Lecture 6

Discrete Time Series

2/06/2018

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Stationary Processes

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In the time series context this means that the joint distribution of $\{y_{t_1}, \ldots, y_{t_n}\}$ must be identical to the distribution of $\{y_{t_1+k}, \ldots, y_{t_n+k}\}$ for any value of n and k.

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Weak Stationary

Strict stationary is unnecessarily strong / restrictive for many applications, so instead we often opt for *weak stationary* which requires the following,

1. The process has finite variance

$$E(y_t^2) < \infty \text{ for all } t$$

2. The mean of the process is constant

$$E(y_t) = \mu \text{ for all } t$$

3. The second moment only depends on the lag

$$Cov(y_t, y_s) = Cov(y_{t+k}, y_{s+k}) \text{ for all } t, s, k$$

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When we say stationary in class we will almost always mean *weakly stationary*.

For a stationary time series, where $E(y_t) = \mu$ and $\text{Var}(y_t) = \sigma^2$ for all t, we define the autocorrelation at lag k as

$$\begin{aligned}
\rho_k &= Cor(y_t,\, y_{t+k}) \\
&= \frac{Cov(y_t, y_{t+k})}{\sqrt{Var(y_t)Var(y_{t+k})}} \\
&= \frac{E\left((y_t - \mu)(y_{t+k} - \mu)\right)}{\sigma^2}
\end{aligned}$$

Autocorrelation

For a stationary time series, where $E(y_t) = \mu$ and Var$(y_t) = \sigma^2$ for all t, we define the autocorrelation at lag k as

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this is also sometimes written in terms of the autocovariance function (γ_k) as

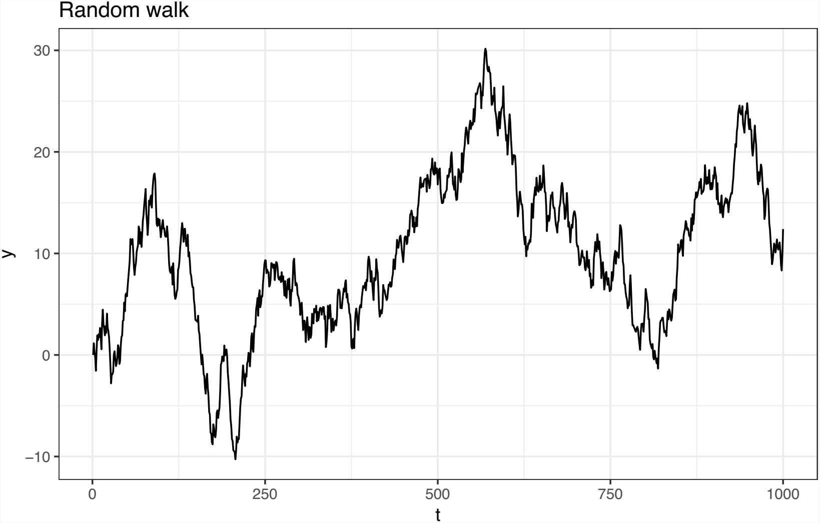
$$\begin{aligned} \gamma_k &= \gamma(t, t+k) = Cov(y_t, y_{t+k}) \\ \rho_k &= \frac{\gamma(t, t+k)}{\sqrt{\gamma(t,t)\gamma(t+k, t+k)}} = \frac{\gamma(k)}{\gamma(0)} \end{aligned}$$

Based on our definition of a (weakly) stationary process, it implies a covariance of the following structure,

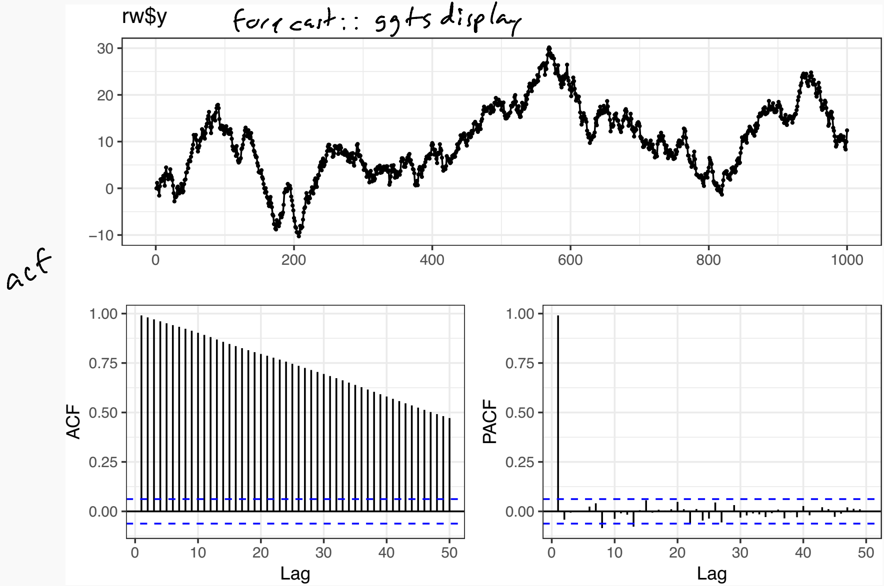
$$\boldsymbol{\Sigma} = \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots & \gamma(n) \\ \gamma(1) & \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(n-1) \\ \gamma(2) & \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(n-2) \\ \gamma(3) & \gamma(2) & \gamma(1) & \gamma(0) & \cdots & \gamma(n-3) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(n) & \gamma(n-1) & \gamma(n-2) & \gamma(n-3) & \cdots & \gamma(0) \end{pmatrix}$$

Example - Random walk

Let $y_t = y_{t-1} + w_t$ with $y_0 = 0$ and $w_t \sim \mathcal{N}(0, 1)$.



ACF + PACF



rw$y
forecast:: ggtsdisplay
acf
30
20
10
0
−10
0
200
400
600
800
1000
ACF
1.00
0.75
0.50
0.25
0.00
0
10
20
30
40
50
Lag
PACF
1.00
0.75
0.50
0.25
0.00
0
10
20
30
40
50
Lag

Stationary?

Is y_t stationary?

$$y_t = y_{t-1} + w_t \qquad w_t \sim N(0,1)$$

$$y_0 = 0$$

$$y_0 = 0$$
$$y_1 = w_1$$
$$y_2 = w_2 + w_1$$
$$y_3 = w_3 + w_2 + w_1$$
$$\vdots$$
$$y_t = \sum_{i=1}^{t} w_i$$

✓ $E(y_t) = E\left(\sum w_i\right) = \sum E(w_i) = \sum 0 = 0$

$$Cov(y_t, y_{t+k}) = E\left((y_t - 0)(y_{t+k} - 0)\right)$$

✗ $= E(y_t \, y_{t+k}) = E\left((w_1 + w_2 + \dots + w_t)(w_1 + \dots + w_t + \dots w_{t+k})\right)$

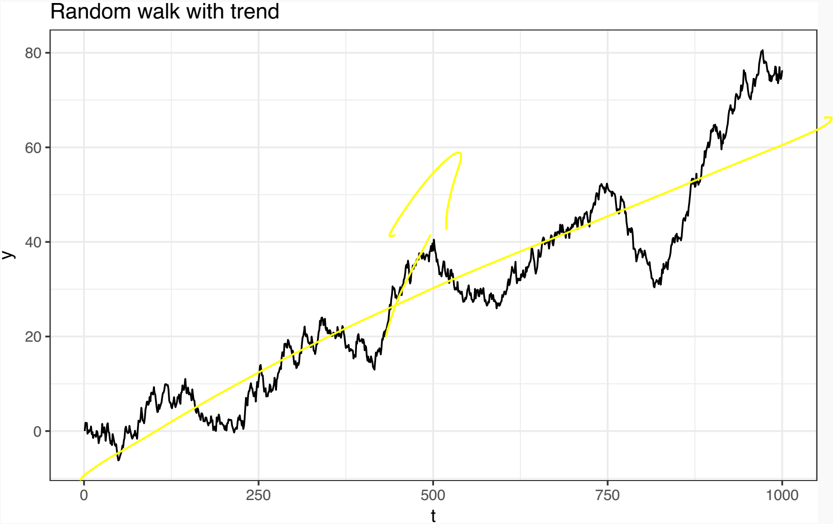
$$= E\left(\left(\sum_{i=1}^{t} w_i\right)\left(\sum_{j=1}^{t+k} w_j\right)\right) = \underbrace{1+1+\dots+1}_{t} = t$$

If $i = j$: $E(w_i w_j) = E(w_i^2) = Var(w_i) + E(w_i)^2$
$$= 1 + 0 = 1$$

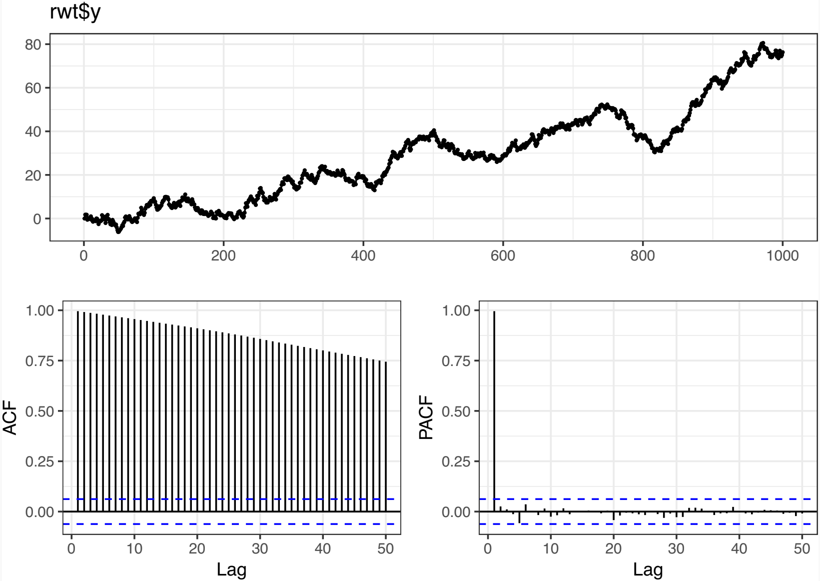
If $i \neq j$: $E(w_i w_j) = E(w_i)E(w_j) = 0 \cdot 0 = 0$

Example - Random walk with drift

Let $y_t = \delta + y_{t-1} + w_t$ with $y_0 = 0$ and $w_t \sim \mathcal{N}(0, 1)$.



ACF + PACF


rwt$y
80
60
40
20
0
0
200
400
600
800
1000
1.00
0.75
0.50
0.25
0.00
ACF
0
10
20
30
40
50
Lag
1.00
0.75
0.50
0.25
0.00
PACF
0
10
20
30
40
50
Lag

Stationary?

Is y_t stationary? No

$y_t = \delta + y_{t-1} + w_t$

$y_0 = 0$

$y_0 = 0$

$y_1 = \delta + w_1$

$y_2 = 2\delta + w_2 + w_1$

$y_3 = 3\delta + w_3 + w_2 + w_1$

$\vdots$

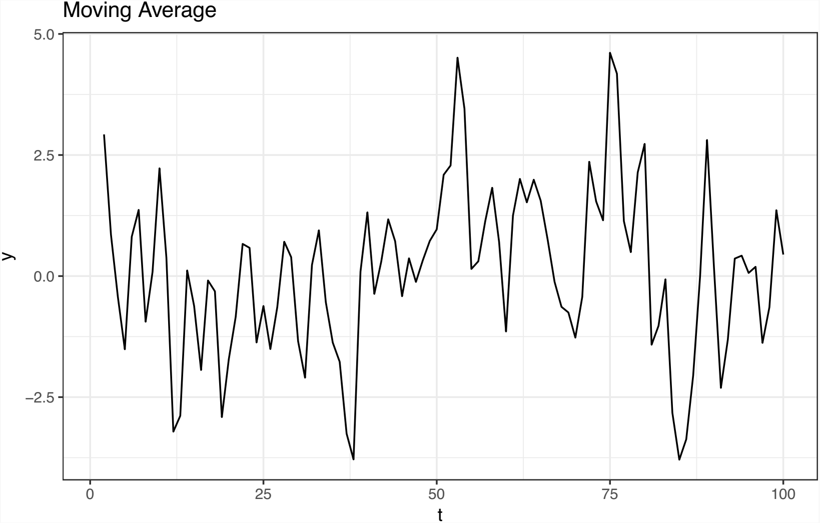
$y_t = \delta t + \sum_{i=1}^{t} w_i$

✗ $Cov(y_t, y_{t+k}) = t$

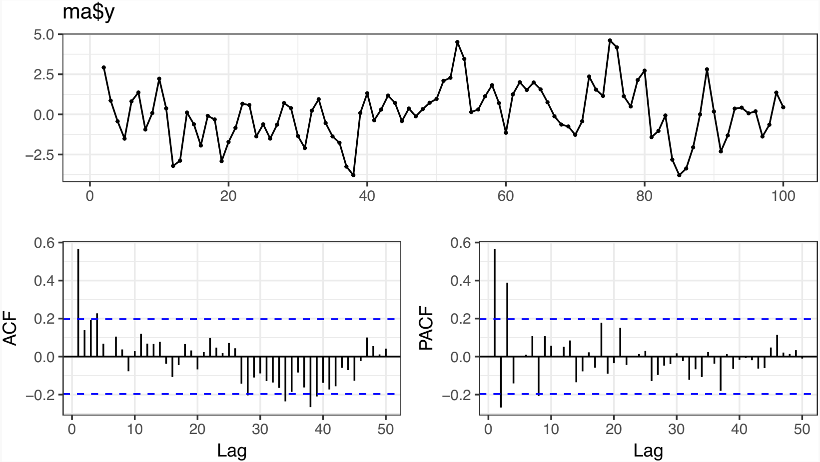
✗ $E(y_t) = \delta t$

Example - Moving Average

Let $w_t \sim \mathcal{N}(0, 1)$ and $y_t = w_{t-1} + w_t$.



ACF + PACF



ma$y
5.0
2.5
0.0
−2.5
0
20
40
60
80
100
0.6
0.4
0.2
0.0
−0.2
ACF
PACF
0
10
20
30
40
50
Lag
Lag

Stationary?

Is y_t stationary?

$y_t = v_{t-1} + w_t$

$y_1 = v_0 + w_1$

$y_2 = v_1 + w_2$

$y_3 = v_2 + w_3$

$\vdots$

$y_t = v_{t-1} + v_t$

$E(y_t) = E(w_{t-1} + w_t)$

$= E(w_{t-1}) + E(w_t) = 0 + 0 = 0$

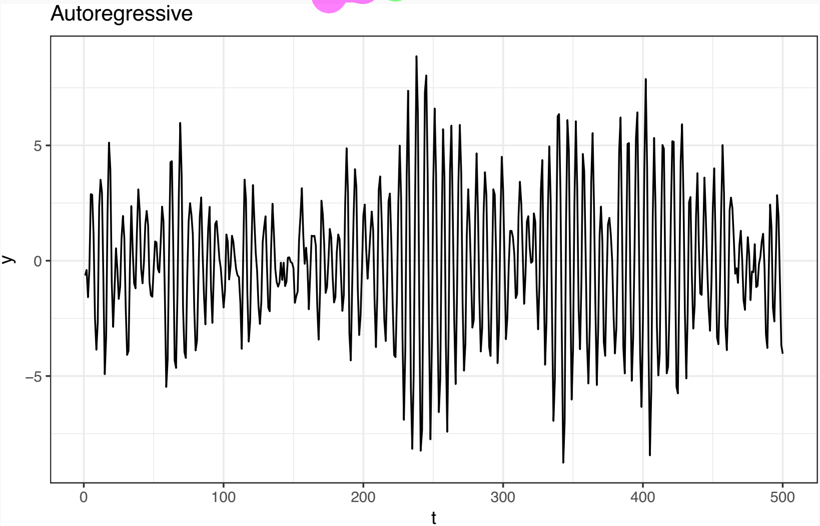
$Cov(y_t, y_{t+k}) = E\left((y_t - 0)(y_{t+k} - 0)\right)$

$= E(y_t \, y_{t+k}) = E\left((v_t + w_{t-1})(w_{t+k} + v_{t+k-1})\right)$

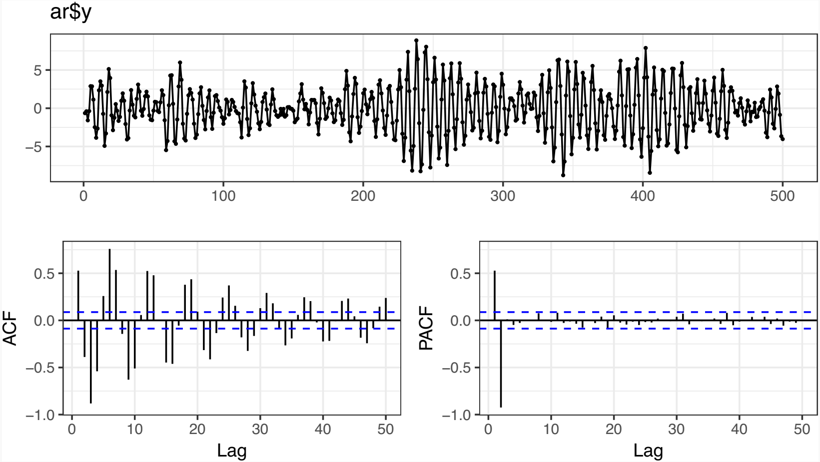
$= \begin{cases} 2 & \text{if } k = 0 \\ 1 & \text{if } k = \pm 1 \\ 0 & \text{if } |k| > 1 \end{cases}$

Autoregressive

Let $w_t \sim \mathcal{N}(0,1)$ and $y_t = y_{t-1} - 0.9y_{t-2} + w_t$ with $y_t = 0$ for $t < 1$.



ACF + PACF



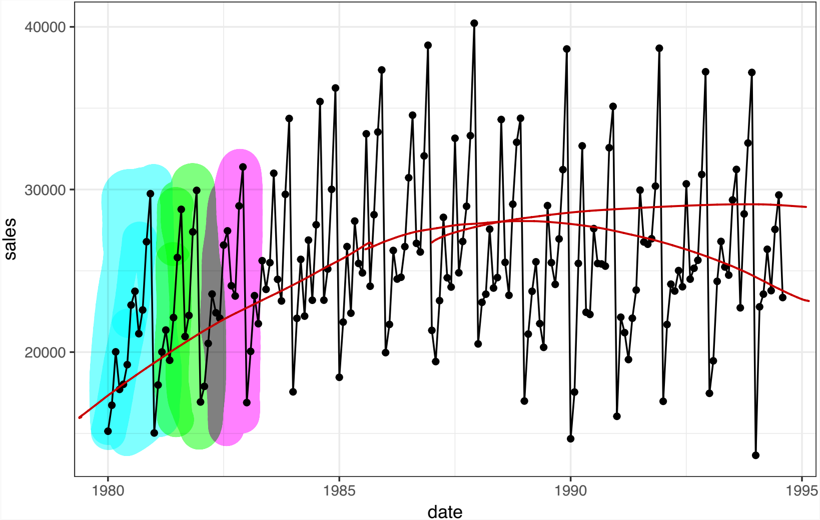
ar$y
5
0
−5
0
100
200
300
400
500
ACF
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
Lag
PACF
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
Lag

Example - Australian Wine Sales

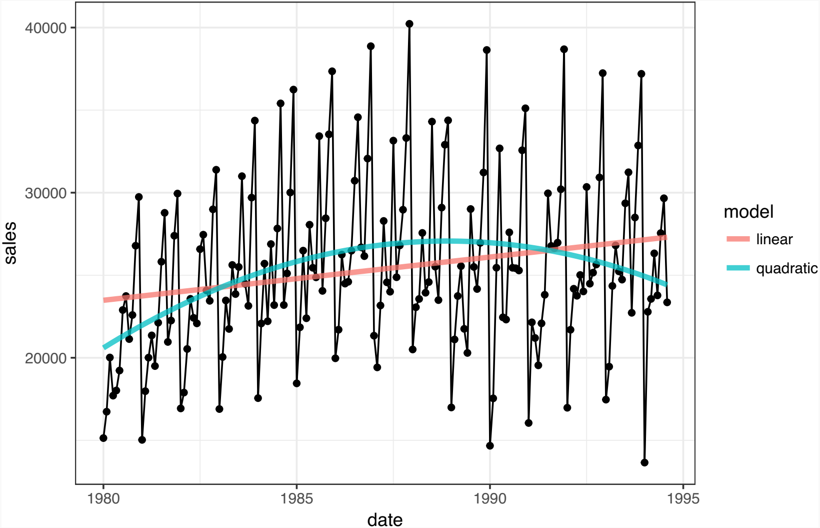
Australian total wine sales by wine makers in bottles <= 1 litre. Jan 1980 – Aug 1994.

```
aus_wine = readRDS("../data/aus_wine.rds")
aus_wine
## # A tibble: 176 x 2
##     date sales
##    <dbl> <dbl>
##  1  1980 15136
##  2  1980 16733
##  3  1980 20016
##  4  1980 17708
##  5  1980 18019
##  6  1980 19227
##  7  1980 22893
##  8  1981 23739
##  9  1981 21133
## 10  1981 22591
## # ... with 166 more rows
```

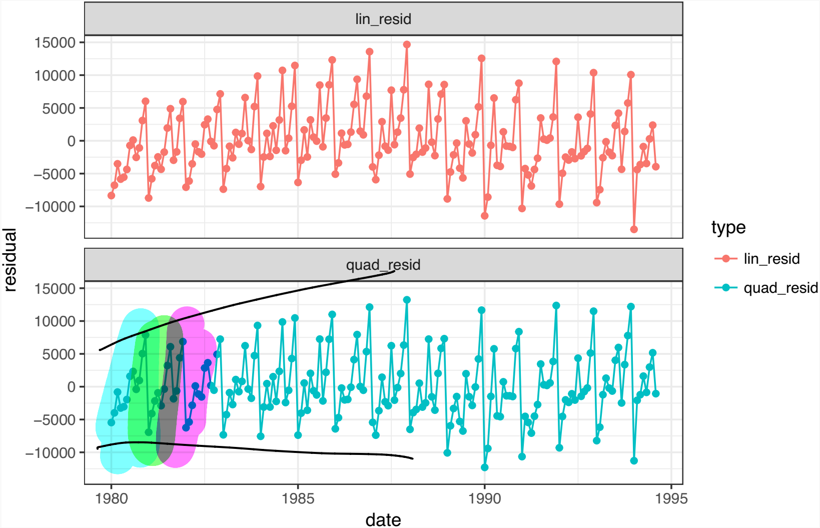
Time series



Basic Model Fit

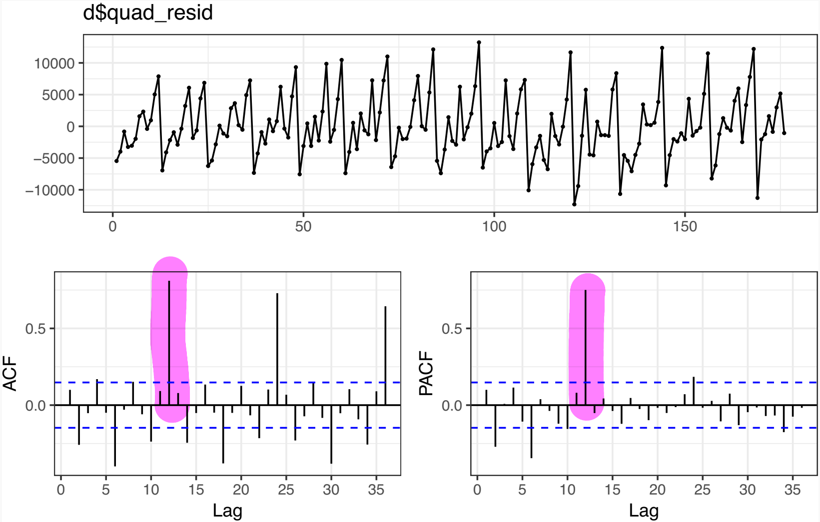


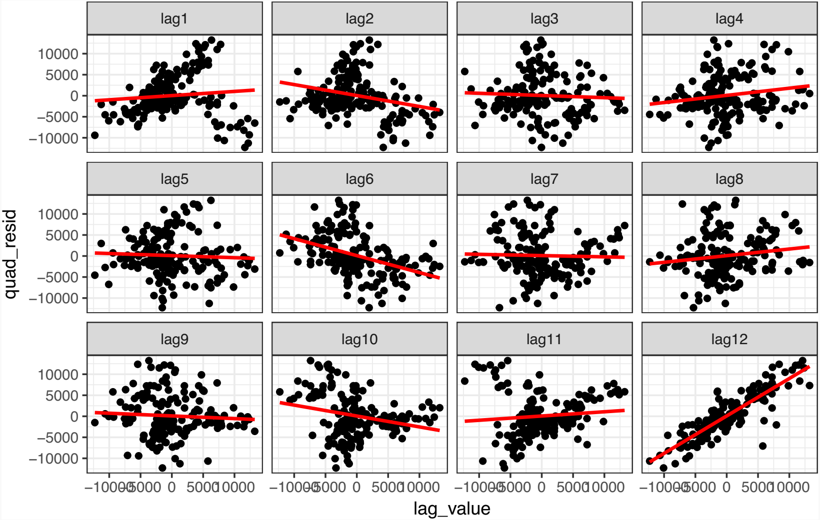
Residuals



lin_resid
quad_resid
15000
10000
5000
0
−5000
−10000
residual
1980
1985
1990
1995
date
type
lin_resid
quad_resid

Autocorrelation Plot

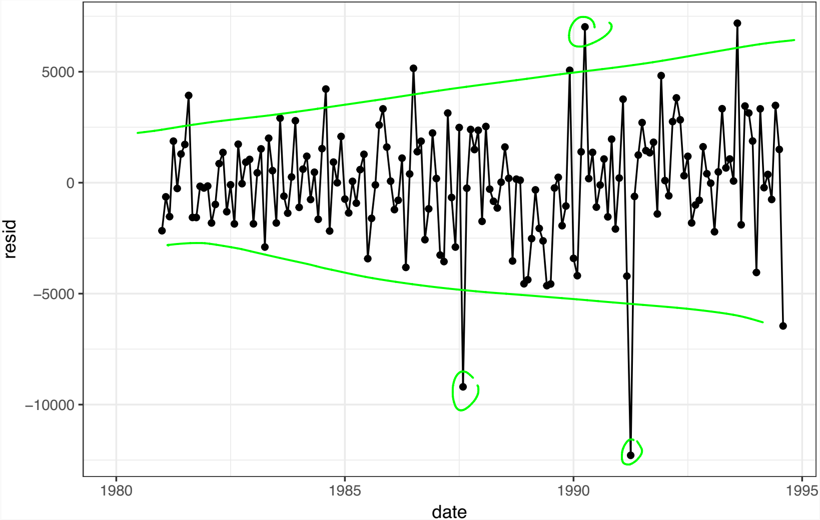



lag1
lag2
lag3
lag4
lag5
lag6
lag7
lag8
lag9
lag10
lag11
lag12
quad_resid
10000
5000
0
−5000
−10000
−10000
−5000
0
5000
10000
lag_value

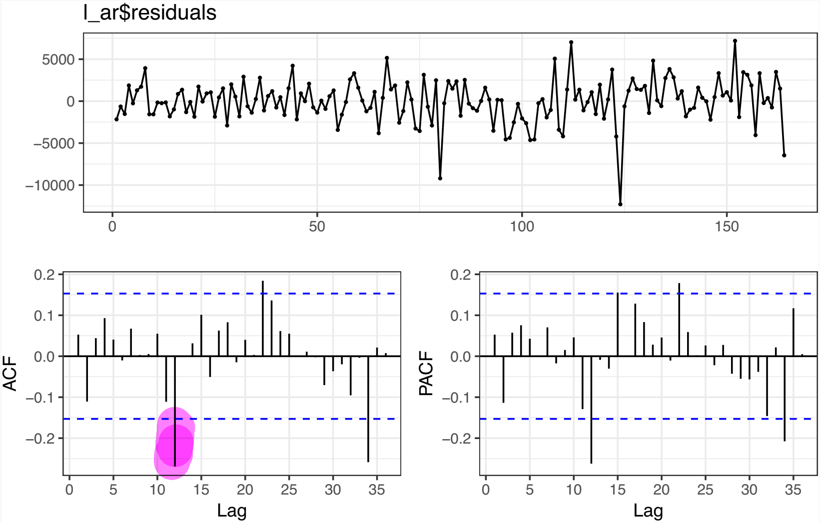
lag(quad_resid, 12)

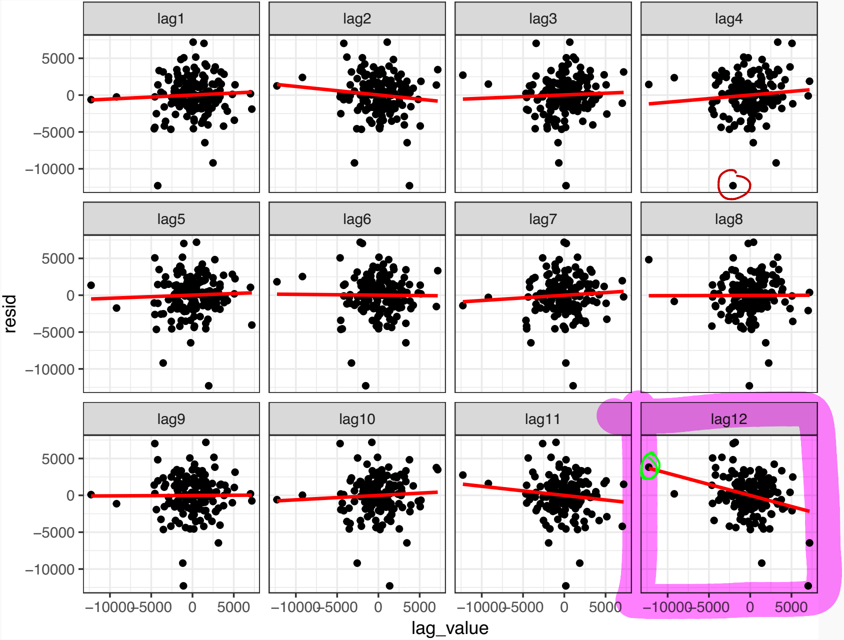
```
##
## Call:
## lm(formula = quad_resid ~ lag_12, data = d_ar)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -12286.5  -1380.5     73.4   1505.2   7188.1
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  83.65080  201.58416   0.415    0.679
## lag_12        0.89024    0.04045  22.006   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2581 on 162 degrees of freedom
##   (12 observations deleted due to missingness)
## Multiple R-squared:  0.7493, Adjusted R-squared:  0.7478
## F-statistic: 484.3 on 1 and 162 DF,  p-value: < 2.2e-16
```

Residual residuals



Residual residuals - acf




lag1
lag2
lag3
lag4
lag5
lag6
lag7
lag8
lag9
lag10
lag11
lag12
resid
lag_value
5000
0
−5000
−10000
−10000
−5000
0
5000

So, is our EDA suggesting that we fit the following model?

$$\text{sales}(t) = \beta_0 + \beta_1\, t + \beta_2\, t^2 + \beta_3\, \text{sales}(t-12) + \epsilon_t$$

$$y'(t) = y(t) - [\beta_0 + \beta_1 t + \beta_2 t^2]$$

$$= \delta\, y'(t-12)$$

. . .

~~the model we actually fit is,~~

$$\text{sales}(t) = \beta_0 + \beta_1\, t + \beta_2\, t^2 + w_t$$

where

$$w_t = \delta\, w_{t-12} + \epsilon_t$$