Lecture 15

Mauna Loa Example & GPs for GLMs

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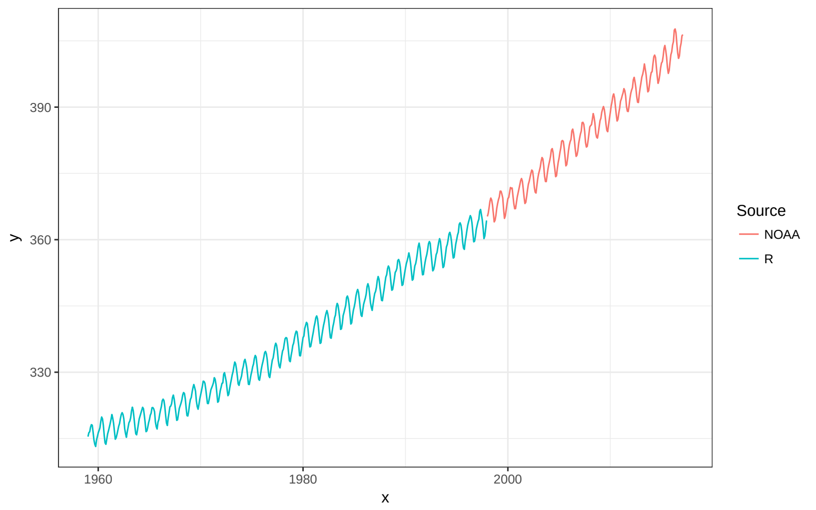
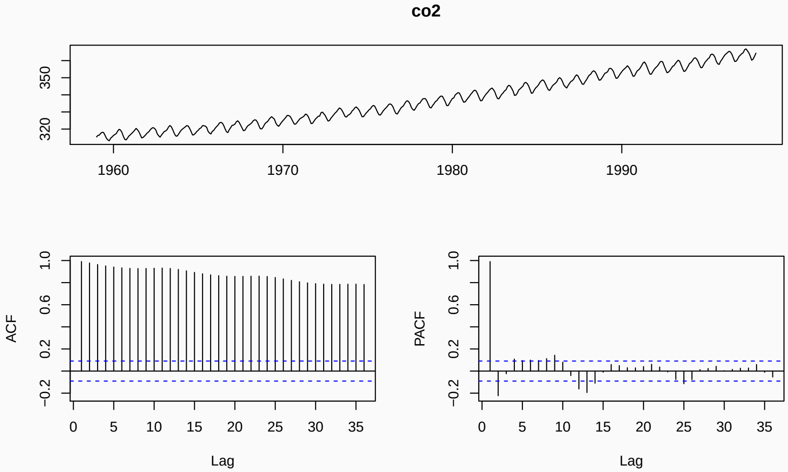
03/08/2017

Mauna Loa Exampel

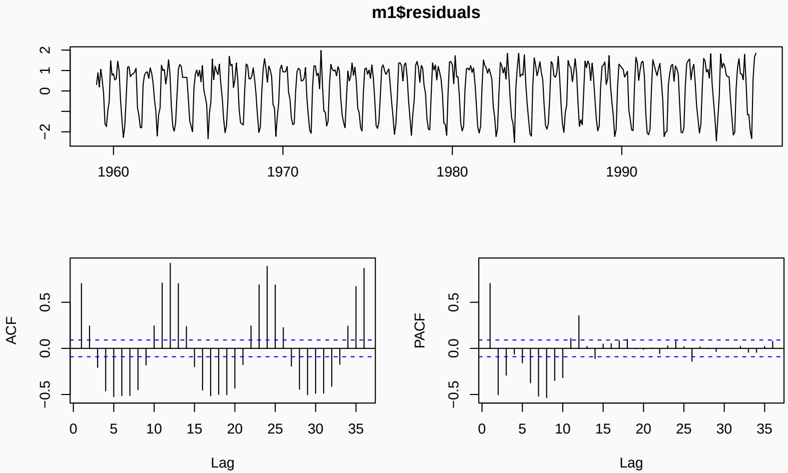
Atmospheric CO_2



ARIMA

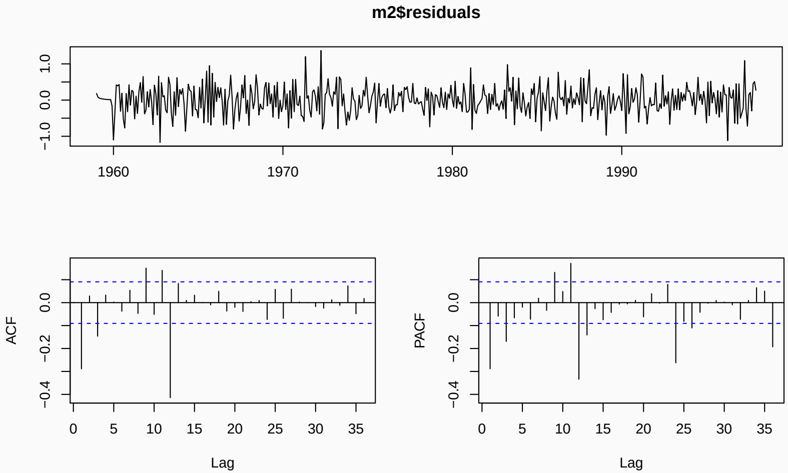

co2
350
320
1960
1970
1980
1990
ACF
1.0
0.6
0.2
−0.2
0
5
10
15
20
25
30
35
Lag
PACF
1.0
0.6
0.2
−0.2
0
5
10
15
20
25
30
35
Lag

ARIMA(0,1,0)×(0,0,0)



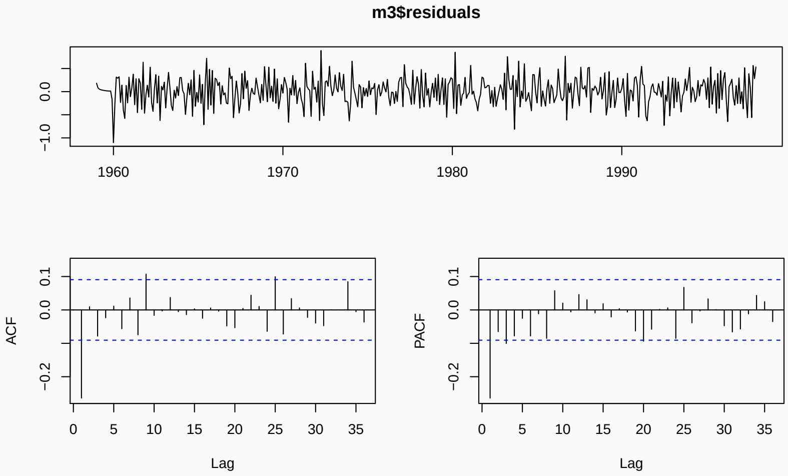

```
## [1] 1505.115
```

ARIMA(0,1,0)×(0,1,0)$_{12}$



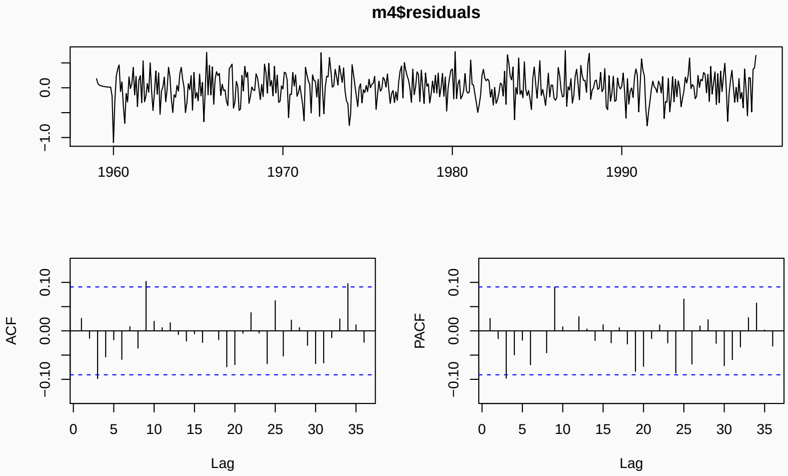

```
## [1] 442.0075
```

ARIMA(0,1,0)$\times$(0,1,1)$_{12}$



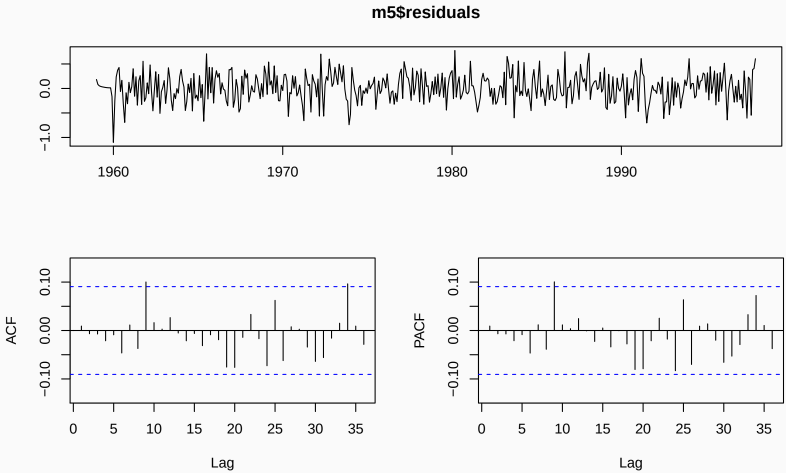

```
## [1] 221.5212
```

ARIMA(0,1,1)×(0,1,1)$_{12}$




```
## [1] 178.2089
```

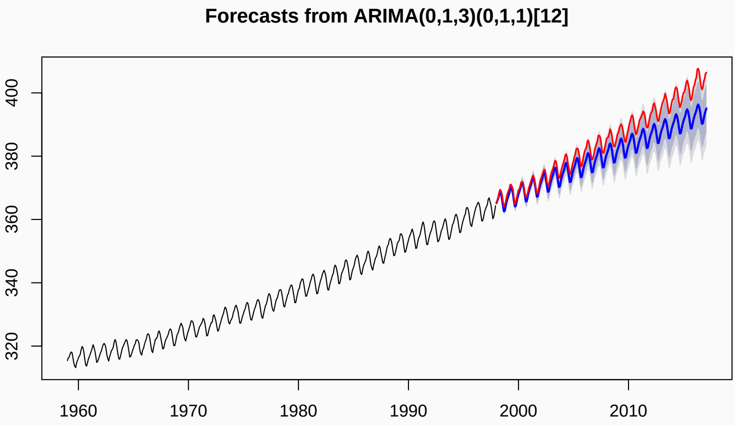
ARIMA(0,1,3)×(0,1,1)$_{12}$



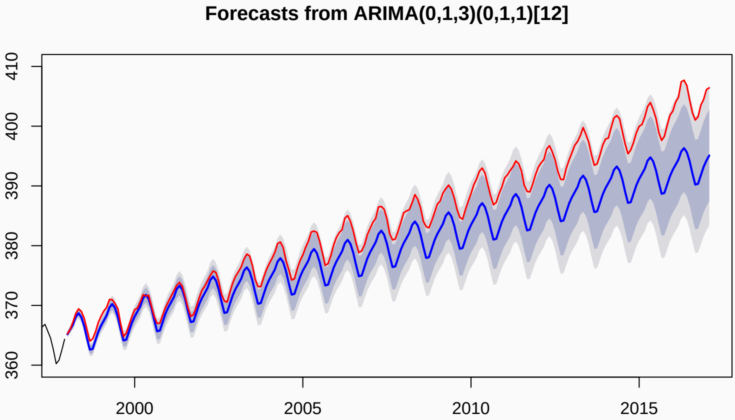

```
## [1] 176.9982
```

auto.arima

```
auto.arima(co2)
## Series: co2
## ARIMA(1,1,1)(1,1,2)[12]
##
## Coefficients:
##          ar1      ma1     sar1     sma1     sma2
##       0.2569  -0.5847  -0.5489  -0.2620  -0.5123
## s.e.  0.1406   0.1203   0.5881   0.5703   0.4820
##
## sigma^2 estimated as 0.08576:  log likelihood=-84.39
## AIC=180.78   AICc=180.97   BIC=205.5
```


Forecasts from ARIMA(0,1,3)(0,1,1)[12]
400
380
360
340
320
1960
1970
1980
1990
2000
2010

Forecasting (zoom)



Based on Rasmussen 5.4.3 (we are using slightly different data and parameterization)

$$y \sim \mathcal{N}(\boldsymbol{\mu},\ \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2 + \boldsymbol{\Sigma}_3 + \boldsymbol{\Sigma}_4 + \sigma_5^2 I\,)$$

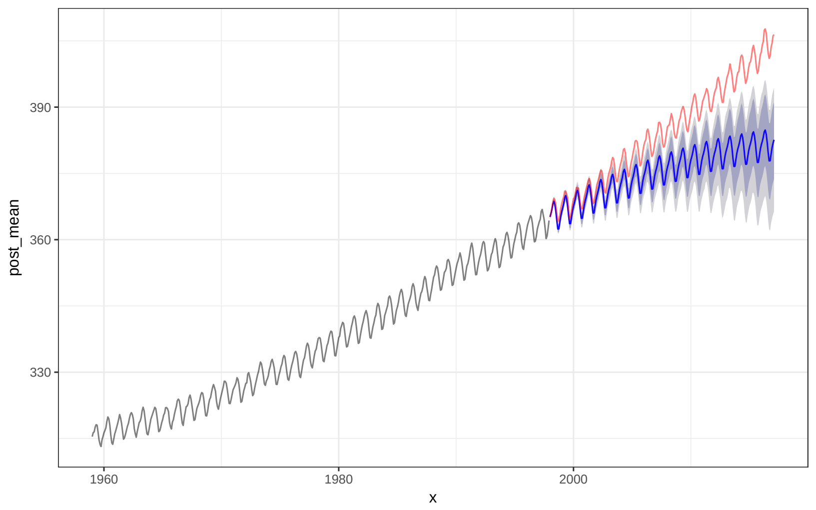
$$\{\boldsymbol{\mu}\}_i = \bar{y}$$

$$\begin{aligned}
\{\boldsymbol{\Sigma}_1\}_{ij} &= \sigma_1^2 \exp\left(-(l_1 \cdot d_{ij})^2\right) \\
\{\boldsymbol{\Sigma}_2\}_{ij} &= \sigma_2^2 \exp\left(-(l_2 \cdot d_{ij})^2\right) \exp\left(-2\,(l_3)^2 \sin^2(\pi\, d_{ij}/p)\right) \\
\{\boldsymbol{\Sigma}_3\}_{ij} &= \sigma_3^2 \left(1 + \frac{(l_4 \cdot d_{ij})^2}{\alpha}\right)^{-\alpha} \\
\{\boldsymbol{\Sigma}_4\}_{ij} &= \sigma_4^2 \exp\left(-(l_5 \cdot d_{ij})^2\right)
\end{aligned}$$

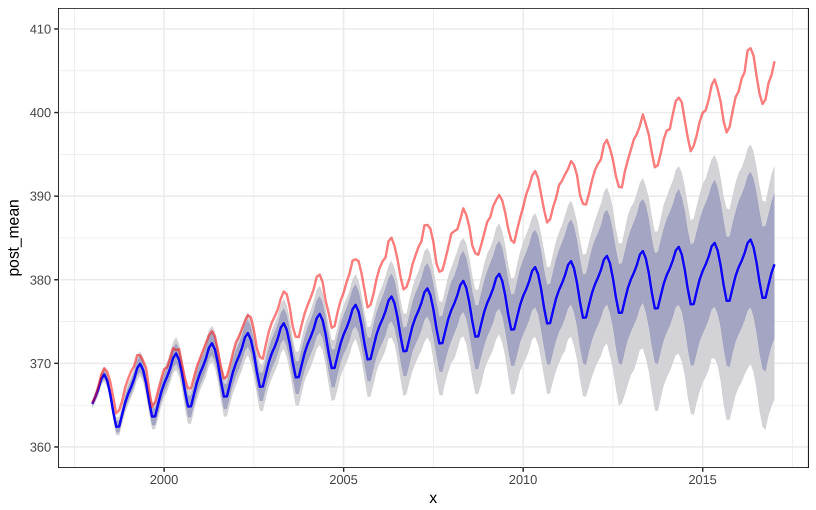
JAGS Model

```
## model{
##   y ~ dmnorm(mu, inverse(Sigma))
##
##   for (i in 1:(length(y)-1)) {
##     for (j in (i+1):length(y)) {
##       k1[i,j] <- sigma2[1] * exp(- pow(l[1] * d[i,j],2))
##       k2[i,j] <- sigma2[2] * exp(- pow(l[2] * d[i,j],2) - 2 * pow(l[3] * sin(pi*d[i
##       k3[i,j] <- sigma2[3] * pow(1+pow(l[4] * d[i,j],2)/alpha, -alpha)
##       k4[i,j] <- sigma2[4] * exp(- pow(l[5] * d[i,j],2))
##
##       Sigma[i,j] <- k1[i,j] + k2[i,j] + k3[i,j] + k4[i,j]
##       Sigma[j,i] <- Sigma[i,j]
##     }
##   }
##
##   for (i in 1:length(y)) {
##     Sigma[i,i] <- sigma2[1] + sigma2[2] + sigma2[3] + sigma2[4] + sigma2[5]
##   }
##
##   for(i in 1:5){
##     sigma2[i] ~ dt(0, 2.5, 1) T(0,)
##     l[i] ~ dt(0, 2.5, 1) T(0,)
##   }
##   alpha ~ dt(0, 2.5, 1) T(0,)
## }
```

Forecasting



Forecasting (zoom)



Forecasting RMSE

dates	RMSE (arima)	RMSE (gp)
Jan 1998 - Jan 2003	1.119	1.911
Jan 1998 - Jan 2008	2.521	4.575
Jan 1998 - Jan 2013	3.839	7.706
Jan 1998 - Mar 2017	5.474	11.395

Rewriting the GP likelihood

From last time, remember that we can view our GP in the following ways,

$$y \sim \mathcal{N}(\boldsymbol{\mu},\ \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2 + \boldsymbol{\Sigma}_3 + \boldsymbol{\Sigma}_4 + \sigma_5^2 I\,)$$

but we can also think of y as being the deterministic sum of 5 independent GPs

$$y = \mu + w_1(\boldsymbol{x}) + w_2(\boldsymbol{x}) + w_3(\boldsymbol{x}) + w_4(\boldsymbol{x}) + w_5(\boldsymbol{x})$$

where

$$\begin{aligned}
w_1(\boldsymbol{x}) &\sim \mathcal{N}(0, \boldsymbol{\Sigma}_1) \\
w_2(\boldsymbol{x}) &\sim \mathcal{N}(0, \boldsymbol{\Sigma}_2) \\
w_3(\boldsymbol{x}) &\sim \mathcal{N}(0, \boldsymbol{\Sigma}_3) \\
w_4(\boldsymbol{x}) &\sim \mathcal{N}(0, \boldsymbol{\Sigma}_4) \\
w_5(\boldsymbol{x}) &\sim \mathcal{N}(0, \sigma_5^2 I\,)
\end{aligned}$$

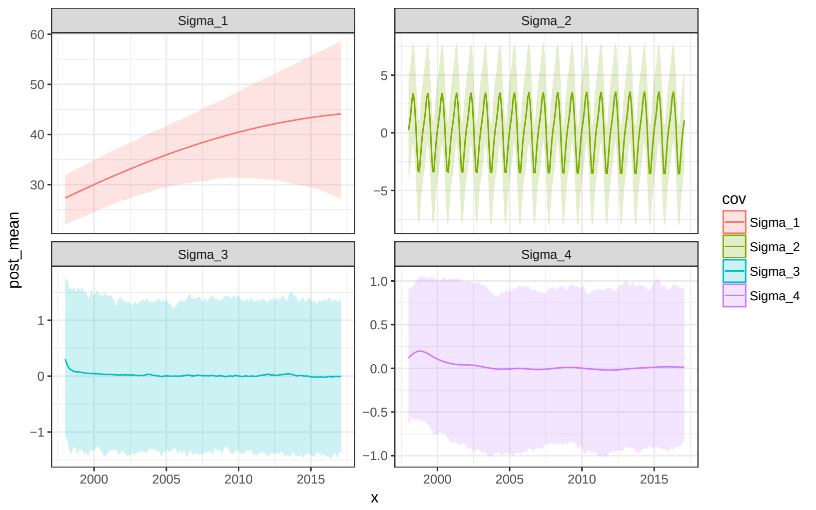
$$\begin{bmatrix} w_1(x) \\ w_1(x^\star) \\ w_2(x) \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{\mu} \end{bmatrix}, \begin{bmatrix} \Sigma_1 & \Sigma_1^\star & 0 & \Sigma_1 \\ \Sigma_1^{\star t} & \Sigma_1^{\star\star} & 0 & \Sigma_1^\star \\ 0 & 0 & \Sigma_2 & \Sigma_2 \\ \Sigma_1 & \Sigma_1^\star & \Sigma_2 & \sum_{i=1}^5 \Sigma_i \end{bmatrix} \right)$$

therefore

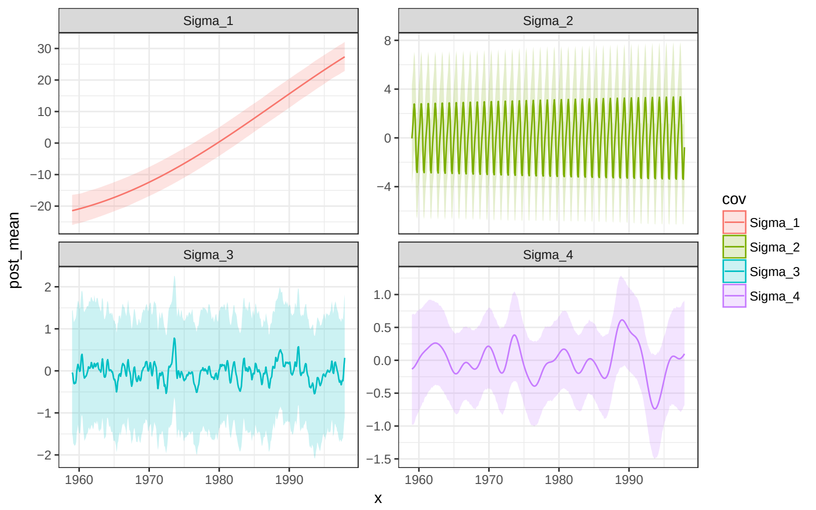
$$w_1(x^\star) \mid y, \boldsymbol{\mu}, \boldsymbol{\theta} \sim \mathcal{N}(\mu_{cond},\ \Sigma_{cond})$$

$$\mu_{cond} = 0 + \Sigma_1^\star(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5)^{-1}(y - \boldsymbol{\mu})$$
$$\Sigma_{cond} = \Sigma_1^{\star\star} - \Sigma_1^\star(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5)^{-1}\Sigma_1^{\star t}$$

Forecasting Components



Fit Components



GPs and Logistic Regression

A typical logistic regression problem uses the following model,

$$
\begin{aligned}
y_i &\sim \text{Bern}(p_i) \\
\text{logit}(p_i) &= \boldsymbol{X}\,\boldsymbol{\beta} \\
&= \beta_0 + \beta_1\, x_{i1} + \cdots + \beta_k\, x_{ik}
\end{aligned}
$$

A typical logistic regression problem uses the following model,

$$
\begin{aligned}
y_i &\sim \text{Bern}(p_i) \\
\text{logit}(p_i) &= X\,\boldsymbol{\beta} \\
&= \beta_0 + \beta_1\, x_{i1} + \cdots + \beta_k\, x_{ik}
\end{aligned}
$$

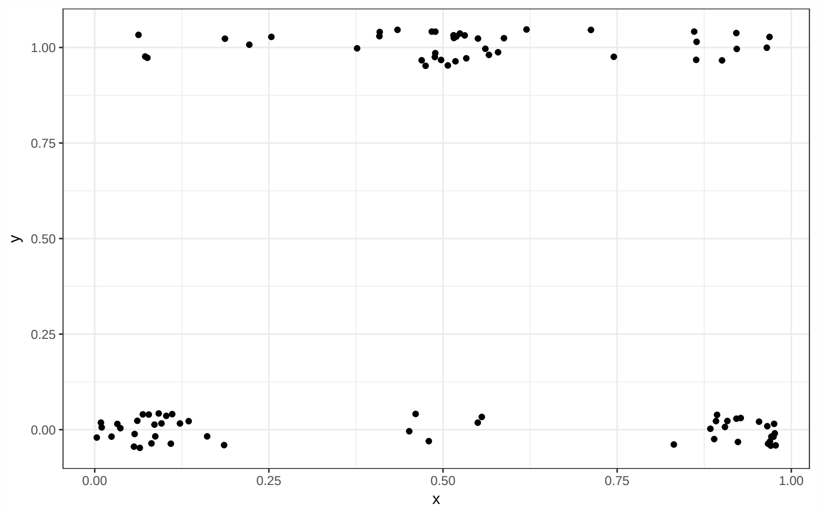
there is no reason that the linear equation above can't contain thing like random effects or GPs

$$
\begin{aligned}
y_i &\sim \text{Bern}(p_i) \\
\text{logit}(p_i) &= X\,\boldsymbol{\beta} + w(\boldsymbol{x})
\end{aligned}
$$

where

$$
w(\boldsymbol{x}) \sim \mathcal{N}(0, \Sigma)
$$

A toy example



Jags Model

```
## model{
##   for(i in 1:N) {
##     y[i] ~ dbern(p[i])
##     logit(p[i]) <- eta[i]
##   }
##   eta ~ dmnorm(rep(0,N), inverse(Sigma))
##
##   for (i in 1:(length(y)-1)) {
##     for (j in (i+1):length(y)) {
##       Sigma[i,j] <- sigma2 * exp(- pow(l * d[i,j],2))
##       Sigma[j,i] <- Sigma[i,j]
##     }
##   }
##
##   for (i in 1:length(y)) {
##     Sigma[i,i] <- sigma2 + 1e-06
##   }
##
##   sigma2 ~ dt(0, 2.5, 1) T(0,)
##   l ~ dunif(sqrt(3),100)
## }
```