Lecture 9

ARIMA Models

Colin Rundel

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$MA(\infty)$

$MA(q)$

From last time,

$$MA(q): \qquad y_t = \delta + w_t + \theta_1 \, w_{t-1} + \theta_2 \, w_{t-2} + \cdots + \theta_q \, w_{t-q}$$

Properties:

$$E(y_t) = \delta$$

$$Var(y_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \, \sigma_w^2$$

$$Cov(y_t, y_{t+h}) = \begin{cases} \theta_h + \theta_1 \, \theta_{1+h} + \theta_2 \, \theta_{2+h} + \cdots + \theta_{q-h} \, \theta_q & \text{if } |h| \le q \\ 0 & \text{if } |h| > q \end{cases}$$

and is stationary for any values of θ_i

$MA(\infty)$

If we let $q \to \infty$ then process will still be stationary if the moving average coefficients (θ 's) are square summable,

$$\sum_{i=1}^{\infty} \theta_i^2 < \infty$$

since this is necessary for $Var(y_t) < \infty$.

Sometimes, a slightly strong condition called absolute summability, $\sum_{i=1}^{\infty} |\theta_i| < \infty$, is necessary (e.g. for some CLT related asymptotic results) .

Invertibility

If a $MA(q)$ process, $y_t = \delta + \theta_q(L)w_t$, can be rewritten as a purely AR process then it is said that the MA process is invertible.

if $|\theta| < 1$

$MA(1)$ w/ $\delta = 0$ example:

$$y_t = w_t + \theta w_{t-1}$$

$$\boxed{w_t = y_t - \theta w_{t-1}}$$

$$= y_t - \theta(y_{t-1} - \theta w_{t-2})$$

$$= y_t - \theta y_{t-1} + \theta^2 w_{t-2}$$

$$= y_t - \theta y_{t-1} + \theta^2(y_{t-2} - \theta w_{t-3})$$

$$= y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 w_{t-3}$$

$$= \sum_{i=0}^{p-1} (-\theta)^i y_{t-i} + (-\theta)^p w_{t-p}$$

$$w_t = y_t - \theta y_{t-1} + \theta^2 y_{t-2} \dots + (-\theta)^p y_{t-p}$$

$$y_t = w_t + \theta y_{t-1} - \theta^2 y_{t-2} + \dots$$

$AR(\infty)$

Invertibility vs Stationarity

A $MA(q)$ process is *invertible* if $y_t = \delta + \theta_q(L)\, w_t$ can be rewritten as an exclusively AR process (of possibly infinite order), i.e. $\phi(L)\, y_t = \alpha + w_t$.

Invertibility vs Stationarity

A $MA(q)$ process is *invertible* if $y_t = \delta + \theta_q(L)\, w_t$ can be rewritten as an exclusively AR process (of possibly infinite order), i.e. $\phi(L)\, y_t = \alpha + w_t$.

Conversely, an $AR(p)$ process is *stationary* if $\phi_p(L)\, y_t = \delta + w_t$ can be rewritten as an exclusively MA process (of possibly infinite order), i.e. $y_t = \delta + \theta(L)\, w_t$.

Invertibility vs Stationarity

A $MA(q)$ process is *invertible* if $y_t = \delta + \theta_q(L)\, w_t$ can be rewritten as an exclusively AR process (of possibly infinite order), i.e. $\phi(L)\, y_t = \alpha + w_t$.

Conversely, an $AR(p)$ process is *stationary* if $\phi_p(L)\, y_t = \delta + w_t$ can be rewritten as an exclusively MA process (of possibly infinite order), i.e. $y_t = \delta + \theta(L)\, w_t$.

So using our results w.r.t. $\phi(L)$ it follows that if all of the roots of $\theta_q(L)$ are outside the complex unit circle then the moving average is invertible.

Differencing

Difference operator

We will need to define one more notational tool for indicating differencing

$$\Delta y_t = y_t - y_{t-1}$$

just like the lag operator we will indicate repeated applications of this operator using exponents

$$\begin{aligned}
\Delta^2 y_t &= \Delta(\Delta y_t) \\
&= (\Delta y_t) - (\Delta y_{t-1}) \\
&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\
&= y_t - 2y_{t-1} + y_{t-2}
\end{aligned}$$

Δ can also be expressed in terms of the lag operator L,

$$\Delta^d = (1 - L)^d$$

Differencing and Stocastic Trend

Using the two component time series model

$$y_t = \mu_t + x_t$$

where μ_t is a non-stationary trend component and x_t is a mean zero stationary component.

We have already shown that differencing can address deterministic trend (e.g. $\mu_t = \beta_0 + \beta_1 t$). In fact, if μ_t is any k-th order polynomial of t then $\Delta^k y_t$ is stationary.

Differencing can also address stochastic trend such as in the case where μ_t follows a random walk.

Stochastic trend - Example 1

Let $y_t = \mu_t + w_t$ where w_t is white noise and $\mu_t = \mu_{t-1} + v_t$ with v_t stationary as well. Is Δy_t stationary?

$$\Delta y_t = y_t - y_{t-1}$$

$$= (\mu_t + v_t) - (\mu_{t-1} + w_{t-1})$$

$$= (\mu_{\cancel{t-1}} + v_t + w_t) - (\mu_{\cancel{t-1}} + w_{t-1})$$

$$= v_t + w_t + v_{t-1}$$

$\Rightarrow$ Stationary

Stochastic trend - Example 2

Let $y_t = \mu_t + w_t$ where w_t is white noise and $\mu_t = \mu_{t-1} + v_t$ but now $v_t = v_{t-1} + e_t$ with e_t being stationary. Is Δy_t stationary? What about $\Delta^2 y_t$, is it stationary?

$$\Delta y_t = (\mu_t + w_t) - (\mu_{t-1} + w_{t-1})$$

$$= (\mu_{t-1} + v_t + w_t) - (\mu_{t-1} + w_{t-1})$$

$$= \underbrace{v_t + w_t - w_{t-1}}$$

$$= \underbrace{\cancel{v_{t-1} + e_t + w_t - w_{t-1}}}$$

$$\Delta^2(y_t) = (v_t + w_t - w_{t-1}) - (v_{t-1} + w_{t-1} - w_{t-2})$$

$$= \cancel{v_{t-1}} + e_t + w_t - w_{t-1} - (\cancel{v_{t-1}} + w_{t-1} - w_{t-2})$$

ARIMA

ARIMA Models

Autoregressive integrated moving average are just an extension of an *ARMA* model to include differencing of degree d to y_t, which is most often used to address trend in the data.

$$ARIMA(p, d, q): \qquad \phi_p(L)\,\Delta^d\,y_t = \delta + \theta_q(L)w_t$$

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Autoregressive integrated moving average are just an extension of an *ARMA* model to include differencing of degree d to y_t, which is most often used to address trend in the data.

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Box-Jenkins approach:

1. Transform data if necessary to stabilize variance
2. Choose order (p, d, and q) of ARIMA model
3. Estimate model parameters (ϕs and θs)
4. Diagnostics

Using `forecast` - random walk with drift

Some of R's base timeseries handling is a bit wonky, the `forecast` package offers some useful alternatives and additional functionality.

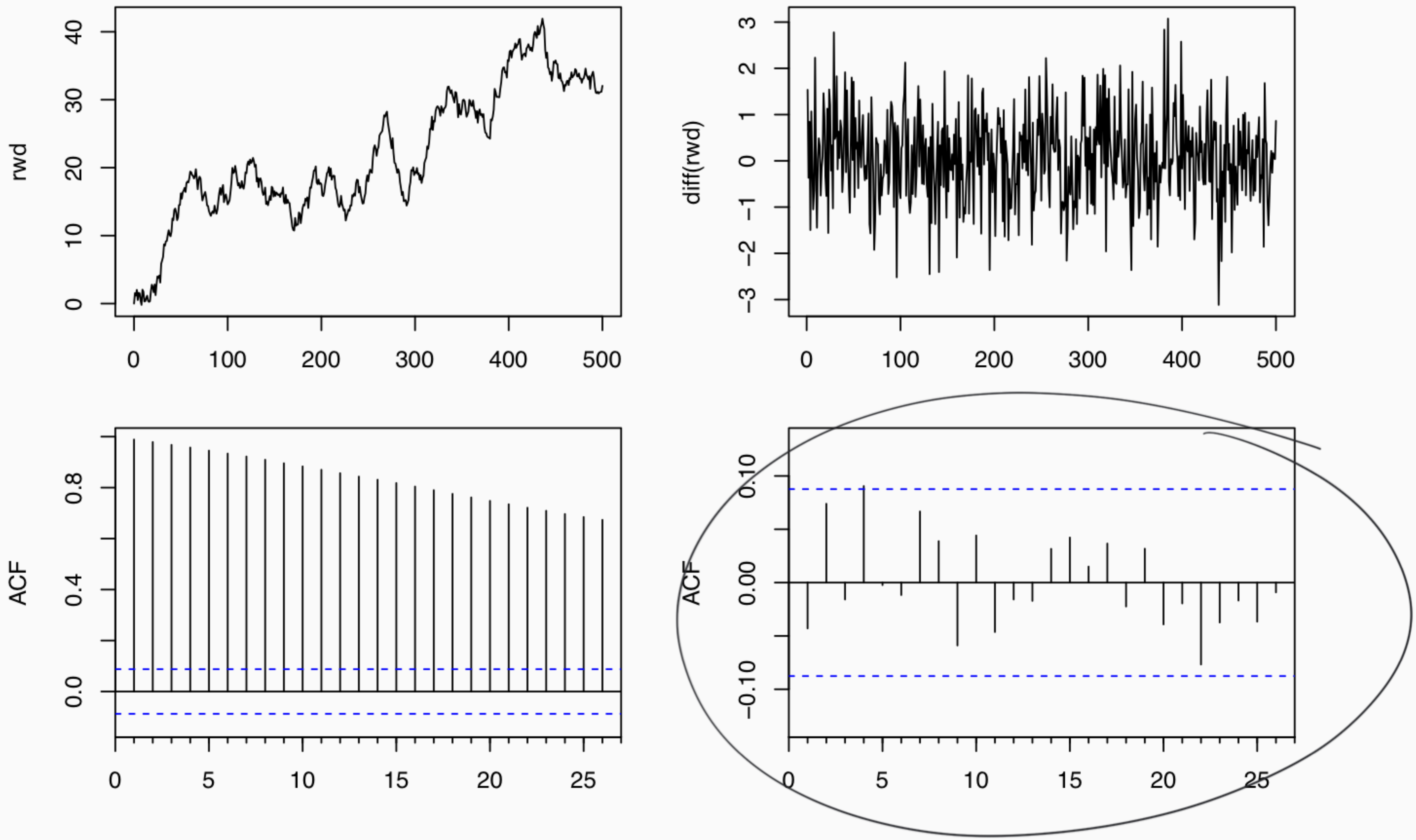
```
rwd = arima.sim(n=500, model=list(order=c(0,1,0)), mean=0.1)

library(forecast)
Arima(rwd, order = c(0,1,0), include.constant = TRUE)
## Series: rwd
## ARIMA(0,1,0) with drift
##
## Coefficients:
##         drift
##        0.0641
## s.e.   0.0431
##
## sigma^2 estimated as 0.9323:  log likelihood=-691.44
## AIC=1386.88   AICc=1386.91   BIC=1395.31
```

$$\Delta y_t = \delta + v_t$$

$$y_t - y_{t-1} = \delta + v_t$$

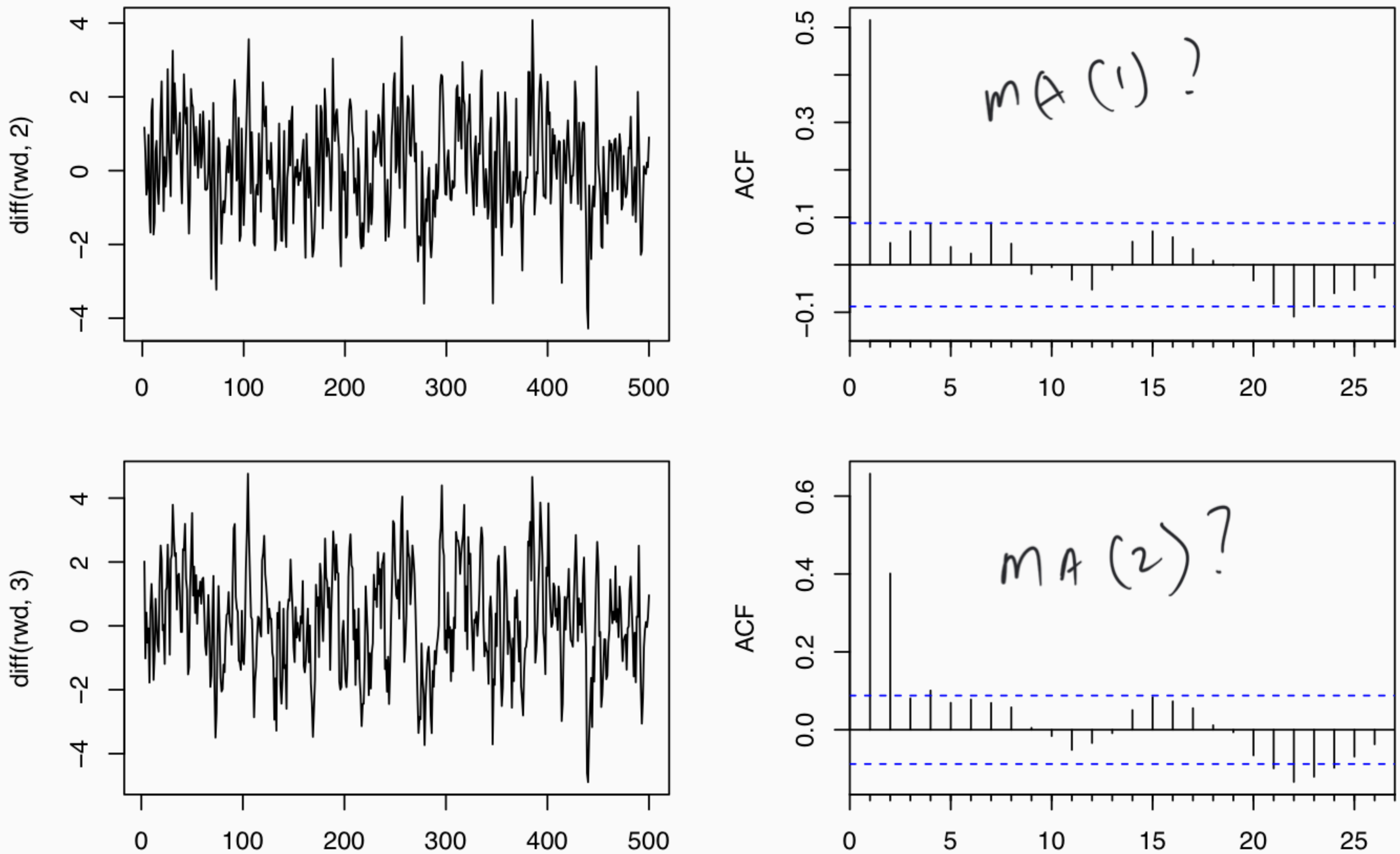
$$y_t = \delta + y_{t-1} + v_t$$

EDA

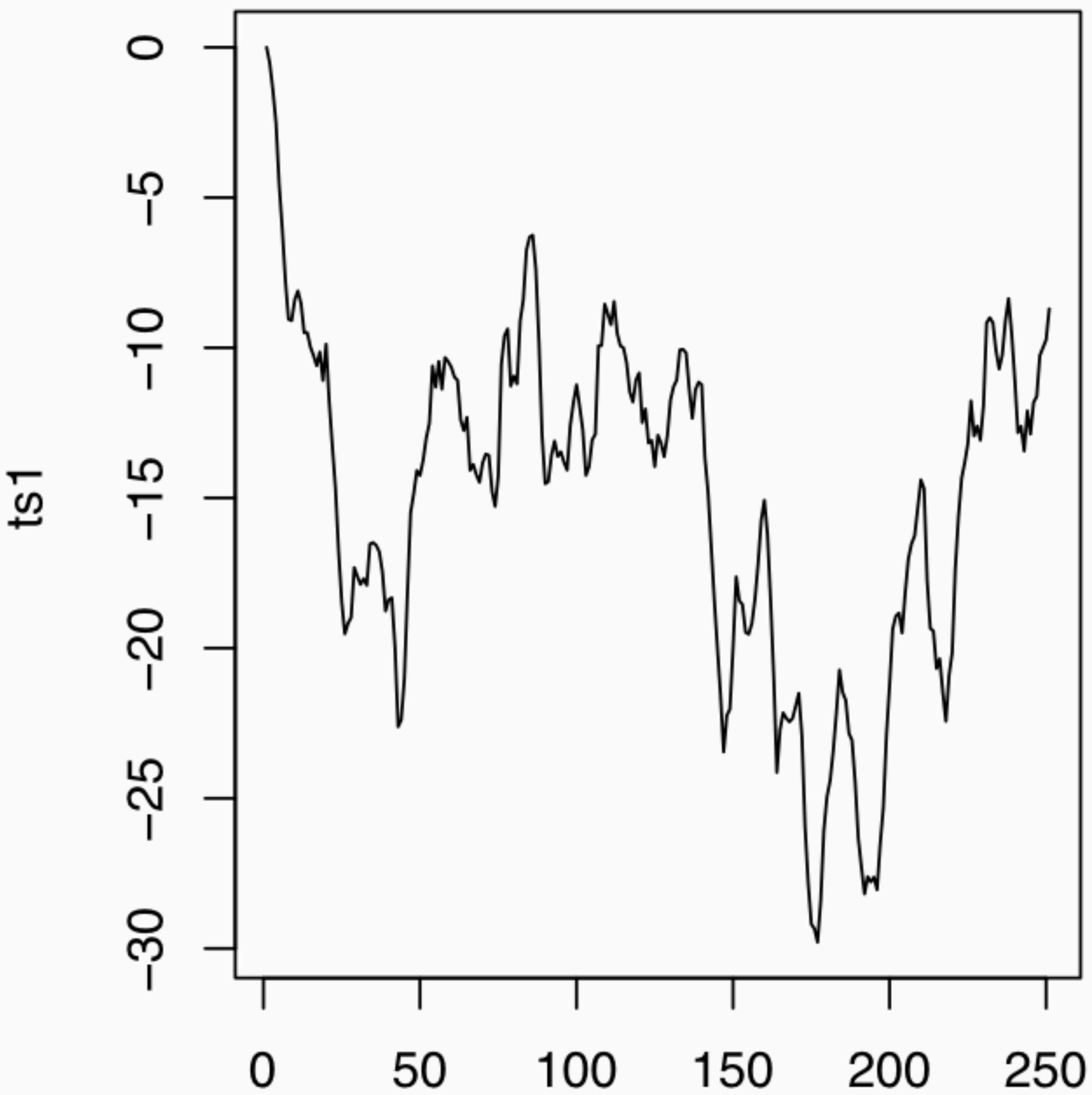


Over differencing

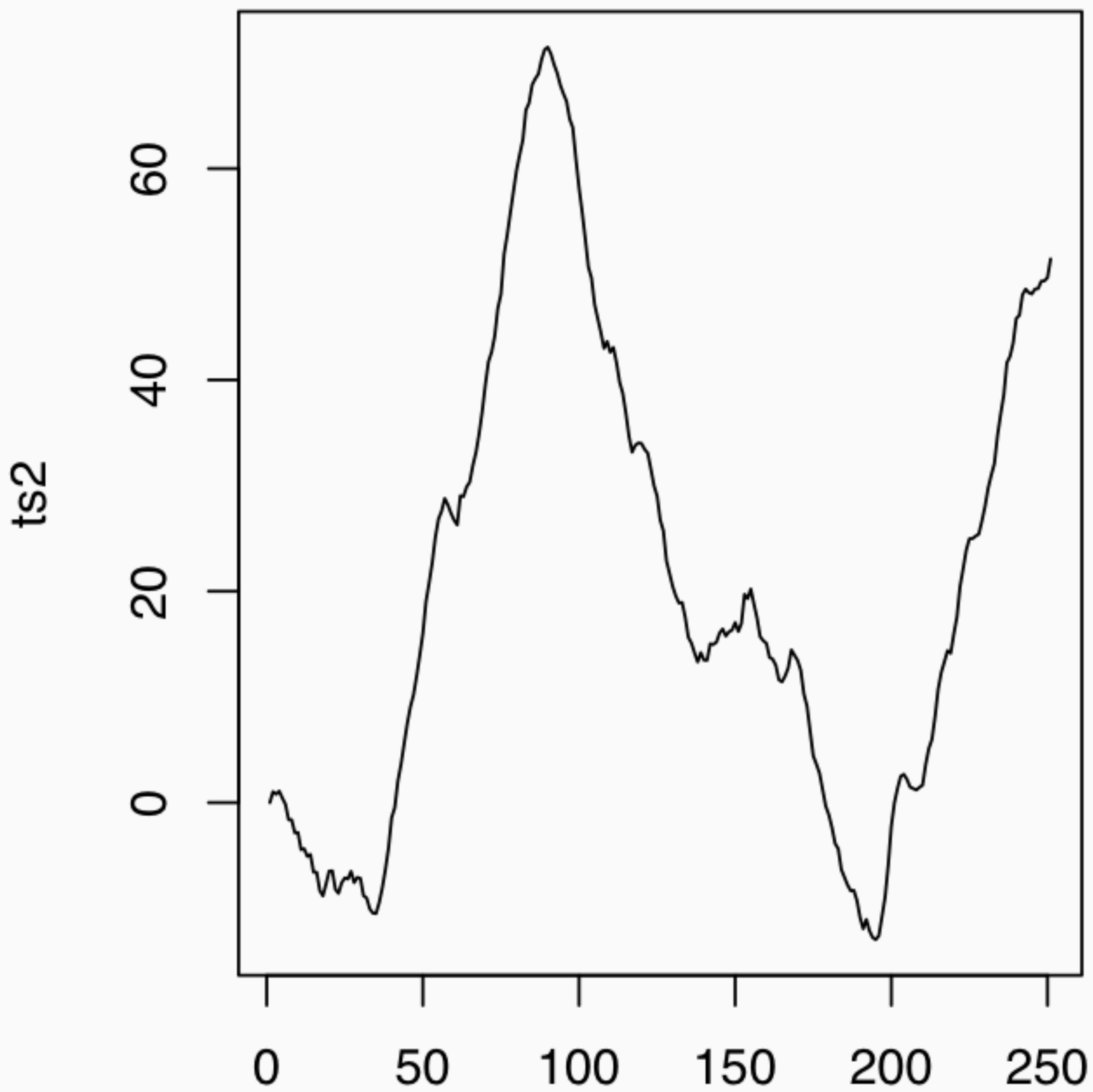
Acf vs acf



AR or MA?

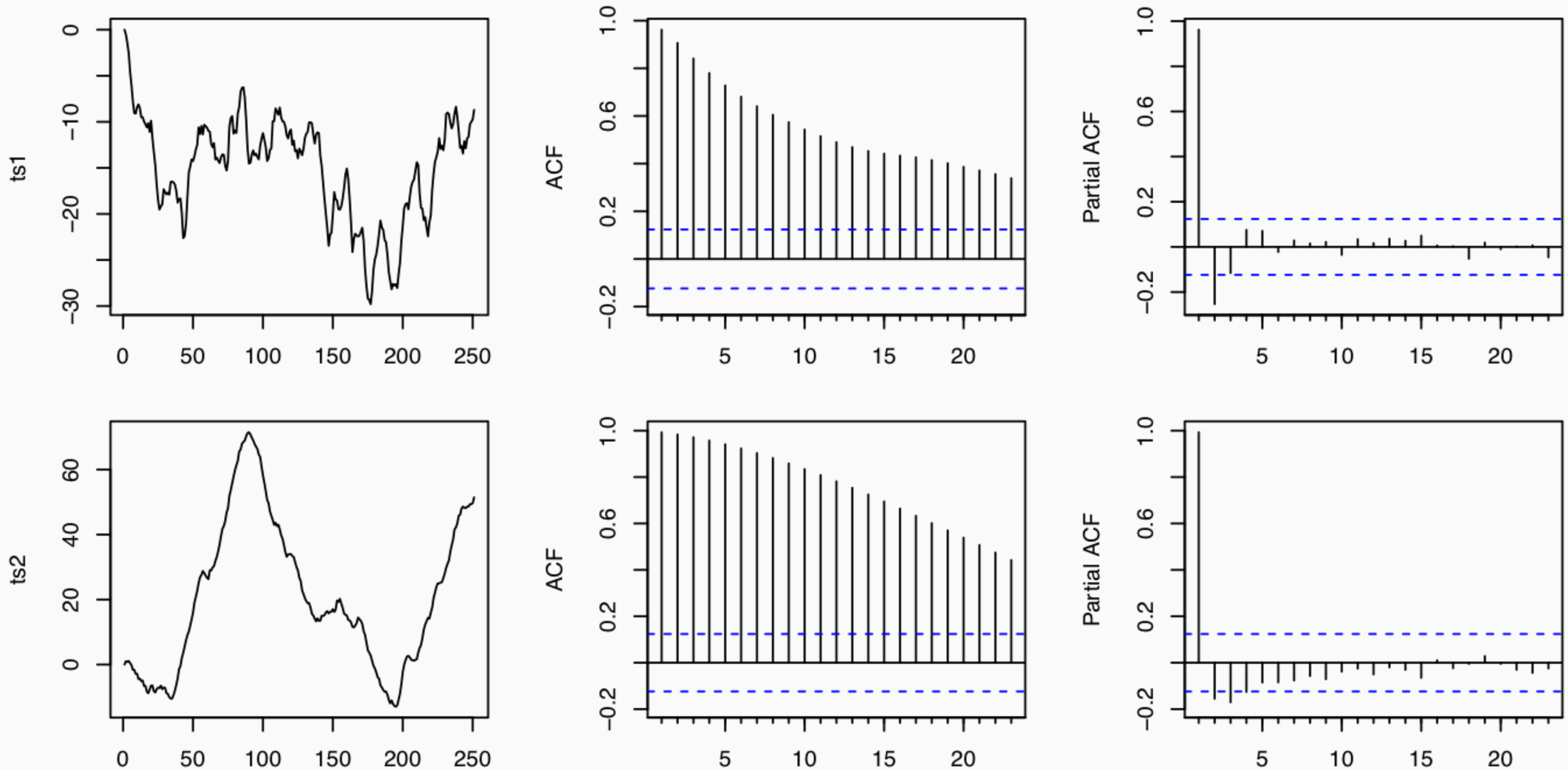


ts1
0
−5
−10
−15
−20
−25
−30
0
50
100
150
200
250



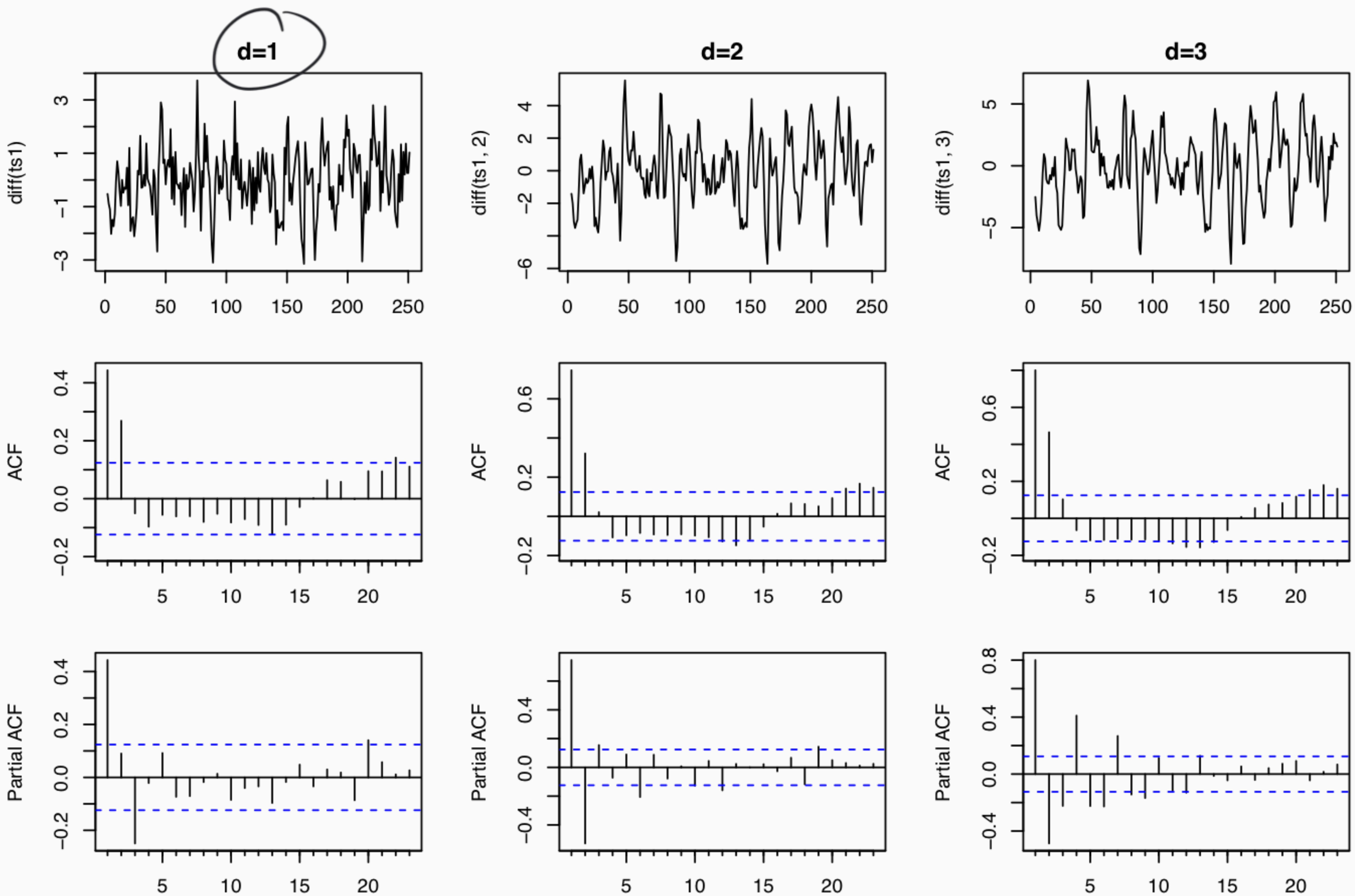
ts2
60
40
20
0
0
50
100
150
200
250

EDA



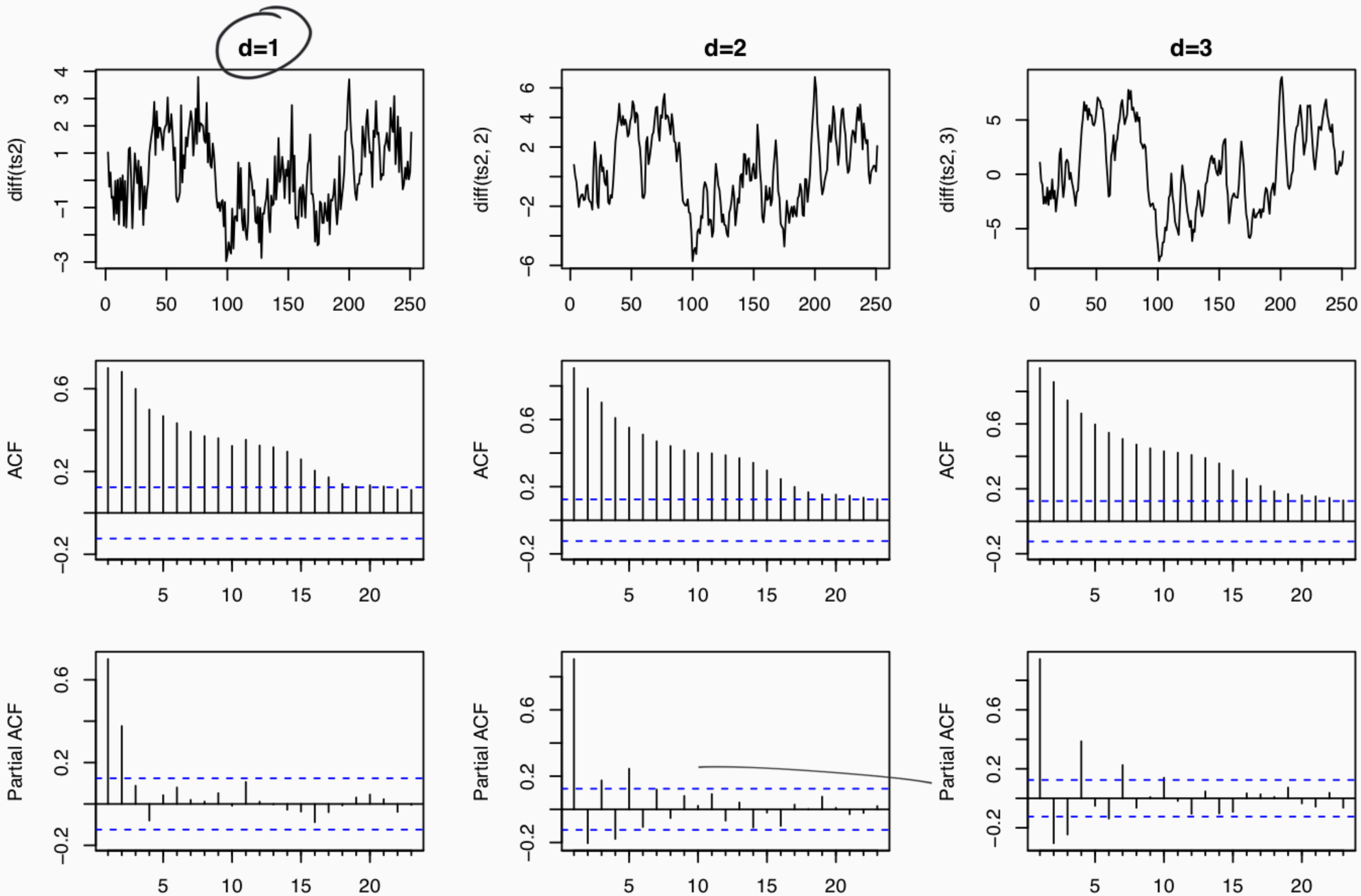
ts1
0
−10
−20
−30
0
50
100
150
200
250
ACF
1.0
0.6
0.2
−0.2
5
10
15
20
Partial ACF
ts2
60
40
20
0

ts1 - Finding d



ARIMA (0,1,2)

ts2 - Finding d



ARIMA(2,1,0)

`ts1` - Models

p	d	q	AIC	BIC
0	1	2	729.43	740.00
1	1	2	731.23	745.31
2	1	2	731.57	749.18
2	1	1	744.29	758.38
2	1	0	747.55	758.12
1	1	0	747.61	754.65
1	1	1	748.65	759.21
0	1	1	764.98	772.02
0	1	0	800.43	803.95

ts2 - Models

p	d	q	AIC	BIC
2	1	0	683.12	693.68
1	1	2	683.25	697.34
2	1	1	683.83	697.92
2	1	2	685.06	702.67
1	1	1	686.38	696.95
1	1	0	719.16	726.20
0	1	2	754.66	765.22
0	1	1	804.44	811.48
0	1	0	890.32	893.85

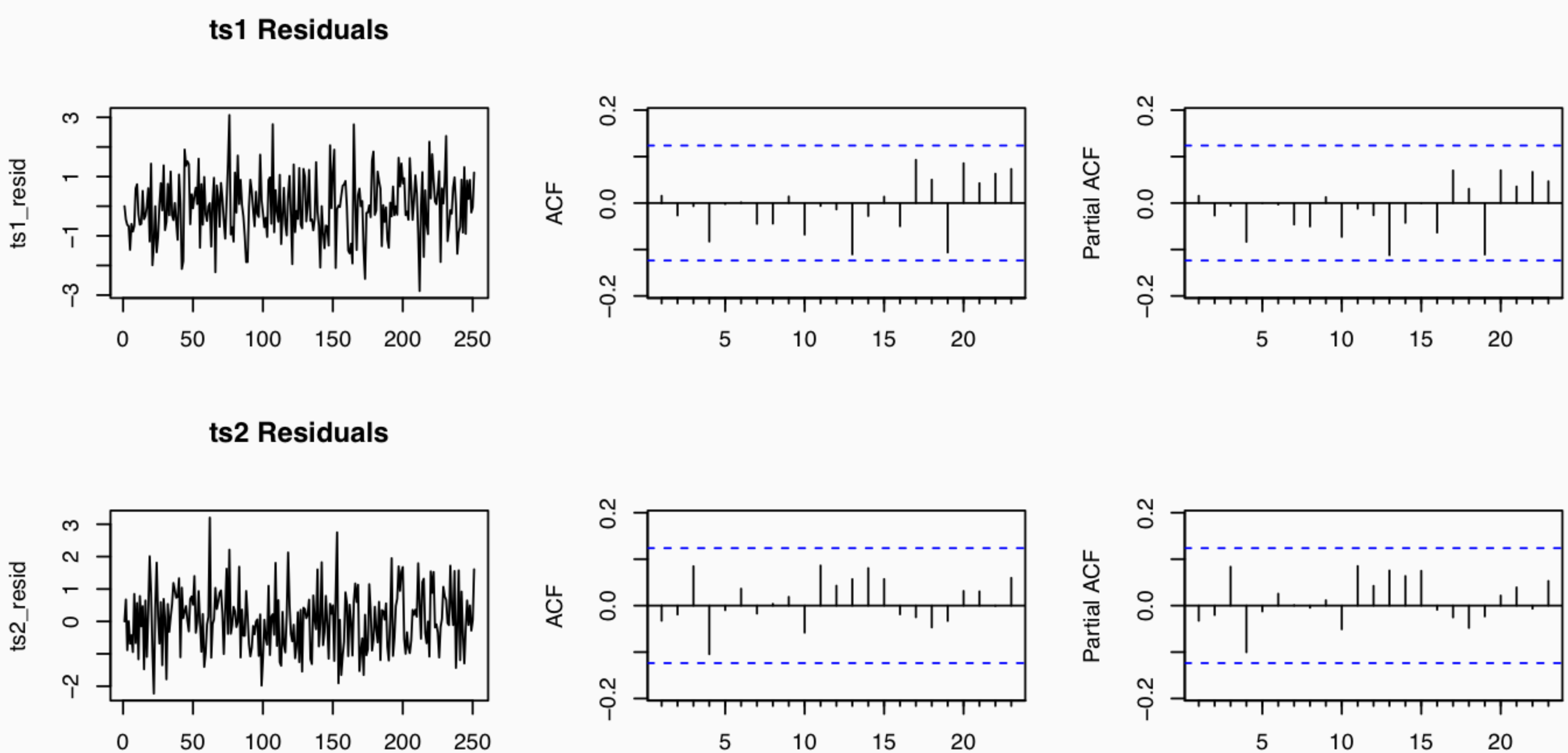
ts1 - Model Choice

```r
Arima(ts1, order = c(0,1,2))
## Series: ts1
## ARIMA(0,1,2)
##
## Coefficients:
##          ma1     ma2
##       0.4138  0.4319
## s.e.  0.0547  0.0622
##
## sigma^2 estimated as 1.064:  log likelihood=-361.72
## AIC=729.43   AICc=729.53   BIC=740
```

ts2 - Model Choice

```
Arima(ts2, order = c(2,1,0))
## Series: ts2
## ARIMA(2,1,0)
##
## Coefficients:
##          ar1     ar2
##       0.4392  0.3770
## s.e.  0.0587  0.0587
##
## sigma^2 estimated as 0.8822:  log likelihood=-338.56
## AIC=683.12   AICc=683.22   BIC=693.68
```

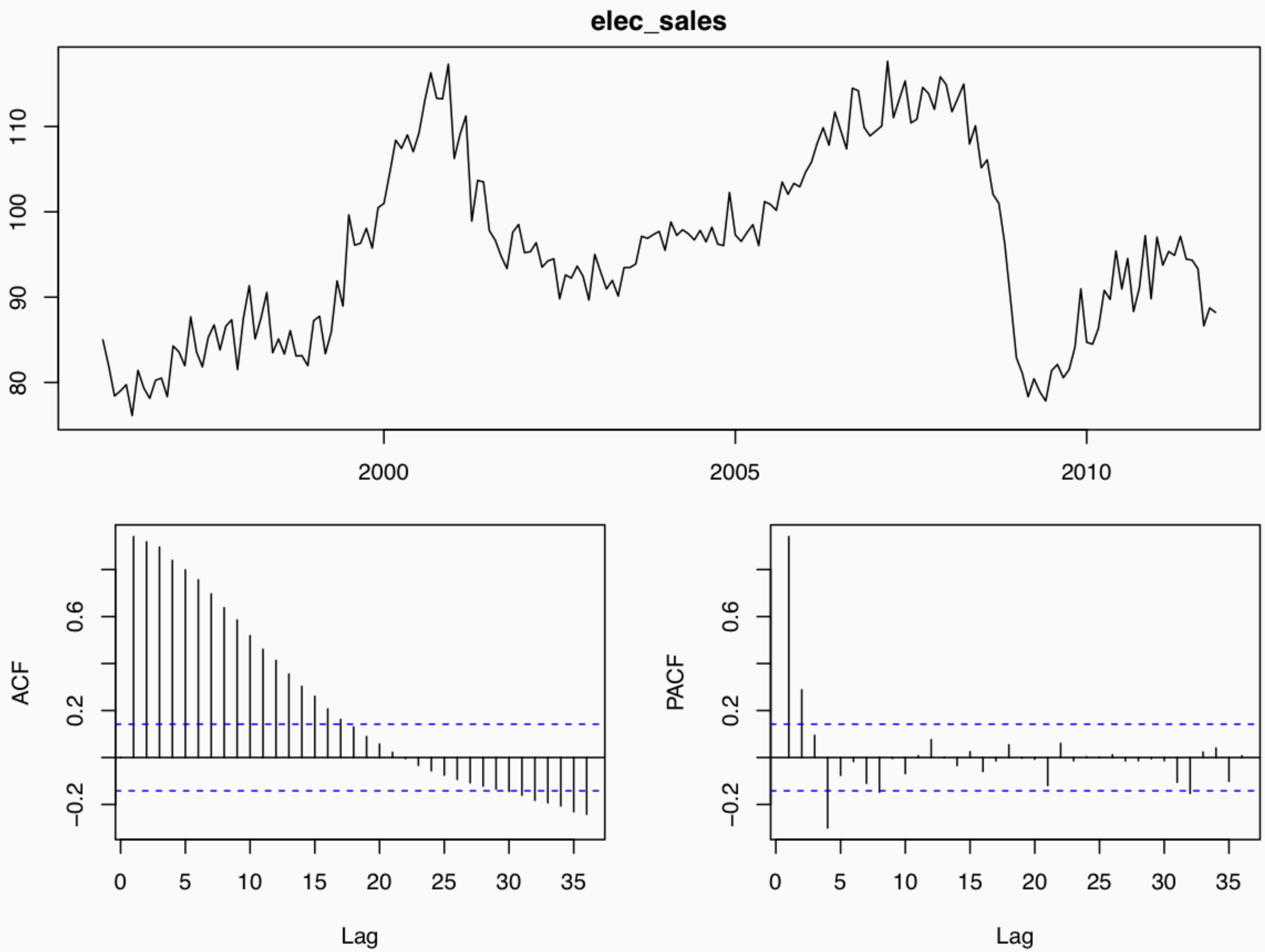
Residuals



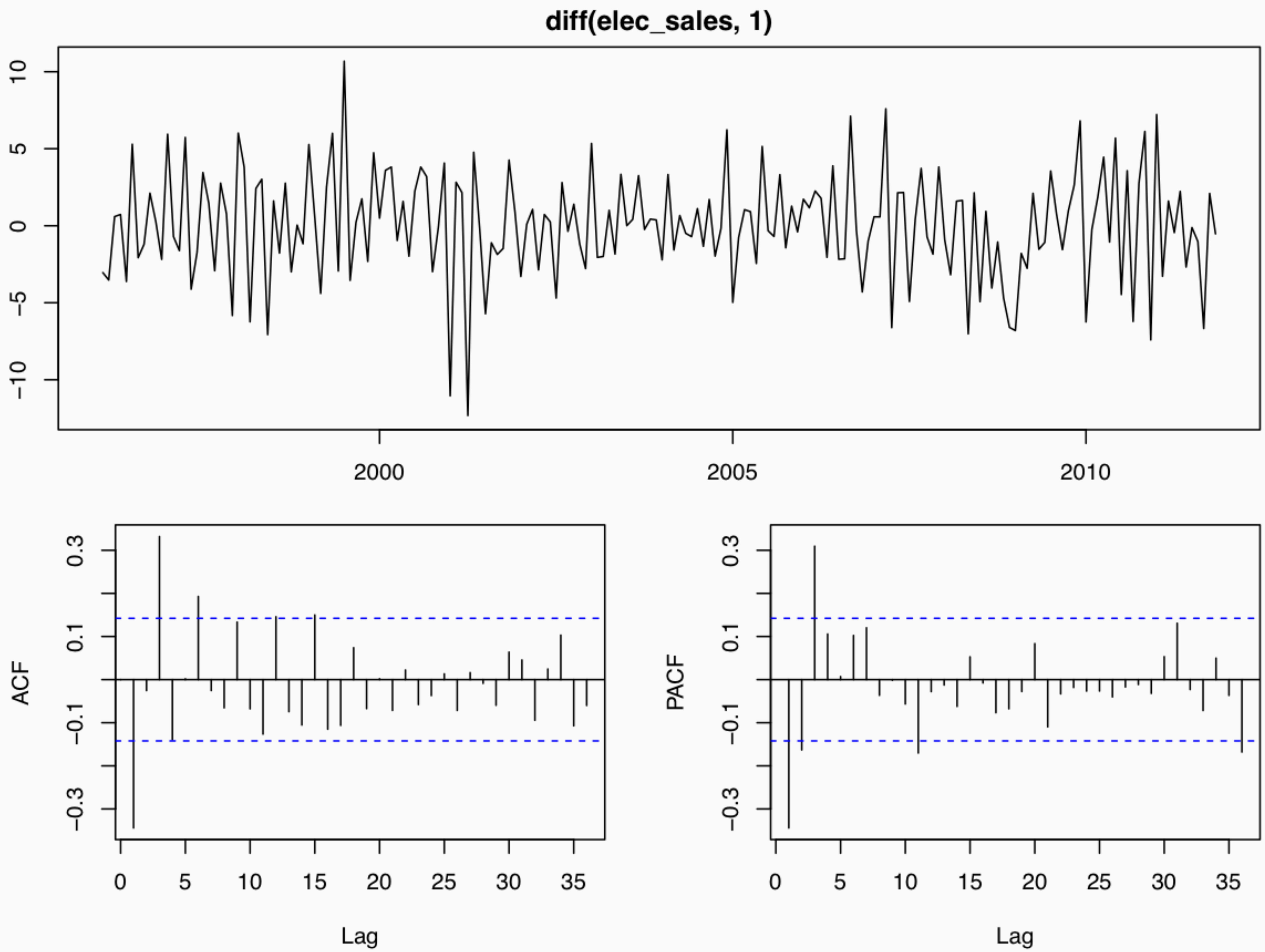
ts1 Residuals
ts1_resid
ACF
Partial ACF
ts2 Residuals
ts2_resid
ACF
Partial ACF

Electrical Equipment Sales

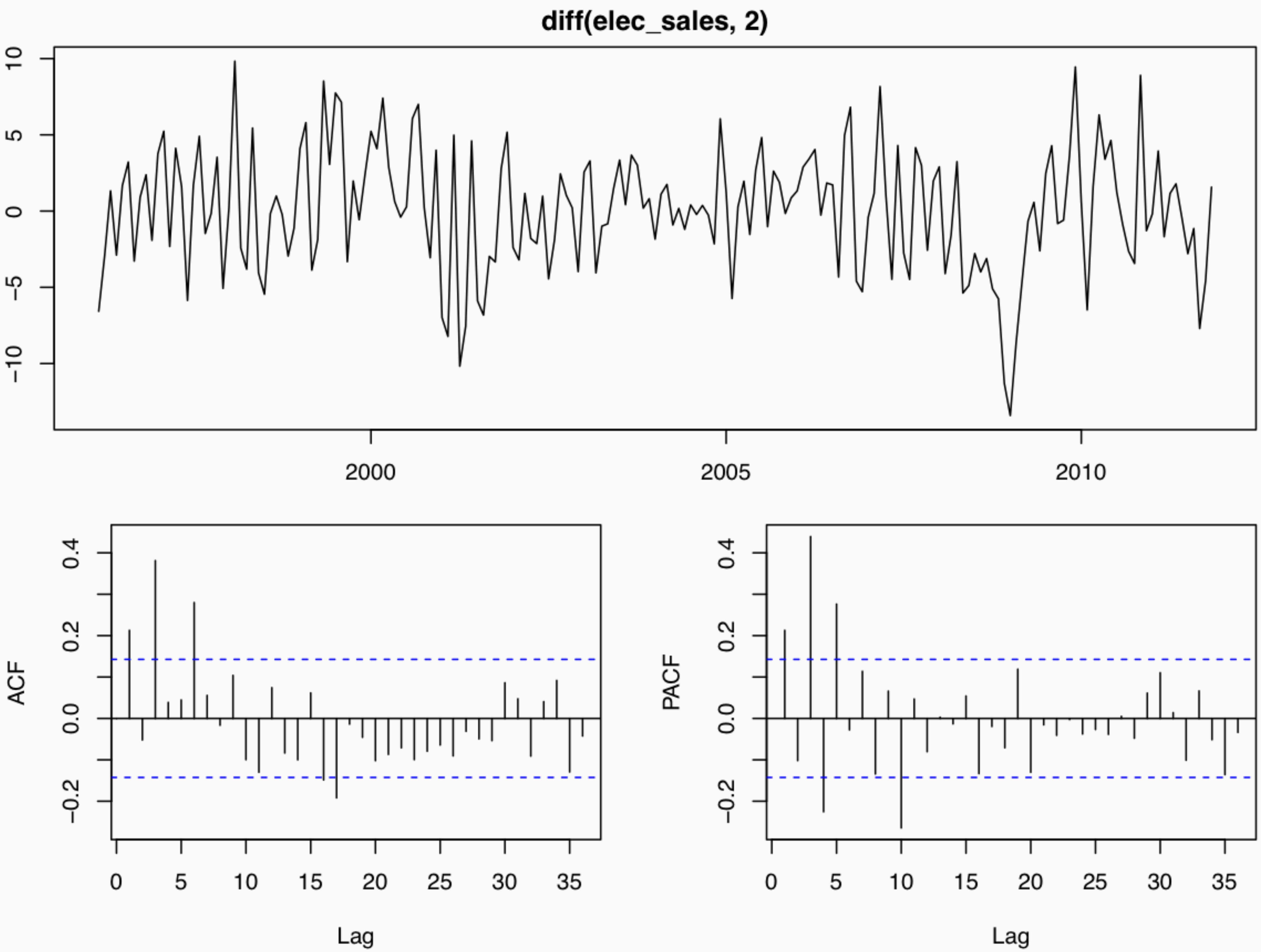
Data


elec_sales
80
90
100
110
2000
2005
2010
ACF
PACF
-0.2
0.2
0.6
0
5
10
15
20
25
30
35
Lag

1st order differencing



2nd order differencing

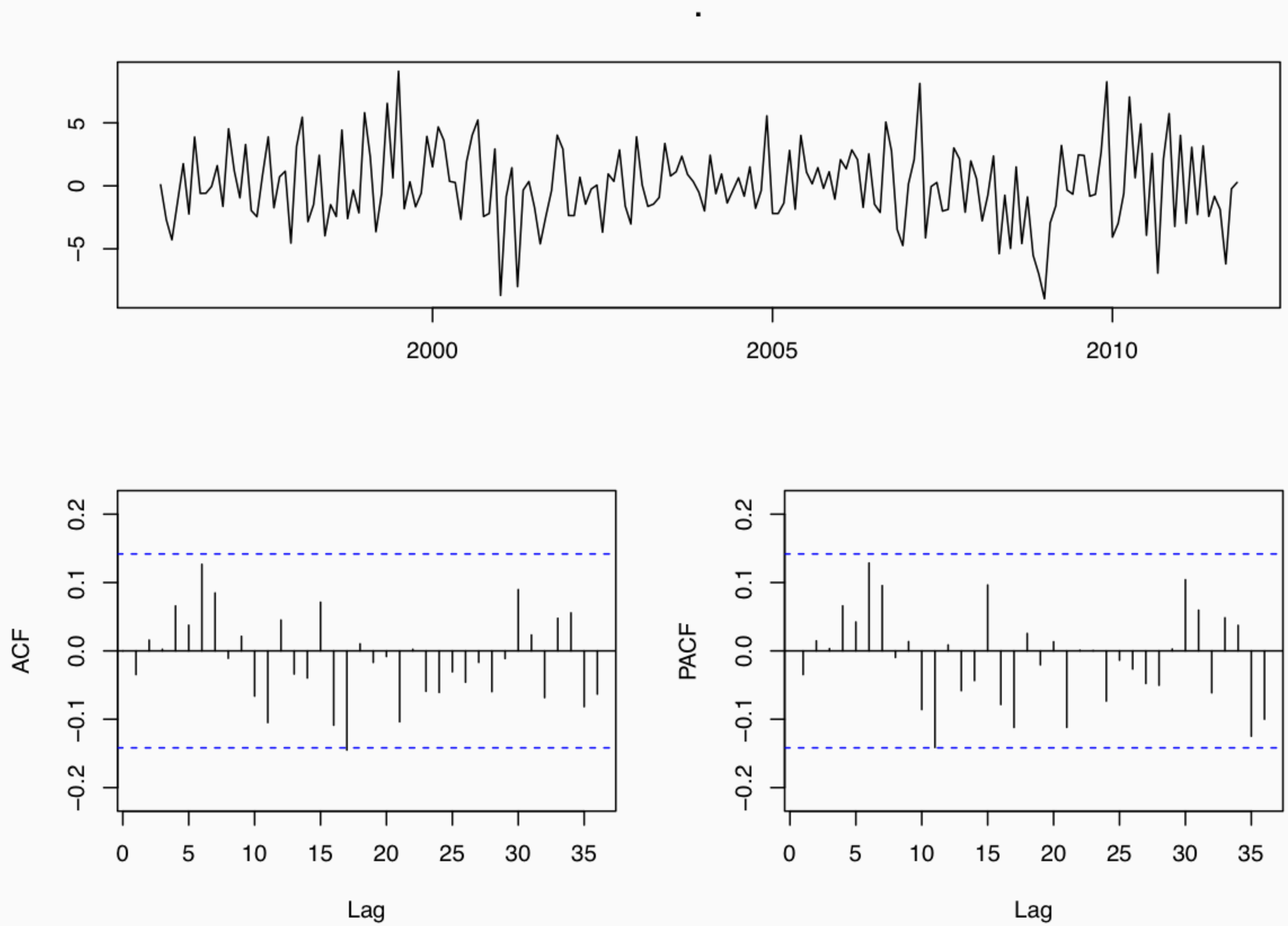


Model

```
Arima(elec_sales, order = c(3,1,0))
## Series: elec_sales
## ARIMA(3,1,0)
##
## Coefficients:
##           ar1      ar2     ar3
##       -0.3488  -0.0386  0.3139
## s.e.   0.0690   0.0736  0.0694
##
## sigma^2 estimated as 9.853:  log likelihood=-485.67
## AIC=979.33   AICc=979.55   BIC=992.32
```

Residuals

```
Arima(elec_sales, order = c(3,1,0)) %>% residuals() %>% tsdisplay(points=FAL
```

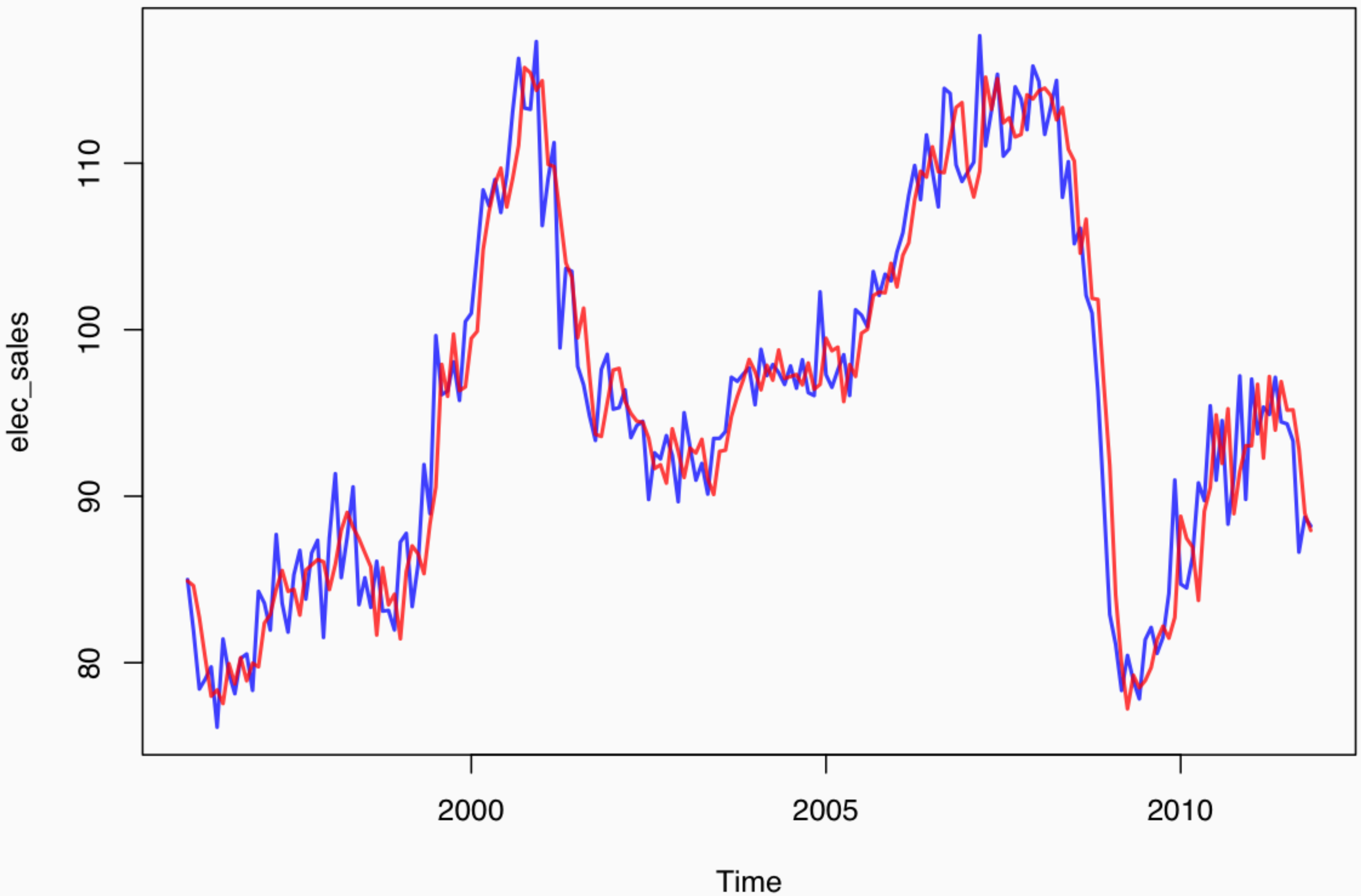


Model Comparison

```r
Arima(elec_sales, order = c(3,1,0))$aic
## [1] 979.3314

Arima(elec_sales, order = c(3,1,1))$aic
## [1] 978.1664

Arima(elec_sales, order = c(4,1,0))$aic
## [1] 978.9048

Arima(elec_sales, order = c(2,1,0))$aic
## [1] 996.6795
```

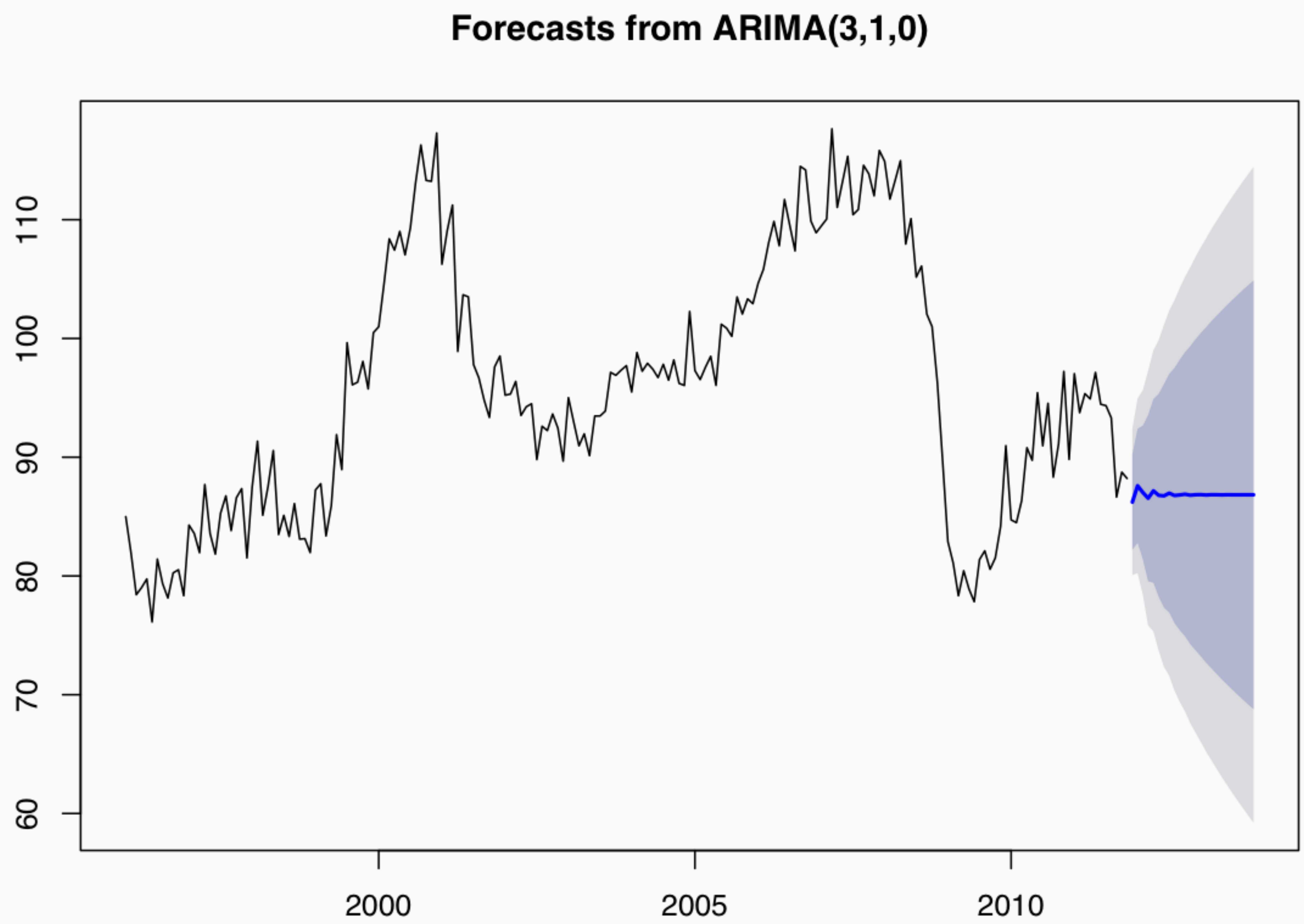
Model fit

```
plot(elec_sales, lwd=2, col=adjustcolor("blue", alpha.f=0.75))
Arima(elec_sales, order = c(3,1,0)) %>% fitted() %>% lines(col=adjustcolor('
```



Model forecast

```
Arima(elec_sales, order = c(3,1,0)) %>% forecast() %>% plot()
```



General Guidance

1. Positive autocorrelations out to a large number of lags usually indicates a need for differencing
2. Slightly too much or slightly too little differencing can be corrected by adding AR or MA terms respectively.
3. A model with no differencing usually includes a constant term, a model with two or more orders (rare) differencing usually does not include a constant term.
4. After differencing, if the PACF has a sharp cutoff then consider adding AR terms to the model.
5. After differencing, if the ACF has a sharp cutoff then consider adding an MA term to the model.
6. It is possible for an AR term and an MA term to cancel each other's effects, so try models with one fewer AR term and one fewer MA term.

Based on rules from https://people.duke.edu/~rnau/411arim2.htm and https://people.duke.edu/~rnau/411arim3.htm