Lec 08 / AR(1) Ex

$$Y_t = \delta + \phi Y_{t-1} + w_t$$

$$Y_t = \delta + \phi L Y_t + w_t$$

$$(1-\phi L)\, Y_t = \delta + w_t$$

$$\Rightarrow \quad 1 - \phi L = 0$$

$$L = 1/\phi$$

$|L| > 1$

$|1/\phi| > 1$

$|\phi| < 1$

AR(2) Ex

$$(1 - \phi_1 L - \phi_2 L^2)\, Y_t = \delta + w_t$$

$$1 - \phi_1 L - \phi_2 L^2 = 0 \qquad \text{Let } \lambda = 1/L$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

$$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

If $||L|| > 1 \Rightarrow$ stationarity

then $||\lambda|| < 1 \Rightarrow$ "

$$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

If $\left(\phi_1^2 + 4\phi_2 > 0\right)$ then

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

$$\phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2$$

$$\phi_1 + \phi_2 < 1$$

$$\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} > -1$$

$$-\sqrt{\phi_1^2 + 4\phi_2} > -2 - \phi_1$$

$$\phi_1^2 + 4\phi_2 < 4 + 4\phi_1 + \phi_1^2$$

$$\phi_2 - \phi_1 < 1$$

If $\left(\phi_1^2 + 4\phi_2 < 0\right)$ then

$$\lambda = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} = \frac{\phi_1}{2} \pm \left(\frac{\sqrt{-(\phi_1^2 + 4\phi_2)}}{2}\right) i$$

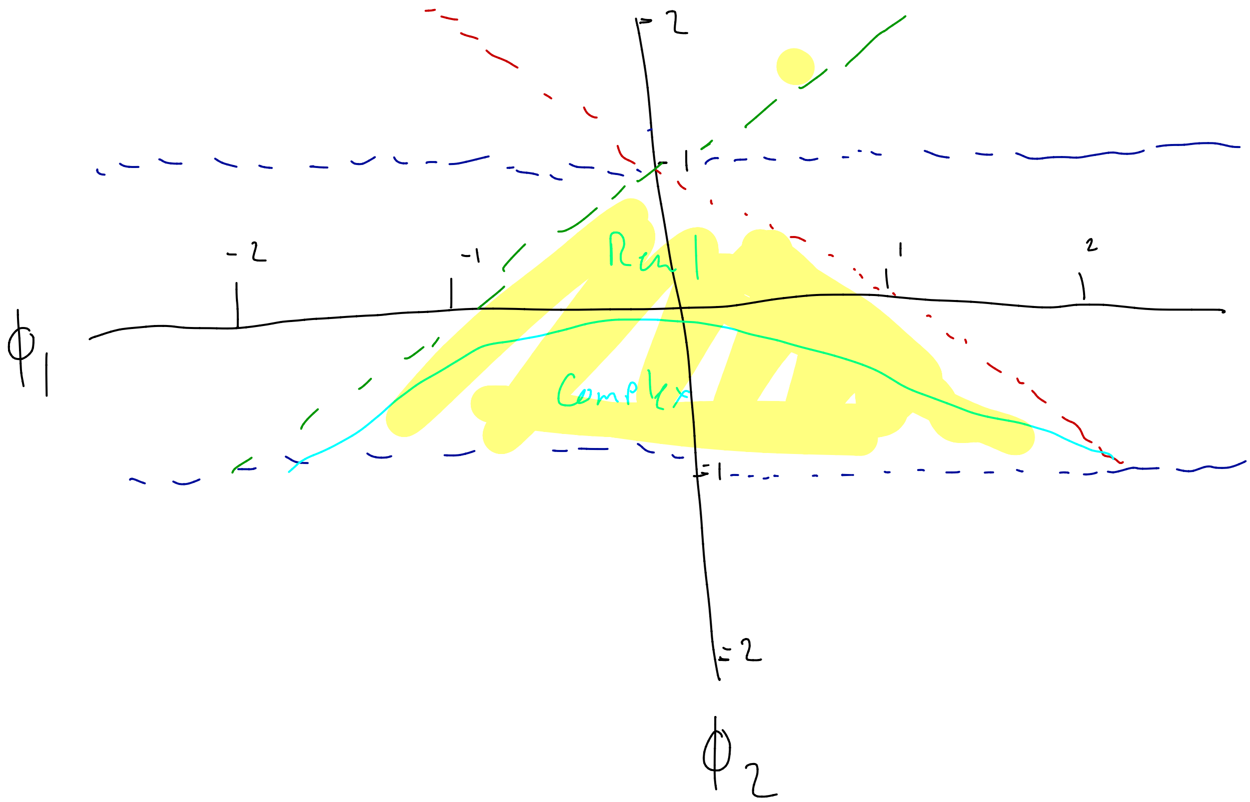
$$|\lambda| < 1$$

$$\sqrt{\left(\frac{\phi_1}{2}\right)^2 - \left(\frac{\sqrt{-(\phi_1^2 + 4\phi_2)}}{2} i\right)^2} < 1$$

$$\sqrt{\frac{\phi_1^2}{4} - \frac{\phi_1^2 + 4\phi_2}{4}} < 1$$

$$\sqrt{\phi_2} < 1$$

$$-1 < \phi_2 < 1$$



$-1 < \phi_2 < 1$

$\phi_1 + \phi_2 < 1$

$\phi_2 < 1 - \phi_1$

$\phi_2 - \phi_1 < 1$

$\phi_2 < 1 + \phi_1$

$\phi_1^2 + 4\phi_2 < 0$

AR(2)

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + v_t$$

$$E(y_t) = \delta + \phi_1 E(y_{t-1}) + \phi_2 E(y_{t-2})$$
$$= \delta + \phi_1 E(y_t) + \phi_2 E(y_t)$$

$$(1-\phi_1-\phi_2)E(y_t) = \delta$$
$$E(y_t) = \delta / 1-\phi_1-\phi_2$$

$\tilde{y}_t = y_t - E(y_t)$ $\quad Cov(y_t, y_{t-h}) = Cov(\tilde{y}_t, \tilde{y}_{t-h})$

$$\gamma(0) = Var(\tilde{y}_t) = \phi_1^2 Var(\tilde{y}_{t-1}) + \phi_2^2 Var(\tilde{y}_{t-2}) + \phi_1\phi_2 E(\tilde{y}_{t-1}\tilde{y}_{t-2}) + \sigma_v^2$$
$$= \phi_1^2\gamma(0) + \phi_2^2\gamma(0) + \phi_1\phi_2\gamma(-1) + \sigma_v^2$$

$$\tilde{y}_t\tilde{y}_{t-h} = \phi_1\tilde{y}_{t-1}\tilde{y}_{t-h} + \phi_2\tilde{y}_{t-2}\tilde{y}_{t-h} + v_t\tilde{y}_{t-h}$$

$$\gamma(h) = E(\tilde{y}_t\tilde{y}_{t-h})$$
$$= \phi_1 E(\tilde{y}_{t-1}\tilde{y}_{t-h}) + \phi_2 E(\tilde{y}_{t-2}\tilde{y}_{t-h}) + E(w_t\tilde{y}_{t-h})$$
$$= \phi_1\gamma(h-1) + \phi_2\gamma(h-2)$$

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(-1) = \phi_1\gamma(0) + \phi_2\gamma(1)$$
$$\gamma(1) = \frac{\phi_1\gamma(0)}{1-\phi_2}$$

$$\gamma(0) = \phi_1^2 \gamma(0) + \phi_2^2 \gamma(0) + \phi_1 \phi_2 \gamma(1) + \sigma_w^2$$

$$\gamma(1) = \frac{\phi_1 \gamma(0)}{1 - \phi_2}$$

$$\gamma(0) = \frac{\sigma_w^2}{1 - \phi_1^2 - \phi_2^2 - \phi_1\left(\frac{\phi_1}{1-\phi_2}\right)} = \frac{(1-\phi_2)\,\sigma_w^2}{(1-\phi_2) - (1-\phi_2)\phi_1^2 - (1-\phi_2)\phi_2^2 - \phi_1^2}$$

$$= \frac{(1-\phi_2)\,\sigma_w^2}{1 - \phi_2 - \phi_1^2 - \phi_1^2\phi_2 - \phi_2^2 + \phi_2^3} = \frac{(1-\phi_2)\,\sigma_w^2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$$

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2)$$

MA (1)

$$Y_t = \delta + w_t + \theta w_{t-1}$$

$$E(Y_t) = \delta$$

$$\gamma(0) = Var(Y_t) = Var(w_t) + \theta^2 Var(w_{t-1})$$
$$= \sigma_w^2 + \theta^2 \sigma_w^2 = \sigma_w^2 (1+\theta^2)$$

$$\gamma(h) = E(Y_t \; Y_{t-h})$$
$$= E\left((\delta + w_t + \theta w_{t-1})(\delta + w_{t-h} + \theta w_{t-h-1})\right)$$
$$= E(w_t w_{t-h}) + E(w_t \theta w_{t-h-1})$$
$$+ E(\theta w_{t-1} w_{t-h}) + E(\theta w_{t-1} \theta w_{t-h-1})$$

$$= \begin{cases} \sigma_w^2 (1+\theta^2) & \text{if } h = 0 \\ \theta \sigma_w^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & \text{if } h = 0 \\ \theta / 1+\theta^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$