Lecture 8

ARMA Models

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02/13/2017

AR(p)

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From last time,

$$AR(p): \quad y_t = \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t$$
$$= \delta + w_t + \sum_{i=1}^{p} \phi_i\, y_{t-i}$$

What are the properities of $AR(p)$,

1. Expected value?
2. Covariance / correlation?
3. Stationarity?

Lag operator

The lag operator is convenience notation for writing out AR (and other) time series models.

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We define the lag operator L as follows,

$$L\,y_t = y_{t-1}$$

this can be generalized where,

$$\begin{aligned} L^2 y_t &= L\,L\,y_t \\ &= L\,y_{t-1} \\ &= y_{t-2} \end{aligned}$$

therefore,

$$L^k\,y_t = y_{t-k}$$

Lag polynomial

An $AR(p)$ model can be rewritten as

$$
\begin{aligned}
y_t &= \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t \\
y_t &= \delta + \phi_1 L y_t + \phi_2 L^2 y_t + \cdots + \phi_p L^p y_t + w_t
\end{aligned}
$$

$$
\begin{aligned}
y_t - \phi_1 L y_t - \phi_2 L^2 y_t - \cdots - \phi_p L^p y_t &= \delta + w_t \\
(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) y_t &= \delta + w_t
\end{aligned}
$$

Lag polynomial

An $AR(p)$ model can be rewritten as

$$y_t = \delta + \phi_1\, y_{t-1} + \phi_2\, y_{t-2} + \cdots + \phi_p\, y_{t-p} + w_t$$
$$y_t = \delta + \phi_1\, L\, y_t + \phi_2\, L^2\, y_t + \cdots + \phi_p\, L^p\, y_t + w_t$$

$$y_t - \phi_1\, L\, y_t - \phi_2\, L^2\, y_t - \cdots - \phi_p\, L^p\, y_t = \delta + w_t$$
$$(1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)\, y_t = \delta + w_t$$

This polynomial of the lags

$$\phi_p(L) = (1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_p\, L^p)$$

is called the lag or characteristic polynomial of the AR process.

Stationarity of $AR(p)$ processes

An $AR(p)$ process is stationary if the roots of the characteristic polynomial lay outside the complex unit circle

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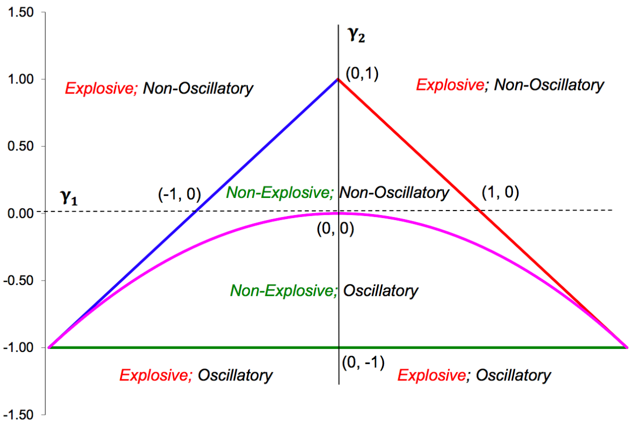
Example AR(1):

Example AR(2)

AR(2) Stationarity Conditions



Source: This diagram is based on Figure 7.1, on p.196 of A. Zellner, *An Introduction to Bayesian Inference in Econometrics*, Wiley, New York, 1971.

From http://www.sfu.ca/~baa7/Teaching/econ818/StationarityAR2.pdf

Proof

We can rewrite the $AR(p)$ model into an $AR(1)$ form using matrix notation

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t$$
$$\boldsymbol{\xi}_t = \boldsymbol{\delta} + F\,\boldsymbol{\xi}_{t-1} + \boldsymbol{w}_t$$

where

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \delta + w_t + \sum_{i=1}^{p} \phi_i y_{t-i} \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}$$

Proof sketch (cont.)

So just like the original $AR(1)$ we can expand out the autoregressive equation

$$\begin{aligned}
\boldsymbol{\xi}_t &= \boldsymbol{\delta} + w_t + F\,\boldsymbol{\xi}_{t-1} \\
&= \boldsymbol{\delta} + w_t + F\left(\boldsymbol{\delta} + w_{t-1}\right) + F^2\left(\boldsymbol{\delta} + w_{t-2}\right) + \cdots \\
&\qquad + F^{t-1}\left(\boldsymbol{\delta} + w_1\right) + F^t\left(\boldsymbol{\delta} + w_0\right) \\
&= \boldsymbol{\delta} \sum_{i=0}^{t} F^i + \sum_{i=0}^{t} F^i\, w_{t-i}
\end{aligned}$$

and therefore we need $\lim_{t\to\infty} F^t \to 0$.

Proof sketch (cont.)

We can find the eigen decomposition such that $F = Q\mathbf{\Lambda}Q^{-1}$ where the columns of Q are the eigenvectors of F and $\mathbf{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues.

A useful property of the eigen decomposition is that

$$F^i = Q\mathbf{\Lambda}^i Q^{-1}$$

Proof sketch (cont.)

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A useful property of the eigen decomposition is that

$$F^i = Q\mathbf{\Lambda}^i Q^{-1}$$

Using this property we can rewrite our equation from the previous slide as

$$\begin{aligned}\boldsymbol{\xi}_t &= \boldsymbol{\delta}\sum_{i=0}^{t} F^i + \sum_{i=0}^{t} F^i w_{t-i} \\ &= \boldsymbol{\delta}\sum_{i=0}^{t} Q\mathbf{\Lambda}^i Q^{-1} + \sum_{i=0}^{t} Q\mathbf{\Lambda}^i Q^{-1} w_{t-i}\end{aligned}$$

Proof sketch (cont.)

$$\mathbf{\Lambda}^i = \begin{bmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^i \end{bmatrix}$$

Therefore,

$$\lim_{t\to\infty} F^t \to 0$$

when

$$\lim_{t\to\infty} \Lambda^t \to 0$$

which requires that

$$|\lambda_i| < 1 \quad \text{for all } i$$

Proof sketch (cont.)

Eigenvalues are defined such that for $\boldsymbol{\lambda}$,

$$\det\big(F - \boldsymbol{\lambda}\, I\big) = 0$$

based on our definition of F our eigenvalues will therefore be the roots of

$$\lambda^{p} - \phi_1\,\lambda^{p-1} - \phi_2\,\lambda^{p-2} - \cdots - \phi_{p_1}\,\lambda^{1} - \phi_p = 0$$

Proof sketch (cont.)

Eigenvalues are defined such that for $\boldsymbol{\lambda}$,

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based on our definition of F our eigenvalues will therefore be the roots of

$$\lambda^p - \phi_1\, \lambda^{p-1} - \phi_2\, \lambda^{p-2} - \cdots - \phi_{p_1}\, \lambda^1 - \phi_p = 0$$

which if we multiply by $1/\lambda^p$ where $L = 1/\lambda$ gives

$$1 - \phi_1\, L - \phi_2\, L^2 - \cdots - \phi_{p_1}\, L^{p-1} - \phi_p\, L^p = 0$$

Properties of $AR(p)$

For a *stationary* $AR(p)$ process where w_t has $E(w_t) = 0$ and $Var(w_t) = \sigma_w^2$

$$E(Y_t) = \frac{\delta}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

$$Var(Y_t) = \gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_w^2$$

$$Cov(Y_t, Y_{t-j}) = \gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \cdots + \phi_p\gamma_{j-p}$$

$$Corr(Y_t, Y_{t-j}) = \rho_j = \phi_1\rho j - 1 + \phi_2\rho j - 2 + \cdots + \phi_p\rho_{j-p}$$

Moving Average (MA) Processes

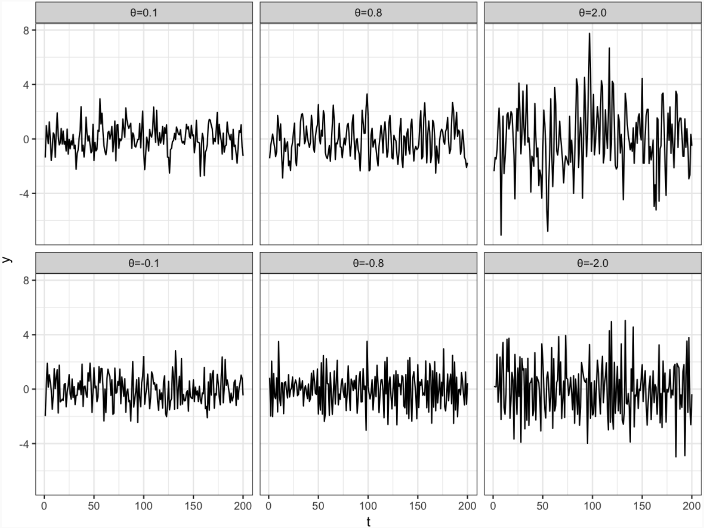
MA(1)

A moving average process is similar to an AR process, except that the autoregression is on the error term.

$$MA(1): \qquad y_t = \delta + w_t + \theta\, w_{t-1}$$

Properties:

Time series



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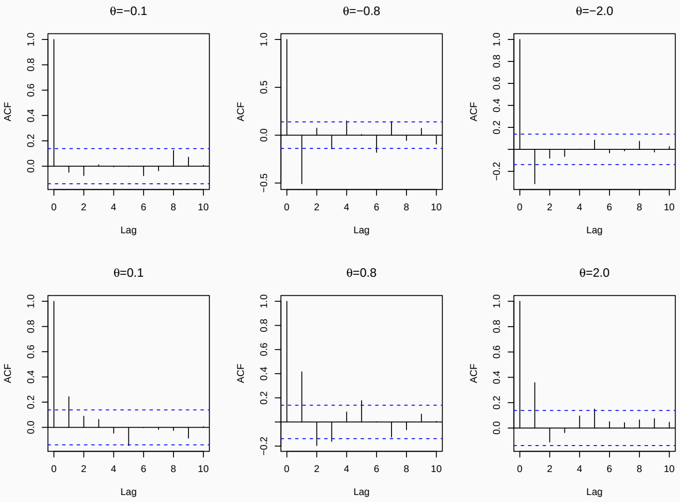
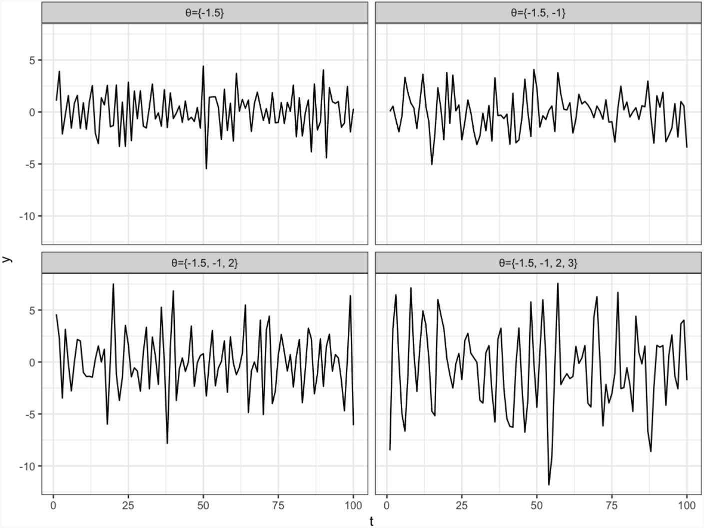


MA(q)

$$MA(q): \qquad y_t = \delta + w_t + \theta_1 \, w_{t-1} + \theta_2 \, w_{t-2} + \cdots + \theta_q \, w_{t-q}$$

Properties:

Time series



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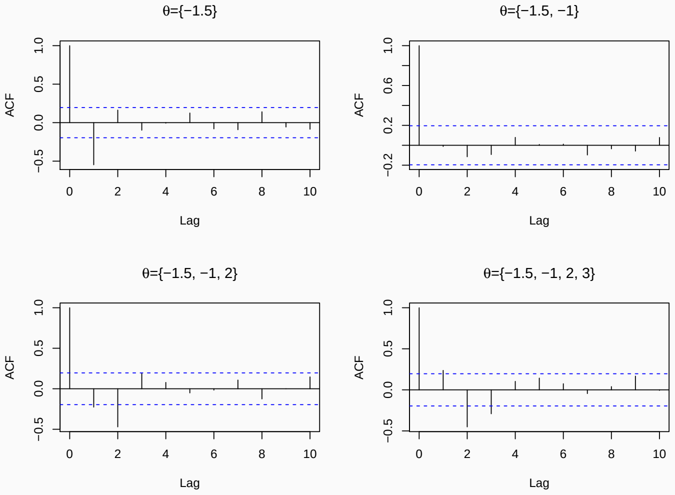


ARMA Model

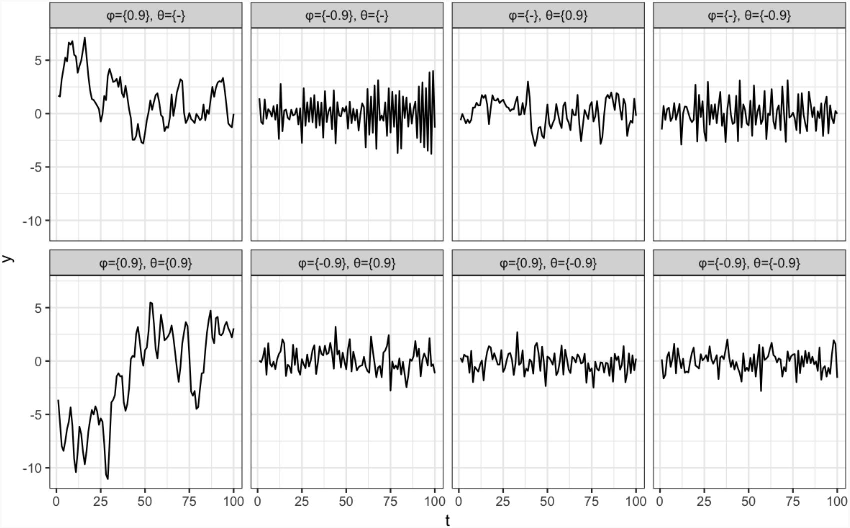
ARMA Model

An ARMA model is a composite of AR and MA processes,

$$
\begin{aligned}
ARMA(p, q) :& \\
y_t &= \delta + \phi_1\, y_{t-1} + \cdots \phi_p\, y_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t_q} \\
\phi_p(L) y_t &= \delta + \theta_q(L) w_t
\end{aligned}
$$

Since all *MA* processes are stationary, we only need to examine the *AR* aspect to determine stationarity (roots of $\phi_p(L)$ lie outside the complex unit circle).

Time series


φ={0.9}, θ={-}
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5
0
-5
-10
y
0 25 50 75 100
t

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