Fifth Kolmogorov Student Olympiad in Probability Theory

**Problem 1.** In a hand are six blades of grass next to each other pointing in the same direction. The tops of the blades are randomly collected into pairs that are linked together, and likewise with the bottoms of the blades. What is the probability that as a result of this operation, the six blades will be connected in a single ring?

**Problem 2.** Let $X$ and $Y$ be independent Gaussian random variables with mean 0 and variance 1. Find $\mathbb{E}(X|XY)$.

**Problem 3.** Is there a probability space and random variables $X_1, X_2, \ldots$ on it with the properties

- all $X_n$ are Gaussian with mean 0 and variance 1;
- $X_n I(X_n \leq 0) = X_m I(X_m \leq 0)$ for all $n, m$;
- the random variables $I(X_n \in [a_n, b_n])$ for $n \in \mathbb{N}$ are independent for all $a_n, b_n \geq 0$.

**Problem 4.** Let $X$ and $Y$ be independent random variables such that $X$ has a continuous distribution, i.e., $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$. Is it true that $X + Y$ has a continuous distribution?

**Problem 5.** Let $X$ be a random variable such that $\mathbb{E}X^2 < \infty$. Find $\inf \mathbb{E}ZX$ for all nonnegative random variables $Z$ with $\mathbb{E}Z^2 \leq 1$.

**Problem 6.**

a) From a pack of 52 cards is taken (without replacement) cards until the ace of hearts appears. Find the expected value of this time (for example, if the ace of hearts occurs right at the start then this time is 1).

b) Find the expected time when the first ace appears.

c) Find the expected time when the first card with the suit hearts appears.

**Problem 7.** Let $X$ be a random variable, and $f$ and $g$ increasing bounded functions. Prove that the random variables $f(X)$ and $g(X)$ have nonnegative correlation.

**Problem 8.** Let $\mathcal{A}$ be a $\sigma$-algebra on the set of natural numbers and $\mathbb{P}$ be a probability measure on $\mathcal{A}$. Is it true that $\mathbb{P}$ can be extended to a probability measure on the $\sigma$-algebra of all subsets of the natural numbers?

**Problem 9.** Let $X_1, X_2, \ldots$ be independent identically distribution strictly positive random variables. Set $S_n = X_1 + \cdots + X_n$. Is it true that $\lim_{n \to \infty} X_n/S_n = 0$ almost surely?

**Problem 10.** Let $B$ be a Brownian motion, starting from 0, and $f : [0, 1] \to \mathbb{R}$ a continuous function with $f(0) \neq 0$. Prove that $\mathbb{P}(\min_{t \in [0,1]} (B_t + f(t)) = 0) = 0$.

**Problem 11.** Let $X_1, X_2, \ldots$ be independent identically distributed random variables such that $\mathbb{P}(X_n \neq 0) > 0$. Set $S_n = X_1 + \cdots + X_n$. Prove that

$$\lim_{n \to \infty} \mathbb{P}(S_n \in [a, b]) = 0$$

for all $a, b$. 

trans. Keith Conrad
Sixth Kolmogorov Student Olympiad in Probability Theory

**Problem 1.** A basket contains $M$ green apples and $N$ red apples. We choose one apple at a time at random, without replacement, until we have removed all the red apples. What is the probability that the basket is empty when we finish?

**Problem 2.** Let $X$ and $Y$ be random variables on a common probability space, such that the distribution of $X + Y$ is the same as the distribution of $X$. (a) Does it follow that $Y = 0$ a.s.? (b) The same question, if we know $Y \geq 0$? (c) (3rd-5th years) The same question, if we know that $X$ and $Y$ are independent?

**Problem 3. (1st-2nd years)** Let $A_1, A_2, A_3, A_4$ be events such that $\mathbb{P}(A_j) = 1/2, j = 1, 2, 3, 4$. Prove the following: (a) $\max_{1 \leq j,k \leq 4; j \neq k} \mathbb{P}(A_j A_k) \geq 1/6$; (b) The bound in the previous part cannot be improved.

**Problem 3. (3rd-5th years)** Suppose the sequence of random variables $(\xi_n)_{n \in \mathbb{N}}$ does not converge to zero a.s. as $n \to \infty$. Prove that there exists an $\varepsilon > 0$, an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$, and a sequence of nested events $(A_k)_{k \in \mathbb{N}}$ (i.e., $A_k \supseteq A_{k+1}, k \in \mathbb{N}$) such that for each $k \in \mathbb{N}$, $\mathbb{P}(A_k) > 0$ and $|\xi_{n_k}(\omega)| \geq \varepsilon$ for all $\omega \in A_k$.

**Problem 4.** (a) Consider two random vectors $(X, Y)$ and $(Z, U)$ which take a finite number of values, and suppose $Y$ and $Z$ have the same distribution. Prove that there exists a random vector $(\xi_1, \xi_2, \xi_3)$ on some probability space such that $(\xi_1, \xi_2)$ has the same distribution as $(X, Y)$ and $(\xi_2, \xi_3)$ has the same distribution as $(Z, U)$. (b) Prove the same result without the assumption that the random vectors take only a finite number of values.

**Problem 5.** Let $\xi \sim \mathcal{N}(0, 1)$. Prove that $\mathbb{P}(|\xi| \geq x) \leq e^{-x^2/2}$ for every $x \geq 0$.

**Problem 6.** 100 locomotives leave a town along a single railroad track, each at a constant random speed. After all are underway, several caravans (groups moving together at the speed of the caravan leader) form. Find the mean and variance of the number of such groups. The distribution function (cdf) of the speed of the locomotives is continuous and strictly increasing on the positive half-line, and the speeds of different locomotives are independent.

**Problem 7.** Twenty people are sitting at a circular table. A plate is placed in front of one of them, who chooses (with equal probability) one of his two neighbors and passes the plate to him. This continues, with independent equal-probability choices at each step, until each person has had the plate passed to them at least once. Each person has a certain probability of being the last to receive the plate; find the number of people whose probability attains the maximum over the group.

**Problem 8.** At each integer point on the real line lives a civilization. Each day, any two civilizations at a distance $n$ from each other will be fighting with probability $p_n$, independently of all other conflicts. Call a segment of the line between two neighboring civilizations safe if there are no ongoing conflicts between parties on opposite sides of the segment. Prove that, with probability one, either there are no safe segments, or there are an infinite number of them.

**Problem 9.** Let $X_1, X_2, \ldots, X_n$ be independent random variables having a Cauchy distribution with location parameter $a \in \mathbb{R}$ and scale parameter $\sigma > 0$ (i.e., $\mathbb{E}e^{itX_1} = e^{iat-\sigma|t|}, t \in \mathbb{R}$). Find estimators of $a$ and $\sigma$ which converge to the true values in probability as $n \to \infty$. 

trans. Jonathan Christensen / jonathan.christensen@stat.duke.edu
Seventh Kolmogorov Student Olympiad in Probability Theory

**Problem 1.** (a) Suppose events $A$ and $B$ are independent. Is it true that $P(AB|C) = P(A|C)P(B|C)$, where $C$ is an arbitrary event? (b) Let $X$ and $Y$ be independent integrable random variables, and $\mathcal{A}$ a $\sigma$-algebra. Is it true that $\mathbb{E}(XY|\mathcal{A}) = \mathbb{E}(X|\mathcal{A})\mathbb{E}(Y|\mathcal{A})$ a.s.?

**Problem 2.** Let $X, Y, Z$ be independent strictly positive random variables. Prove that for all $x > 0$, $P(X/Z < x, Y/Z < c) \geq P(X/Z < x)P(Y/Z < c)$.

**Problem 3.** Peter must bring home from the well each day a full bucket’s worth of water. Having drawn a full bucket from the well, on the way home he spills a proportion uniformly distributed on the interval $[0, 1]$. How many times, on average, must he go to the well each day?

**Problem 4.** Let $X$ and $Y$ be integrable random variables such that $\mathbb{E}X^+/\mathbb{E}X^- \geq z$ and $\mathbb{E}Y^+/\mathbb{E}Y^- \geq z$ for some $z > 0$. Does it follow that $\mathbb{E}(X + Y)^+/\mathbb{E}(X + Y)^- \geq z$? As usual, $t^+ = \max\{t, 0\}$ and $t^- = (-t)^+$.

**Problem 5.** Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary sequence of random variables (i.e., the covariance $\text{cov}(X_j, X_k)$ exists and depends only on $j-k$). Suppose that $\text{cov}(X_0, X_j) \leq 0$ for every $j \in \mathbb{Z} \setminus \{0\}$. Is it possible for the series $\sum_{j=0}^{\infty} \text{cov}(X_0, X_j)$ to diverge?

**Problem 6.** Prove that there exists a square $11 \times 11$ matrix $A$, each of whose elements is equal to either 1 or $-1$, such that $\det A > 4000$.

**Problem 7.** Let $X_1, X_2, \ldots, X_{100}$ be independent random variables uniformly distributed on the interval $[-1, 1]$, and let $X_{(k)}$, $k = 1, 2, \ldots, 100$ be the corresponding order statistics in increasing order. Find the expectation of the random variable $X_{(17)}$.

**Problem 8.** (a) The random variable $X$ has characteristic function $\mathbb{E}e^{itX} = 1/\sqrt{1 + t^2}$, and is distributed like the product of two independent random variables which have density functions. Find their densities. (b) The same, but for the characteristic function $1/(1 + |t|)$.

**Problem 9.** Let the matrices $A = (a_{ij})_{i,j=1}^{n}$ and $B = (b_{ij})_{i,j=1}^{n}$ be symmetric and nonnegative definite (positive semidefinite). Prove that the matrix $C = (c_{ij})_{i,j=1}^{n}$, where $c_{ij} = a_{ij}b_{ij}$, $i, j \in \{1, \ldots, n\}$, is also symmetric and nonnegative definite.

**Problem 10.** Given a sample of size one from the random variable $\xi \sim N(\mu, \sigma^2)$, both of whose parameters are unknown, give a confidence interval for $\sigma^2$ with confidence level at least 99%.
Eighth Kolmogorov Student Olympiad in Probability Theory

Problem 1. Let $X$ be a random variable with finite variance which is not identically zero. Prove that $\mathbb{P}(X = 0) \leq \text{var}(X)(\mathbb{E}(X^2))^{-1}$.

Problem 2. Let $A_1, \ldots, A_{2000} \subset A$ be sets each containing at least six elements, and not all identical. Prove that there exit 100 distinct partitions of $A$ into 5 disjoint subsets $E_1, \ldots E_5$ such that each $A_i$ contains elements belonging to at least two of the $E_i$.

Problem 3. Container $A$ initially contains 1000 green and 3000 red apples and container $B$ contains 3000 green and 1000 red apples. We randomly choose 2000 apples from container $A$ and place them in container $B$. One apple is then chosen at random from container $B$. What is the probability that it is green?

Problem 4. Each resident of the city N. gets off work at a random time, after which they love to go fishing. The lake in which they fish is home to carp and bream, of which the bream make up a proportion $p$. The city has a law which forbids catching more than one bream in a day, so the strictly law-abiding citizens stop fishing as soon as they catch their first bream. What proportion of all fish caught in the city are bream?

Problem 5. The sequence of random variables $(X_n)_{n \in \mathbb{N}}$ converges in probability to the random variable $X$. If $X_n$ and $X$ are independent for every $n \in \mathbb{N}$, does it follow that $X$ is constant a.s.?

Problem 6. Let $X_1, X_2, \ldots$ be a sequence of Poisson-distributed random variables with rate 1. Prove that $\mathbb{E}\max\{X_1, X_2, \ldots, X_n\} = O(\ln n), n \to \infty$.

Problem 7. Let $(S_n)_{n \in \mathbb{N}}$ be a simple random walk, with $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, where the $(\xi_j)_{j \in \mathbb{N}}$ are independent and take on the values 1 and $-1$ with probability 1/2. Let $\tau = \inf\{n \in \mathbb{N} : S_n = 0\}$ be the time of first return to zero. For each $a \in \mathbb{N}$ find $\mathbb{E}N_a$, where $N_a = |\{j < \tau : S_j = a\}|$ is the amount of time the random walk spent at the point $a$ before time $\tau$.

Problem 8. Let $W = (W_t)_{t \geq 0}$ be a Wiener process. (a) Find the expected amount of time the process spends above the line $y = t$; (b) Write the variance of this time as an integral of elementary functions; (c) calculate this variance.

Problem 9. You are given a sample containing a single observation $X$ from a Normal distribution with mean $\mu$ and variance 1. Let $f(X)$ be an estimator of $\mu$ such that $f$ is continuous and $\mathbb{E}f(X)^2 < \infty$ for all $\mu$. Prove that $\sup_{\mu \in \mathbb{R}} \mathbb{E}(f(X) - X)^2$ takes its minimum over all such functions $f$ when $f(x) = x$.

Problem 10. Let $p > 0$, and let $X_1, X_2, \ldots$ be random variables such that for every $\varepsilon > 0$ the series $\sum_{n=1}^{\infty} n^{-1} \mathbb{P}(\max_{k=1,\ldots,n} |X_k| > \varepsilon n^{1/p})$ converges. Prove that $X_n/n^{1/p}$ converges a.s. as $n \to \infty$.

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trans. Jonathan Christensen / jonathan.christensen@stat.duke.edu