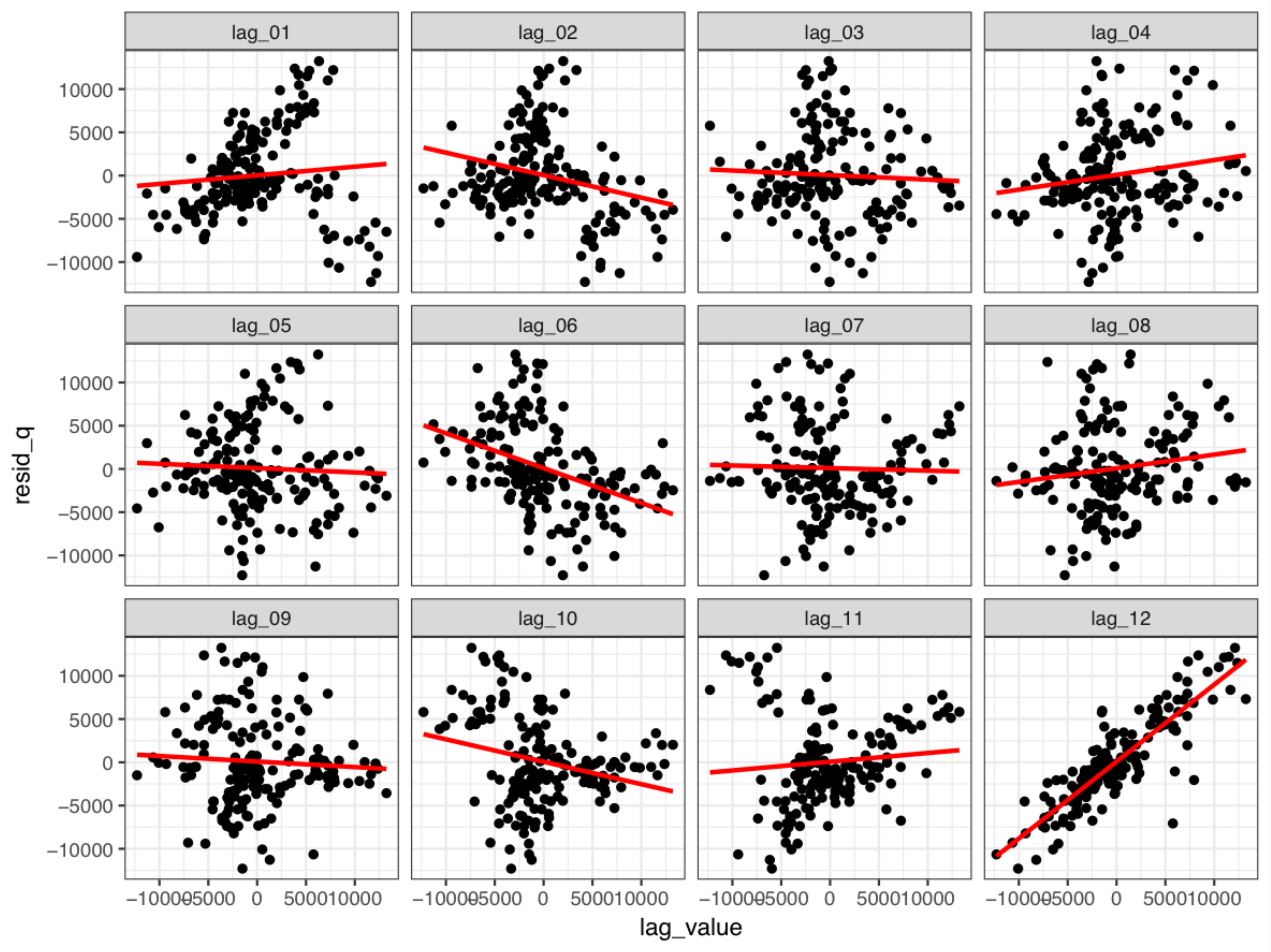
resid_l
resid_q
15000
10000
5000
0
−5000
−10000
residual
1980
1985
1990
1995
date
type
resid_l
resid_q

Autocorrelation Plot



Partial Autocorrelation Plot



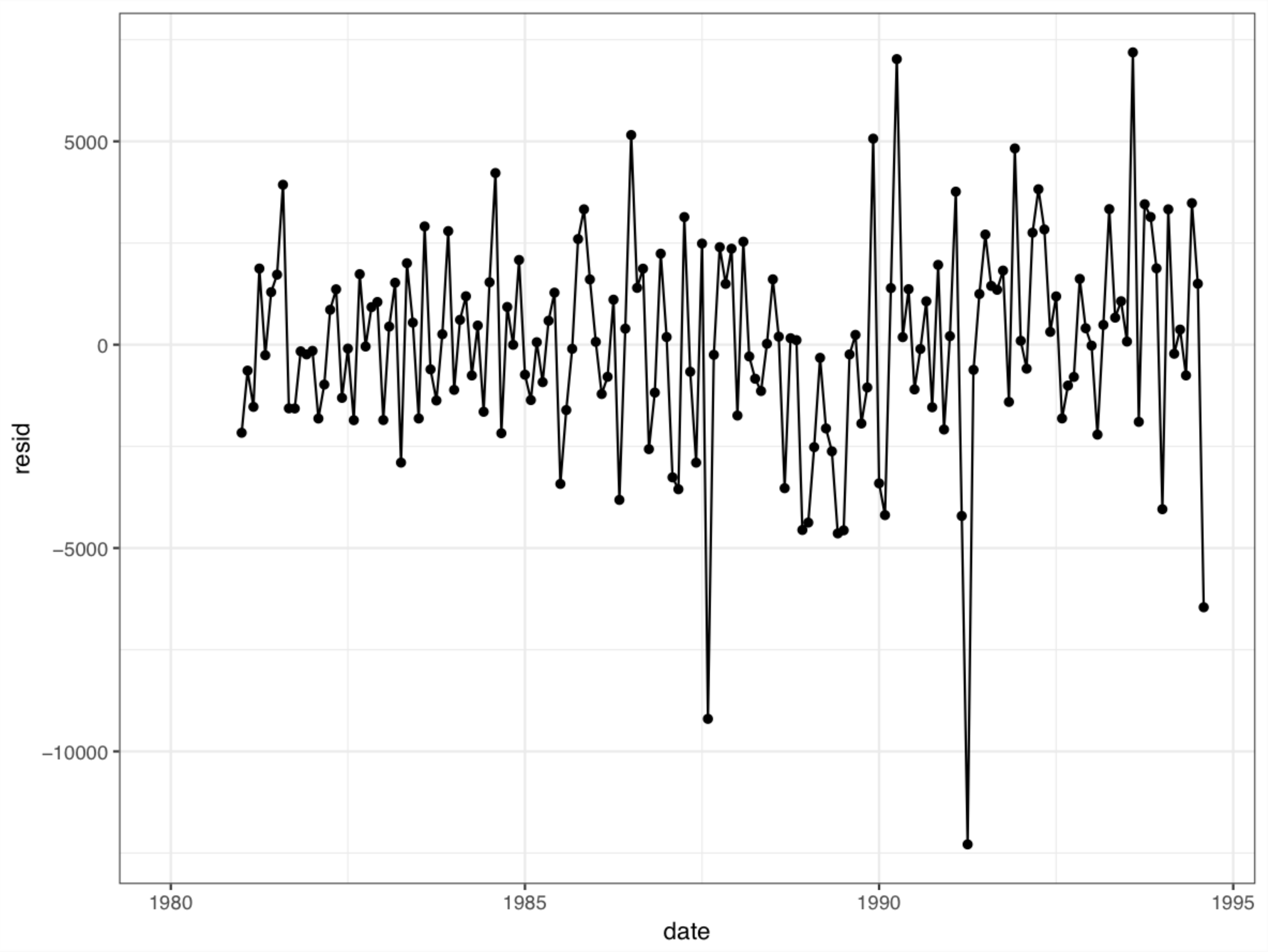


lag_01
lag_02
lag_03
lag_04
lag_05
lag_06
lag_07
lag_08
lag_09
lag_10
lag_11
lag_12
10000
5000
0
−5000
−10000
resid_q
−10000−5000 0 500010000
lag_value

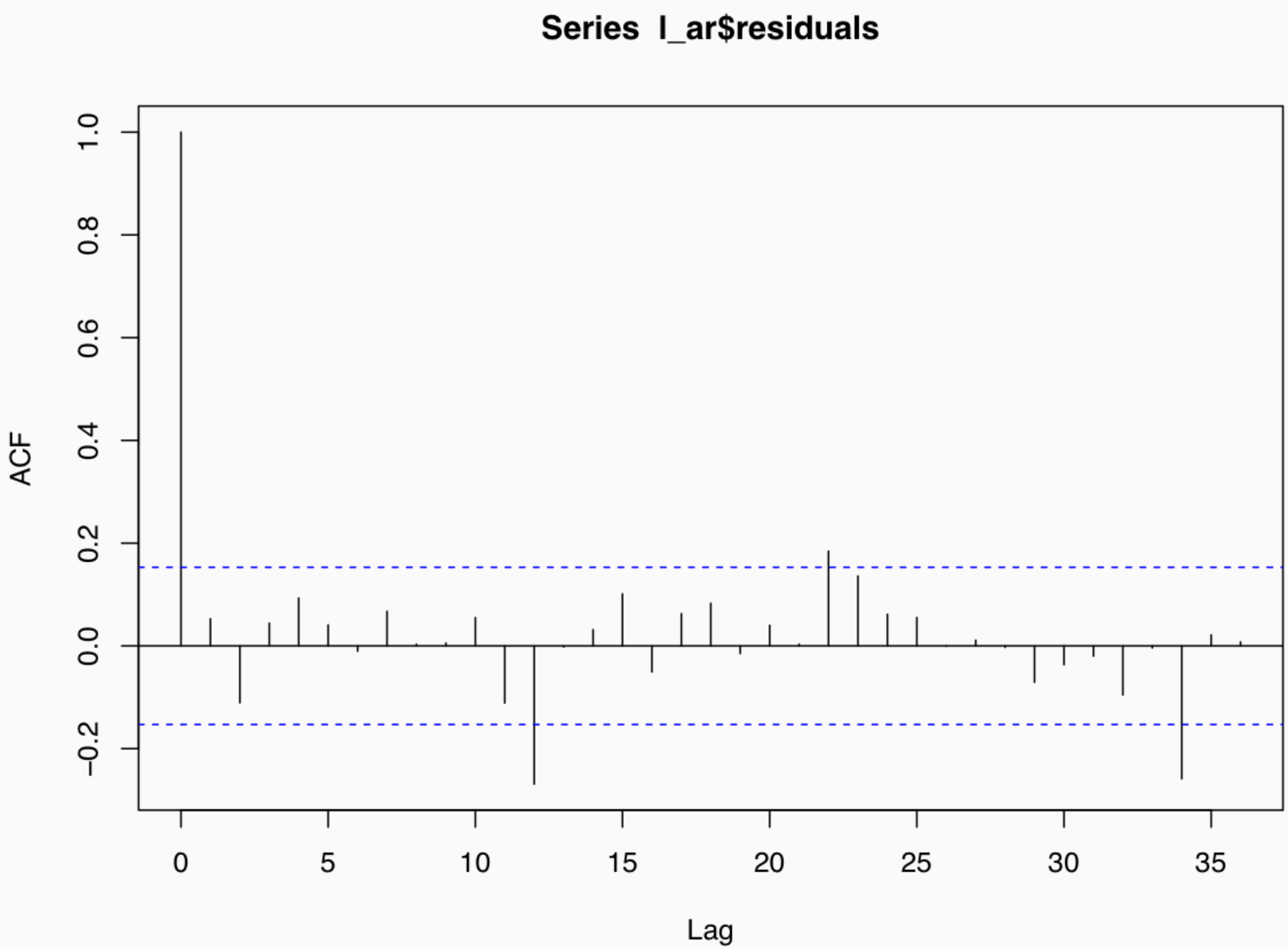
Auto regressive errors

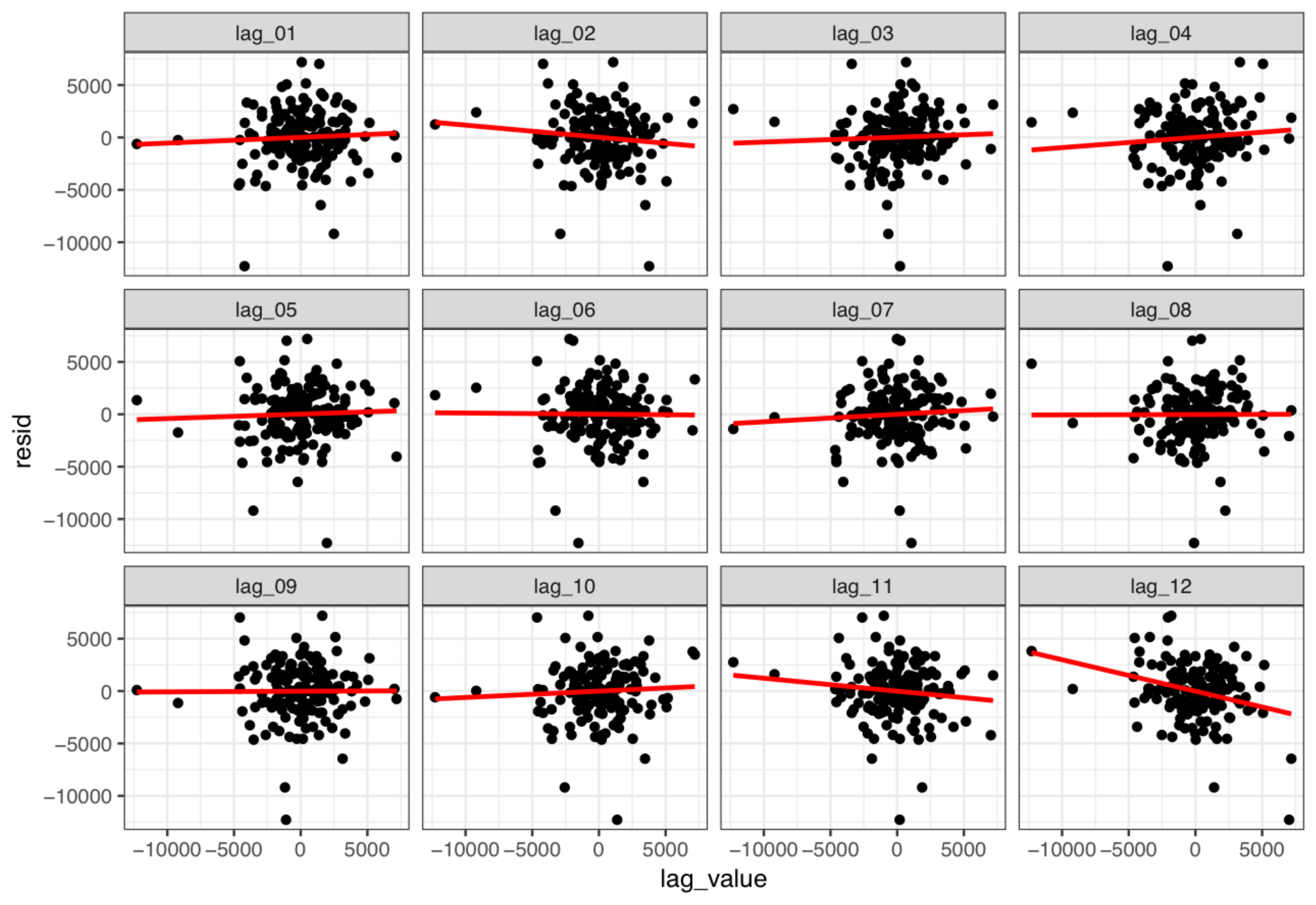
```
##
## Call:
## lm(formula = resid_q ~ lag_12, data = d_ar)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -12286.5  -1380.5     73.4   1505.2   7188.1
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 83.65080  201.58416   0.415    0.679
## lag_12       0.89024    0.04045  22.006   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2581 on 162 degrees of freedom
##   (12 observations deleted due to missingness)
## Multiple R-squared:  0.7493, Adjusted R-squared:  0.7478
## F-statistic: 484.3 on 1 and 162 DF,  p-value: < 2.2e-16
```

Residual residuals



Residual residuals - acf





lag_01
lag_02
lag_03
lag_04
lag_05
lag_06
lag_07
lag_08
lag_09
lag_10
lag_11
lag_12
resid
5000
0
−5000
−10000
−10000 −5000 0 5000
lag_value

Writing down the model?

So, is our EDA suggesting that we then fit the following model?

$$\text{sales}(t) = \beta_0 + \beta_1\, t + \beta_2\, t^2 + \beta_3\, \text{sales}(t-12) + \epsilon_t$$

. . .

the implied model is,

$$\text{sales}(t) = \beta_0 + \beta_1\, t + \beta_2\, t^2 + w_t$$

where

$$w_t = \delta\, w_{t-12} + \epsilon_t$$

$$S_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 S_{t-12} + \varepsilon_t$$

$$S_{13} = \beta_0 + \beta_1 (13) + \beta_2 (13^2) + \beta_3 \left(\beta_0 + \beta_1 (1) + \beta_2 (2) + \beta_3 0 + \varepsilon_1 \right) + \varepsilon_{13}$$

$$= \beta_0 (1 + \beta_3) + \beta_1 \left(13 + \beta_3 (1) \right) + \ldots$$