DISCOVERING EXACTLY WHEN A RATIONAL IS A BEST APPROXIMATE OF AN IRRATIONAL

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I. MOTIVATION - THE SEARCH FOR BEST APPROXIMATES

Irrational numbers are complicated, hard to understand, and hard to work with. Rational numbers are easily understood and easier to work with. For this reason, number theorists have long worked in the field of *Diophantine Approximation*, the process of approximating irrational numbers with rational ones. What we are specifically concerned with here is determining when a rational number is a *best rational approximate* of an irrational number.

Suppose we want to approximate the irrational number $e = 2.71828....$ Intuitively, one might use the rational $\frac{2718}{1000}$, and observe

$$\left| e - \frac{2178}{1000} \right| = .00028... \ .$$

At a glance, this appears to be a good approximation of e, but how do we know if it is a *best* approximation? How do we know we can't do better? What about the (not so intuitive) rational approximation, $\frac{1650}{607}$? One may compute

$$\left| e - \frac{1650}{607} \right| = .0000048... \ ,$$

so in some sense, this is a "better" approximation. It brings us closer to the irrational we are approximating, and it is a "simpler" fraction, in that it uses lower numbers. Yet how do you come up with the number $\frac{1650}{607}$, and once you have it, how do you determine if it really is a best approximate? The first question was answered in the 18th century, the second question is the driving force behind this paper.

Clearly, there is no "closest rational approximation" to any irrational number. You can always keep increasing the denominator of the rational approximation, and in doing so, keep getting closer and closer to the irrational number you are trying to approximate. However, since the purpose of a rational approximate is to give us a number that is easier to work with, often we do not want our approximations to have extremely large denominators. Also, although in theory denominators can be arbitrarily large, computers only have a finite amount of memory, and so even if only for this reason alone, it is necessary to restrict the size of the denominator. Most commonly in practice, the denominator is restricted far before the limits of computer memory, restricted to a size which humans are comfortable working with. More importantly, and in the true spirit of number theory, regardless of whether we need to restrict denominators or not, we use the size of a denominator as an indicator for the complexity of a rational, and in our search for best approximates we are looking for rationals that are both close to the irrational, and relatively simple. For whatever reason, we often want/need to restrict the size of the denominator of our rational approximates, in which case a closest approximation to any irrational number does exist. We will call this closest rational approximation given a restriction on the size of the denominator a *best approximate*.

To give a better sense for what a best approximate is, let's work for a second with money. Pretend you have to purchase your rational approximate, and the price is the size of the denominator. Naturally, you would like to get the best deal for your money. The best approximates to any irrational number are precisely these "best deals". Alternatively, the amount of money you have to spend, or the amount you are willing to spend limits your options. The closet rational to your irrational number that still lies within your price range will be a best approximate (you can't do any better without paying more).

For each irrational number, there is more than one best approximate (in fact, as we will see, there are infinitely many). If you are willing to pay more, or if you put a higher limitation on the size of the denominator, you very well might find a closer best approximate. The best approximate is dependent on whatever you set your limit on the denominator to be. Interestingly enough, just because you allow for a larger denominator, the best approximate you find won't necessarily be any different. In other words, the denominator of a best approximate is not always the limit you have placed on the size of your denominator. For example, let's consider the irrational number π. If you require a denominator less than or equal to 7, the best rational approximate to π would be $\frac{22}{7} = 3.1428...$ (if you check each denominator less than or equal to 7 you will find that each corresponding rational closest to π is not as close as $\frac{22}{7}$). However, if you set the limit on the denominator to be 100, the best rational approximate to π is still $\frac{22}{7}$. In fact, for those who are curious, the next best approximate to π is $\frac{333}{106} = 3.141509....$ Given that the best approximate depends on a usually arbitrary choice of how much to limit the denominator, we are not typically concerned with *the* best approximate in a particular situation, but rather in finding the complete sequence of best approximates for a given irrational number.

Due to work by Lagrange in 1770, we know exactly how to compute this sequence of best approximates. However, as we will see, the method following from Lagrange's discovery uses recursion, requiring a knowledge of each and every previous best approximate in the sequence in order to compute the next. This was an incredibly significant step in the right direction, yet wouldn't it be wonderful if one did not have to use recursion to determine whether or not a rational is in the sequence of best approximates? Wouldn't it be nice if we could simply look at a rational number, do a simple computation, and determine precisely whether or not this rational is a best approximate of a certain irrational number? This is the motivation for our work here. We have attempted to completely classify best approximates without requiring the entire list of best approximates up to that point. Our accomplishment of this goal rests heavily on the above-mentioned result of Lagrange, which in turn requires a journey into the land of continued fractions.

II. BACKGROUND AND PREVIOUS RESULTS

Consider a real number, α. Denote the integer part of this number by $\lfloor \alpha \rfloor$, and the fractional part by $\{\alpha\}$. Clearly,

$$\alpha = \lfloor \alpha \rfloor + \{\alpha\}.$$

We now present an algorithm for generating the *continued fraction expansion* of α. Let a_0 be the integer part of α and α_0 be the fractional part of α. If $\alpha_0 = 0$ we are done and $\alpha = a_0$. Otherwise, we flip α_0 twice (bringing us back to the original) and observe

$$\alpha = a_0 + \frac{1}{\frac{1}{\alpha_0}}.$$

Since α_0 is the fractional part, we must have $0 < \alpha_0 < 1$, and thus the reciprocal of α_0 must be greater than 1. We now take the integer part of $\frac{1}{\alpha_0}$ and denote it by a_1, and take the fractional part of $\frac{1}{\alpha_0}$ and denote it by α_1, hence we have

$$\alpha = a_0 + \frac{1}{a_1 + \alpha_1}.$$

Again, if $\alpha_1 = 0$ we are done, otherwise we flip α_1 twice and break its reciprocal into its integer and fractional parts, denoted by a_2 and α_2, getting the following:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2+\alpha_2}}.$$

We continue in this fashion, flipping each fractional part α_i twice and replacing its reciprocal with its integer part a_{i+1} and its fractional part α_{i+1}. We either continue with this process forever (as is the case for irrational α), or until we reach a fractional part equal to 0 (as will be the case for rational α).

This algorithm generates what is known as the *continued fraction expansion* of a real number. The continued fraction expansion for a real number α,

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2+\frac{1}{a_3+\ldots}}}$$

is denoted by $\alpha = [a_0, a_1, a_2, a_3, \ldots]$. Note that since the a_i's are integer parts of numbers, clearly we have $a_i \in \mathbf{Z}$ for all i. The first term a_0 may be positive or negative depending on the sign of the initial number, α, but since fractional parts are always positive, their reciprocals, and hence the integer parts of their reciprocals are always positive, thus we observe $a_i > 0$ for all $i > 0$. If the continued fraction expansion for a real number terminates, you can keep getting common denominators, and by unravelling the denominators, you will observe a rational number. However, if the continued fraction expansion for a real number α is infinite, then α is irrational. Moreover, following the above algorithm, every irrational number can be expressed as a unique continued fraction expansion and this continued fraction expansion will continue forever.

To demonstrate, we will find the continued fraction expansions of two different numbers. We start with the rational number $\frac{7}{5}$ and observe

$$\frac{7}{5} = 1 + \frac{2}{5}.$$

We now flip the fractional part twice to get:

$$\begin{aligned} \frac{7}{5} &= 1 + \frac{1}{\frac{1}{\frac{2}{5}}} \\ &= 1 + \frac{1}{\frac{5}{2}}, \end{aligned}$$

and now dividing $\frac{5}{2}$ into its integer and fractional parts we find

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}}.$$

We could stop here since this is in the desired form, yet we will continue with the algorithm for completeness. Again we flip the fractional part twice:

$$\begin{aligned} &= 1 + \frac{1}{2 + \frac{1}{\frac{1}{\frac{1}{2}}}} \\ &= 1 + \frac{1}{2 + \frac{1}{2}}, \end{aligned}$$

and since the fractional part of 2 is 0, we are done. Hence we have deduced $\frac{7}{5} = [1, 2, 2]$. Observe that the rational number $\frac{7}{5}$ does indeed have a finite continued fraction expansion, as expected.

We now compute the continued fraction expansion of a very popular irrational number, $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio:

$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \left(\frac{1+\sqrt{5}}{2} - 1 \right),$$

and by getting a common denominator we see

$$\varphi = 1 + \frac{\sqrt{5}-1}{2}.$$

Continuing with our algorithm, and rationalizing the denominator, we get the following:

$$\begin{aligned}
\varphi &= 1 + \frac{1}{\frac{1}{\frac{\sqrt{5}-1}{2}}} \\
&= 1 + \frac{1}{\frac{2}{\sqrt{5}-1}} \\
&= 1 + \frac{1}{\frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}} \\
&= 1 + \frac{1}{\frac{\sqrt{5}+1}{2}} \\
&= 1 + \frac{1}{\varphi}.
\end{aligned}$$

Thus we have discovered that

$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{\varphi} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\ldots}}},$$

so the continued fraction expansion for the golden ratio is periodic, and $\varphi = [\,\overline{1}\,]$. We will return to the golden ratio as the irrational with the simplest continued fraction expansion. For now however, we turn to terminating these infinite continued fractions.

If you take the continued fraction expansion $[a_0, a_1, a_2, a_3, ...]$ of a real number α and cut if off after a certain a_n, this results in the rational number $[a_0, a_1, a_2, ..., a_n]$. Such a number is known as the n^{th} *convergent of* α. Since the continued fraction for an irrational number can be stopped at infinitely many a_i's, there are infinitely many convergents to each irrational number. By a very useful theorem by Lagrange (hinted at previously), these convergents are directly related to our original quest for best approximates (Theorem 5.9 in [1]).

The Law of Best Approximates. *The best approximates of an irrational number* α *are precisely* *the convergents of* α.

Thus determining whether or not a rational is a best approximate of an irrational number is equivalent to determining whether or not it is a convergent of that irrational, a much more approachable problem.

Given any irrational number, we could find the continued fraction expansion, cut off the continued fraction at each point, and observe the sequence of convergents (a tedious task). Luckily, convergents are somewhat easier to find. Given an irrational number $\alpha = [a_0, a_1, a_2, ...]$, the n^{th} convergent of α, $\frac{p_n}{q_n}$, is defined by $p_0 = a_0, q_0 = 1$, $p_1 = a_1 a_0 + 1, q_1 = a_1$ and recursively for $n > 1$ by

$$p_n = a_n p_{n-1} + p_{n-2} \text{ and } q_n = a_n q_{n-1} + q_{n-2}. \tag{1}$$

(See for example, Corollary 4.2 in [1]). This makes calculating convergents much easier, however it is still a recursive definition, requiring the complete list of previous convergents to determine whether or not a rational number is a convergent of α. Moreover, this is an infinite process... we could never create the entire list of convergents since there are infinitely many. Thus, *our quest becomes discovering a way to completely classify the convergents (and hence best approximates) of an irrational number, without needed to compute each and every convergent.*

Enormous progress was made on this problem in 1808 with the following theorem by Legendre: (Theorem 5.12 in [1]):

Legendre's Theorem. *Suppose that* p *and* q *are relatively prime integers with* $q > 0$ *such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}. \tag{2}$$

Then $\frac{p}{q}$ *is a convergent of* α.

This accomplishes our goal for many rational numbers. We automatically now know that any rational within $\frac{1}{2q^2}$ of an irrational α is a convergent, and hence a best approximate of α. Moreover, and even more intrinsic to our specific goal, we know this without needing to compute any other convergents of α, and without even mentioning the concept "continued fraction expansion". We simply find the distance between the rational and the irrational, and if it is small enough, we know the rational is a best approximate. In this sense, this theorem proves to be extremely useful in determining when a rational number is a convergent (or best approximate) of an irrational number. Alas however, the above theorem is not if and only if. It is possible for rational numbers to be convergents of an irrational number without satisfying inequality (2). Thus Legendre's Theorem fails to classify all convergents of an irrational number, and so falls short of completing our quest.

The pessimist would frown and sigh that unfortunately determining whether or not a rational is a best approximate is still a hard task that often requires the complete sequence of convergents to be computed. The optimist on the other hand would view this as an opportunity, for the shortcomings of Legendre's theorem leaves our quest alive and gives us something to work towards. Our goal thus becomes improving Legendre's theorem... creating a similar theorem that classifies precisely when a rational number is a convergent of an irrational number. We want a theorem that not only can determine exactly when a rational is a convergent, but gives some criterium that holds for every convergent of an irrational number α, and for which every rational number satisfying the criterium is indeed a convergent of α. More specifically, we seek to find an interval (like Legendre, based on the denominator of the rational, q) for which $\alpha - \frac{p}{q}$ lies within the interval *if and only if* the rational $\frac{p}{q}$ is in fact a convergent of α.

For some specific values of α, this problem has already been tackled, leading to the inspiration for our work. The problem of completely classifying convergents was completely answered (although in a more cryptic form) by M. Mobius in 1998 for the irrational number $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio. In addition to being a simple number to consider (as we already observed, $\varphi = [\,\overline{1}\,]$), the golden ratio also has the unique property that the denominators of its sequence of convergents are exactly the Fibonacci numbers. Recall (or see equation (1)) that the denominators of convergents are defined recursively. For the golden ratio, $a_i = 1$ for all i, so clearly the sequence of denominators turns out to be the fibonacci numbers. So, in classifying the Fibonacci numbers (as was Mobius's incentive), Mobius was also completely classifying the denominators of the convergents of φ. Mobius developed the following theorem (see [7]):

Theorem (Mobius). *Let z be an integer with $z \geq 2$. Then z is a Fibonacci number if and only if the interval*

$$\left[\varphi z - \frac{1}{z}, \varphi z + \frac{1}{z}\right]$$

contains exactly one integer, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

This theorem is stated in the terms of Mobius himself, however, we will analyze the theorem a little and rephrase it for our purposes without affecting the validity of the theorem. First, note that since φ is irrational, φ times an integer and then plus or minus a rational number is certainly still irrational. Hence, the closed bounds of the interval may be replaced with open bounds. For $z \geq 2$ the interval has a maximum width of 1, and since the integer lying within the interval (if there is one) cannot exist on the boundaries, the phrase "exactly one integer" is superfluous, since the interval may contain at most one integer. For our concerns, we will also restate this theorem in terms of convergents rather than Fibonacci numbers, keeping in sight our goal of further generalization to irrationals beyond the golden ratio. Hence, the above theorem is equivalent to stating that a positive integer $q > 1$ is the denominator of a convergent of φ if and only if there exists an integer in the above interval (replacing z with q to follow conventional terminology).

The denominators of the convergents of an irrational number α are known as the *continuants* of α. We remark that completely classifying the continuants of any irrational α (as Mobius has done here for φ) is equivalent to completely classifying the convergents, since the corresponding numerators will simply be whatever numerator makes the fraction closest to α. In fact, here when q is a continuant of φ, the integer existing in the above interval is exactly the numerator of the convergent for which q is the denominator. Assuming q is a continuant of φ, by the above theorem there exists an integer in the above interval. Call this integer p and examine the inequality that results:

$$\varphi q - \frac{1}{q} < p < \varphi q + \frac{1}{q}.$$

Dividing through by q we observe

$$\varphi - \frac{1}{q^2} < \frac{p}{q} < \varphi + \frac{1}{q^2},$$

and subtracting φ throughout yields

$$-\frac{1}{q^2} < \frac{p}{q} - \varphi < \frac{1}{q^2}.$$

From this inequality we deduce that if q is in fact a continuant of φ we have

$$\left| \varphi - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{3}$$

Similarly, we can show that if there exists a rational $\frac{p}{q}$ satisfying inequality (3), then working backwards through what we have just done, p falls within the interval

$$\left(\varphi q - \frac{1}{q}, \varphi q + \frac{1}{q} \right),$$

and hence q is a continuant of φ (by Mobius's original theorem). Therefore, we may write the following as a direct corollary of Mobius's theorem. In doing so, we determine exactly when a number is a convergent of φ, the first step of our journey.

Corollary. *A rational number* $\frac{p}{q}$ *in lowest terms with* $q > 1$ *is a convergent of* $\varphi = \frac{1+\sqrt{5}}{2} = [\overline{1}\ ]$ *(the golden ratio) if and only if*

$$\left| \varphi - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Notice the similarity between this result and Legendre's theorem. The advantage of the above result is that it has what Legendre's lacked, namely it is if and only if. Not only does satisfying the above inequality brand a rational as a convergent, but also *every* convergent satisfies the inequality. However, there is a reason that Legendre's result is incredibly well known, this reason being that his result applies to all irrationals. The above corollary to Mobius's theorem applies to only one. Yet it is a first step - for one specific irrational number it completely classifies when a rational number is a best approximate. The next natural step is to try and generalize this result to other irrational numbers. This step was taken in 2003 by T. Komatsu. Where Mobius deals with the golden ratio ($\varphi = [\ \overline{1}\ ]$) and classifying the Fibonacci numbers, Komatsu deals with the generalized golden ratio ($\varphi_a = [\ \overline{a}\ ]$) and the generalized Fibonacci numbers. He produced the following result (see [4]):

Theorem (Komatsu). *Let $\varphi_a = \frac{a+\sqrt{a^2+4}}{2} = [\ \overline{a}\ ]$ with $a \geq 2$. Then q is a generalized Fibonacci number if and only if the interval*

$$\left[q\varphi_a - \frac{1}{aq}, q\varphi_a + \frac{1}{aq}\right]$$

contains exactly one integer p.

The generalized Fibonacci numbers are the numbers given by the recursion $q_0 = 1$, $q_1 = a$, and for $n > 1$, $q_n = aq_{n-1} + q_{n-2}$, which are exactly the continuants of the generalized golden ratio, φ_a. Thus we may rewrite Komatsu's theorem in terms of convergents, as we did with Mobius's. Before doing this however, we make a few other observations. Observe first that by combining Mobius's result with Komatsu's, Komatsu's result holds for all positive integers a. Also, we may again replace the closed interval with an open interval. We may again change the phrase "contains exactly one integer" to "contains an integer," with one caveat; if $a = 1$ and $q = 1$ the interval would clearly contain more than one integer (note that Komatsu requires $a \geq 2$ and Mobius requires $q \geq 2$). With two integers in the interval, in terms of convergents this would mean that two integers are both convergents of α, which is clearly impossible. Thus we require either $q > 1$ or $a > 1$, which we accomplish with the added criteria $qa > 1$. We then have the following corollary to Komatsu's theorem:

Corollary. *A rational number $\frac{p}{q}$ in lowest terms with $aq > 1$ is a convergent of $\varphi_a = \frac{a+\sqrt{a^2+4}}{2} = [\overline{a}\ ]$*

(the generalized golden ratio) if and only if

$$\left|\varphi_a - \frac{p}{q}\right| < \frac{1}{aq^2}.$$

Thus for $\varphi_a = [\ \overline{a}\ ]$, it is known exactly when a rational number is a convergent, and hence a best approximate of φ_a. This is where the previous results on the topic end, so we take the next logical step and further generalize this result.

III. GENERALIZATION: A FIRST STEP

While Mobius looked at the number $[\ \overline{1}\ ]$ and Komatsu generalized this for the numbers $[\ \overline{a}\ ]$, we seek to generate a similar result for numbers of the form $[\ \overline{a,b}\ ]$. This seems to be the next simplest case, and due to an important fact about convergents, it turns out that this case is indeed the easiest way to further generalize the work of Mobius and Komatsu. Before stating our result however, we require a deeper excursion into the theory of convergents.

We first mention two well-known facts about convergents (see [1]).

Fact 1. *If* $\frac{p_n}{q_n}$ *is the* n^{th} *convergent of* $\alpha = [a_0, a_1, a_2, ...]$*, then*

$$\left|\ \alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})},$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, ...]$.

By observing that $\alpha_{n+1} > a_{n+1}$ (since $a_i > 0$ for $i > 0$) and that $q_{n-1} > 0$ (recall continuants of α

are defined recursively as in (1)), we are lead to the inequality

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}. \tag{4}$$

Since we hope to ultimately produce an interval that contains $\alpha - \frac{p}{q}$ if and only if $\frac{p}{q}$ is a convergent, this fact will be particularly relevant.

If $\frac{p_n}{q_n}$ is a convergent, it is called an even convergent when n is even, and an odd convergent when n is odd. This second fact deals with even and odd convergents.

Fact 2. *For a convergent $\frac{p_n}{q_n}$ of an irrational number α, $\frac{p_n}{q_n}$ underestimates α when n is even and overestimates α when n is odd.*

This fact will be especially valuable for the specific case currently being considered, when the irrational we are dealing with is of the form $[\,\overline{a, b}\,]$. It is primarily this fact that makes irrationals of this form the next logical case to consider after the generalized golden ratios. More importantly, fact 1 used in conjunction with fact 2 allows us to determine the exact value of the difference between an irrational α and a convergent of α (we gain the value from fact 1 and the sign from fact 2). Besides the above two facts about convergents, it is important to recall that both the numerators and denominators are each defined recursively, as in equation (1).

Although Lagrange has shown the best approximates of an irrational number α to be exactly the convergents of α, and hence we wish for our theorem to classify each and every convergent and only convergents, we must also gain familiarity with another type of “convergent” before we delve into our result.

Given convergents $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$, consider the two “Farey-esque” fractions

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} \quad \text{and} \quad \frac{p_n - p_{n-1}}{q_n - q_{n-1}}.$$

Such rational numbers are known as *secondary convergents* of α. For convenience we will denote the former a secondary convergent of type 1 and the latter a secondary convergent of type 2. A theorem reminiscent of Legendre holds for secondary convergents. (see [8]).

Theorem. *If a rational number $\frac{p}{q}$ with p and q relatively prime satisfies inequality*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}, \tag{5}$$

Then $\frac{p}{q}$ must be either a convergent or a secondary convergent of α.

The above theorem gives us important information on any rationals that lie within $\frac{1}{q^2}$ of α, namely that they must take one of three forms; convergent, type 1 secondary convergent, or type 2 secondary convergent. The key to our work lies in this crucial fact.

Before moving ahead, some terminology must be introduced. If we have a continued fraction expansion $[a_0, a_1, a_2, ...]$, we denote the n^{th} *complete quotient of α* by α_n, where $\alpha_n = [a_n, a_{n+1}, a_{n+2}, ...]$ (the "tail"). Going the other direction, we denote the rational number $[a_n, a_{n-1}, ..., a_1]$ by γ_n. We now present the following fact on secondary convergents: (see [8]):

Fact 3. *If $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ are convergents of $\alpha = [a_0, a_1, a_2, ...]$, then*

$$\left|\alpha - \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right| = \frac{1}{\lambda_n (q_n + q_{n-1})^2}, \text{ where } \lambda_n = 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1},$$

and

$$\left|\alpha - \frac{p_n - p_{n-1}}{q_n - q_{n-1}}\right| = \frac{1}{\lambda'_n (q_n - q_{n-1})^2}, \text{ where } \lambda'_n = 1 + \frac{1}{\gamma_n - 1} - \frac{1}{\alpha_{n+1} + 1}.$$

Exploiting the union of the above theorem and facts, we are now able, for every rational within $\frac{1}{q^2}$ of α, to determine its exact distance from α. Armed with this information, we are ready to state our result concerning the specific irrationals $\alpha = [\ \overline{a, b}\ ]$.

Theorem 1. *Let* $\alpha = [a_0, a_1, a_2, ...] = [\ \overline{a, b}\ ] = \frac{ab+\sqrt{a^2b^2+4ab}}{2b}$*, where* $\alpha \neq [\ \overline{1, 2}\ ]$*,* $\alpha \neq [\ \overline{1, 3}\ ]$*,* $\alpha \neq [\ \overline{2, 1}\ ]$*. A rational number,* $\frac{p}{q}$ *in lowest terms with* $q > 1$ *is a convergent of* α *if and only if*

$$\frac{-1}{aq^2} < \alpha - \frac{p}{q} < \frac{1}{bq^2}. \tag{6}$$

Before embarking on the proof of this result, we first make note of the fact that the theorems of both Mobius and Komatsu are specific cases of the above result, so this is indeed a generalization. As for the three values which α is hypothesized not to take, for now we simply remark that the theorem does not hold for each of those specific values of α (when α takes on those values, there exists a rational in interval (6) that is not a convergent). We will examine these cases in more detail later. More importantly for now, we need to verify the above result. Proving that all convergents lie within the above interval is not hard, it is in proving that each rational within the interval is a convergent that the proof becomes more delicate and involved.

Proof. Let $\alpha = [a_0, a_1, a_2, ...]$ be an irrational number of the form $\alpha = [\ \overline{a, b}\ ]$ and let $\frac{p_n}{q_n}$ be a convergent of α. To show that this convergent lies within interval (6), we consider two cases, depending on the parity of n.

<u>Case 1</u>: Suppose that n is even. Then $a_{n+1} = b$, so by inequality (4) we see

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{bq_n^2}.$$

Since n is even, we recall that $\frac{p_n}{q_n} < \alpha$ (by fact 2). Thus we observe

$$0 < \alpha - \frac{p_n}{q_n} < \frac{1}{bq_n^2},$$

which implies inequality (6).

Case 2: Suppose that n is odd, so $a_{n+1} = a$. Hence inequality (4) reveals

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{aq_n^2}.$$

As n is odd, we now have that $\frac{p_n}{q_n} > \alpha$ and deduce

$$0 < \frac{p_n}{q_n} - \alpha < \frac{1}{aq_n^2}.$$

Therefore, every convergent of α satisfies inequality (6), as desired.

We now assume that $\frac{p}{q}$ is a rational number in lowest terms that satisfies inequality (6), and wish to show that $\frac{p}{q}$ is a convergent of α. By inequality (6) we have

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\min\{a, b\}q^2}. \tag{7}$$

We know that every rational satisfying inequality (5) must be either a convergent or a secondary convergent. The challenge (excepting the trivial case) lies in showing that each secondary convergent satisfying inequality (6) is in fact a convergent. We consider several cases, depending on the values of a and b.

Case 1: $a \geq 2$ and $b \geq 2$.

In this case we note that $\min\{a, b\} \geq 2$, and thus

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\min\{a, b\}q^2} \leq \frac{1}{2q^2}.$$

Hence by Legendre's Theorem, $\frac{p}{q}$ is a convergent of α. We now move on to the nontrivial cases, when either a or b equals 1.

Case 2: $a = b = 1$.

Then $\alpha = \varphi$ and this case is given to us by our corollary to Mobius's theorem.

Case 3: $a = 1$, $b \geq 4$. Recall that by our initial hypothesis, when $a = 1$ we require $b \neq 2$ and $b \neq 3$. Thus in proving this case we complete the proof for when $a = 1$.

We first assume that $\frac{p}{q}$ is a type 1 secondary convergent; that is, there exists an n such that

$$\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

If n is odd, then $a_{n+1} = a = 1$, so

$$\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}$$

so $\frac{p}{q}$ is a convergent to α, as desired. Thus, we may now assume n is even. In this case $\alpha_{n+1} = [\ \overline{b, a}\ ] = b + \frac{1}{\alpha}$. Recall λ_n as defined in fact 3. It is easy to verify that (see for example Lemma 6.1 of [1]) $\gamma_n = [a_n, a_{n-1}, a_{n-2}, ..., a_1] = \frac{q_n}{q_{n-1}}$. Hence we find

$$\begin{aligned}
\lambda_n &= 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1} \\
&= 1 + \frac{1}{b + \frac{1}{\alpha} - 1} - \frac{1}{\frac{q_n}{q_{n-1}} + 1} \\
&= 1 + \frac{\alpha}{(b-1)\alpha + 1} - \frac{q_{n-1}}{q_n + q_{n-1}} \\
&= \frac{(q_n + q_{n-1})(b-1)\alpha + (q_n + q_{n-1}) + (q_n + q_{n-1})\alpha - q_{n-1}(b-1)\alpha - q_{n-1}}{(q_n + q_{n-1})(b-1)\alpha + (q_n + q_{n-1})} \\
&= \frac{bq_n\alpha + q_n + q_{n-1}\alpha}{\alpha(q_n + q_{n-1})(b-1) + (q_n + q_{n-1})}.
\end{aligned}$$

We now claim that for $a = 1$, $b \geq 4$, and n even, we have $\lambda_n < 1$. Since $\alpha = [\ \overline{1, b}\ ]$, $1 < \alpha < 2$, so $b > \alpha$. Also as n is even, we have that $q_{n-1} = bq_{n-2} + q_{n-3}$. Hence we note that $q_{n-1} > bq_{n-2} > \alpha q_{n-2}$. In view of our assumption that $b \geq 4$, we also have that $b\alpha q_{n-1} \geq 3\alpha q_{n-1}$. Thus we have the following:

$$\begin{aligned}
b\alpha q_{n-1} + q_{n-1} &> 3\alpha q_{n-1} + \alpha q_{n-2} \\
&= 2\alpha q_{n-1} + \alpha(q_{n-1} + q_{n-2})
\end{aligned}$$

$$= \quad 2\alpha q_{n-1} + \alpha q_n.$$

Adding $b\alpha q_n + q_n$ to both sides yields

$$b\alpha q_{n-1} + q_{n-1} + b\alpha q_n + q_n > 2\alpha q_{n-1} + \alpha q_n + b\alpha q_n + q_n,$$

or equivalently,

$$\alpha(q_n + q_{n-1})(b-1) + (q_n + q_{n-1}) > bq_n\alpha + q_n + q_{n-1}\alpha.$$

This immediately implies that

$$\lambda_n = \frac{bq_n\alpha + q_n + q_{n-1}\alpha}{\alpha(q_n + q_{n-1})(b-1) + (q_n + q_{n-1})} < 1,$$

and establishes our claim. So by fact 3 we have

$$\left| \alpha - \frac{p}{q} \right| = \frac{1}{\lambda_n q^2} > \frac{1}{q^2},$$

which contradicts our original assumption that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Therefore when $a = 1$, $b \geq 4$, and n is even, there are no type 1 secondary convergents that satisfy inequality (6) that are not convergents themselves.

Now assume that $\frac{p}{q}$ is a type 2 secondary convergent, so there exists an n such that

$$\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}}.$$

If n is even, then $a_n = a = 1$, so $p_n = p_{n-1} + p_{n-2}$, $q_n = q_{n-1} + q_{n-2}$. Therefore,

$$\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}} = \frac{p_{n-2}}{q_{n-2}}.$$

Since $q > 1$, we must have $n > 0$ and since n is even we must have $n > 1$, thus q_{n-2} exists. So for all even n, $\frac{p}{q}$ is a convergent of α, as desired. Thus we may now assume that n is odd, hence $\alpha_{n+1} = [\,\overline{a,b}\,] = \alpha$. In this case we have

$$\begin{aligned}\lambda'_n &= 1 + \frac{1}{\frac{q_n}{q_{n-1}} - 1} - \frac{1}{\alpha+1} \\ &= \frac{(q_n - q_{n-1})(\alpha+1) + q_{n-1}(\alpha+1) - q_n + q_{n-1}}{(q_n - q_{n-1})(\alpha+1)} \\ &= \frac{\alpha q_n + q_{n-1}}{(\alpha+1)(q_n - q_{n-1})}.\end{aligned}$$

We now claim that for $a = 1$, $b \geq 4$, and n odd, we have $\lambda'_n < 1$. Since n is odd, $q_n = bq_{n-1} + q_{n-2}$. We have that $\alpha = [\,\overline{1,b}\,]$ and $b \geq 4$, hence $1 < \alpha < 2$ and $b > \alpha + 2$. Therefore, $q_n = bq_{n-1} + q_{n-2} > bq_{n-1} > (\alpha+2)q_{n-1} = \alpha q_{n-1} + 2q_{n-1}$. Thus we have

$$q_n > \alpha q_{n-1} + 2q_{n-1}.$$

Adding $\alpha(q_n - q_{n-1}) - q_{n-1}$ to both sides yields

$$(\alpha+1)(q_n - q_{n-1}) > \alpha q_n + q_{n-1},$$

which implies

$$\lambda'_n = \frac{\alpha q_n + q_{n-1}}{(\alpha+1)(q_n - q_{n-1})} < 1$$

as claimed. Hence, in view of fact 3 and our above observation,

$$\left|\, \alpha - \frac{p}{q} \right| = \frac{1}{\lambda'_n q^2} > \frac{1}{q^2},$$

and once again this contradicts our original assumption. Thus when $a = 1$, $b \geq 4$, and n is odd, the only type 2 secondary convergents that satisfy inequality (6) are convergents. Therefore, for all $\alpha = [\,\overline{1,b}\,]$ with $b \geq 4$ we have that if $\frac{p}{q}$ satisfies inequality (6), then $\frac{p}{q}$ must be a convergent of α.

<u>Case 4</u>: $a \geq 3$, $b = 1$. This case follows very similarly to Case 3, but is included for completeness.

We will first assume that $\frac{p}{q}$ is a type 1 secondary convergent; thus there exists an n such that

$$\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

If n is even, then $a_{n+1} = b = 1$, so

$$\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}},$$

and $\frac{p}{q}$ is a convergent to α, as desired.

Thus we may assume n is odd, so $\alpha_{n+1} = [\ \overline{a, b}\ ] = \alpha$, and we have

$$\begin{aligned}
\lambda_n &= 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1} \\
&= 1 + \frac{1}{\alpha - 1} - \frac{1}{\frac{q_n}{q_{n-1}} + 1} \\
&= \frac{(\alpha - 1)(q_n + q_{n-1}) + (q_n + q_{n-1}) - (\alpha - 1)q_{n-1}}{(\alpha - 1)(q_n + q_{n-1})} \\
&= \frac{\alpha q_n + q_{n-1}}{(\alpha - 1)(q_n + q_{n-1})}.
\end{aligned}$$

Here we claim that for $a \geq 3$, $b = 1$, and n odd, we have $\lambda_n < 1$. First, consider $a \geq 4$. Then $\alpha > 4$ so $\alpha q_{n-1} > 4q_{n-1} > 3q_{n-1} + q_{n-2}$. Now, consider $a = 3$. Here n is odd so $q_{n-1} = aq_{n-2} + q_{n-3}$. So when $a = 3$, $q_{n-2} = \frac{1}{3}q_{n-1} - \frac{1}{3}q_{n-3}$. Now examine $3q_{n-1} + q_{n-2}$:

$$\begin{aligned}
3q_{n-1} + q_{n-2} &= 3q_{n-1} + \left(\frac{1}{3}q_{n-1} - \frac{1}{3}q_{n-3}\right) \\
&= \frac{10}{3}q_{n-1} - \frac{1}{3}q_{n-3} \\
&< \frac{10}{3}q_{n-1}.
\end{aligned}$$

When $a = 3$ and $b = 1$, then

$$\alpha = [\ \overline{3, 1}\ ] = \frac{3 + \sqrt{21}}{2} > \frac{21}{6} > \frac{10}{3}.$$

Thus, we have for all $a \geq 3$,

$$\alpha q_{n-1} > 3q_{n-1} + q_{n-2}.$$

Since n is odd and $b = 1$, $q_n = q_{n-1} + q_{n-2}$. So we have

$$\alpha q_{n-1} > 3q_{n-1} + q_{n-2} = 2q_{n-1} + q_n.$$

Now adding $(\alpha - 1)q_n - q_{n-1}$ to both sides yields

$$(\alpha - 1)(q_n + q_{n-1}) > \alpha q_n + q_{n-1},$$

and it follows that

$$\lambda_n = \frac{\alpha q_n + q_{n-1}}{(\alpha - 1)(q_n + q_{n-1})} < 1,$$

as claimed. Therefore, once again we have

$$\left| \alpha - \frac{p}{q} \right| = \frac{1}{\lambda_n q^2} > \frac{1}{q^2},$$

which is a contradiction. So when $b = 1$, $a \geq 3$, and n is odd, unless $\frac{p}{q}$ is a convergent, no type 1 secondary convergents satisfy inequality (6).

We now assume that $\frac{p}{q}$ is a type 2 secondary convergent, so there exists an n such that

$$\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}}.$$

We will again consider two subcases: when n is odd and when n is even. First, we suppose that n is odd. In this case we have $p_n = p_{n-1} + p_{n-2}$ and $q_n = q_{n-1} + q_{n-2}$, hence

$$\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}} = \frac{p_{n-2}}{q_{n-2}}.$$

For all $n > 1$, $\frac{p}{q}$ is a convergent, and since $q_1 = b = 1$, when $n = 1$ this is undefined (and so cannot exist).

Now suppose n is even, so $\alpha_{n+1} = [\ \overline{b, a}\ ] = b + \frac{1}{\alpha}$, and thus we have

$$\begin{aligned}
\lambda'_n &= 1 + \frac{1}{\gamma_n - 1} - \frac{1}{\alpha_{n+1} + 1} \\
&= 1 + \frac{1}{\frac{q_n}{q_{n-1}} - 1} - \frac{1}{b + \frac{1}{\alpha} + 1} \\
&= \frac{(q_n - q_{n-1})[(b+1)\alpha + 1] + q_{n-1}[(b+1)\alpha + 1] - \alpha(q_n - q_{n-1})}{(q_n - q_{n-1})[(b+1)\alpha + 1]} \\
&= \frac{b\alpha q_n + q_n + \alpha q_{n-1}}{(q_n - q_{n-1})[(b+1)\alpha + 1]}.
\end{aligned}$$

We now claim that for $a \geq 3$, $b = 1$, and n even, we have $\lambda'_n < 1$. Then $\alpha q_{n-2} > 2q_{n-2} > q_{n-2} + q_{n-3} = q_{n-1}$. Also, we have $a\alpha q_{n-1} \geq 3\alpha q_{n-1}$. Putting these inequalities together, we see

$$a\alpha q_{n-1} + \alpha q_{n-2} > 3\alpha q_{n-1} + q_{n-1} = (b+2)\alpha q_{n-1} + q_{n-1}.$$

Since n is even, $a\alpha q_{n-1} + \alpha q_{n-2} = \alpha q_n$, so we have

$$\alpha q_n > (b+2)\alpha q_{n-1} + q_{n-1}.$$

Adding $b\alpha q_n + q_n - \alpha q_{n-1} - b\alpha q_{n-1}$ to both sides of the equation, we get

$$(q_n - q_{n-1})[(b+1)\alpha + 1] > b\alpha q_n + q_n + \alpha q_{n-1}.$$

Therefore,

$$\lambda'_n = \frac{b\alpha q_n + q_n + \alpha q_{n-1}}{(q_n - q_{n-1})[(b+1)\alpha + 1]} < 1.$$

Thus once again we have

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{\lambda'_n q^2} > \frac{1}{q^2},$$

which is a contradiction. Thus when $b = 1$, $a \geq 3$, and n is even, there are no type 2 secondary convergents (not themselves convergents) that satisfy inequality (6). Therefore, for all $\alpha = [\ \overline{a, 1}\ ]$ with $a \geq 3$, we have that if $\frac{p}{q}$ satisfies inequality (6), then $\frac{p}{q}$ is a convergent. As we have reached this conclusion in each of the possible cases, our proof is complete.

□

We remark again that when $\alpha = [\ \overline{1,2}\ ], [\ \overline{1,3}\ ]$, or $[\ \overline{2,1}\ ]$, the theorem does not hold. When α takes on each of these values, then α has secondary convergents that are not themselves convergents, but that satisfy inequality (6). In fact, for $\alpha = [\ \overline{1,2}\ ]$ every secondary convergent satisfies inequality (6) and type 1 secondary convergents are only themselves convergents when n is odd and type 2 secondary convergents are only convergents when n is even. Likewise, when $\alpha = [\ \overline{1,3}\ ]$ every type 2 secondary convergent satisfies inequality (6), yet these are only convergents when n is even. Also, when $\alpha = [\ \overline{2,1}\ ]$, every secondary convergent satisfies equation (6), but these are only convergents when n is even for the type 1 and when n is odd for the type 2. Thus, for these particular values of α, not only does the theorem not hold, but for each of these three values of α, there are infinitely many rationals $\frac{p}{q}$ that violate the theorem. However, it should be noted that if $\frac{p}{q}$ is a convergent of α for any of these α's, then $\frac{p}{q}$ will satisfy inequality (6). More insight will be shed onto these cases in the further generalization to follow in the next chapter.

Excepting these three specific cases, we have shown that for all $\alpha = [\ \overline{a,b}\ ]$, if a reduced rational number $\frac{p}{q}$ satisfies inequality (6), then $\frac{p}{q}$ is a convergent of α. Moreover, every convergent of α satisfies inequality (6). Thus for irrational numbers of this form we have completely classified convergents (even more importantly, without needing to compute each and every convergent!), and hence have accomplished our goal of being able to determine precisely when a rational number is a best approximate.

IV. GENERALIZATION: THE REST OF THE WAY

If we can so classify the best approximates of certain irrational numbers as we have done above, is it possible to produce a similar result for all irrational numbers? Or if not for all (as in the above case our theorem did not hold for three specific cases), perhaps we may find a theorem that holds for a much larger class of irrational numbers. Thus our quest now involves finding a theorem similar to the one given above (classifying a rational, $\frac{p}{q}$, as a convergent of α if and only if $\left|\alpha - \frac{p}{q}\right|$ falls within a given interval) that holds for as many different irrational numbers as possible.

In creating such an interval containing all convergents of an irrational α, since the interval is concerned with the distance of the convergent from α, we turn again to fact 1. Since this fact is crucial to our argument, we recall

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})},$$

from which we deduced inequality (4), stating

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}.$$

Granted, in reducing fact 1 to inequality (4) we are letting a lot of information go. Nonetheless, if we were to work solely with fact 1 in creating interval, we would not be accomplishing our goal of simplicity. In order to create an interval for each convergent using fact 1 (without the bridge to inequality (4)) we would have to know the denominator of the convergent directly prior to the one we are working with (to use q_{n-1}). Since we are already able to find all the convergents recursively, this would render our theorem completely superfluous. If we take out the q_{n-1} but leave in the α_{n+1} we would have to deal with an irrational number in our interval, moreover an irrational number that changes for each value of n. The beauty and simplicity of the interval we have created for the specific case above lies in the fact that the interval can be easily found and stated once one has found the continued fraction expansion. After the interval has been stated, one may test any rational they desire to determine if it is a best approximate or not, and they may do so without even knowing what a continued fraction expansion is. If we leave α_{n+1} in the interval, you would need the continued fraction expansion each time you wanted to use the theorem (and although this may be trivial for $\alpha = [\ \overline{a, b}\ ]$, this is not desirable for more complicated irrationals). So although one may be able to achieve stronger results working directly with fact 1, we believe the complexity that would arise counteracts the whole purpose we began with. Thus in striving for simplicity, for our work here we have reduced fact 1 to inequality (4).

In the above specific case we were extremely lucky (or chose our initial example wisely, depending on how you look at it) because we were able to combine inequality (4) with the fact that odd convergents overestimate and even convergents underestimate, and thus create our interval. In the more general case however, this is not so easy. If you consider $\alpha = [a_0, a_1, a_2, a_3, ...]$, then for the 2^{nd} convergent inequality (4) tells us

$$\left|\alpha - \frac{p_2}{q_2}\right| < \frac{1}{a_3 q_n^2},$$

while the 4^{th} convergent inequality (4) gives us

$$\left|\alpha - \frac{p_4}{q_4}\right| < \frac{1}{a_5 q_n^2}.$$

Unfortunately, we know nothing about either a_3 or a_5, or their relationship. All we can determine is that both the 2^{nd} and 4^{th} convergents satisfy

$$0 < \alpha - \frac{p_n}{q_n} < \frac{1}{\min\{a_3, a_5\} q_n^2}.$$

We are certainly able to determine more for each individual convergent, but as ideally we would like an interval that works for all convergents, we do not have the liberty of referring to specific n's. We only know that odd convergents overestimate α and even convergents underestimate α, thus we define two values, the minimum of all the even coefficients and the minimum of all the odd coefficients. Let

$$m_e = \min\{a_{2n}\} \text{ and } m_o = \min\{a_{2n+1}\}.$$

Then, as in the above example with the 2^{nd} and 4^{th} convergents, the most information we are able to extract from inequality (4) is that all even convergents satisfy

$$0 < \alpha - \frac{p_n}{q_n} < \frac{1}{m_o q_n^2},$$

and all odd convergents satisfy

$$\frac{-1}{m_e q_n^2} < \alpha - \frac{p_n}{q_n} < 0.$$

This leads us to a conjecture for an interval classifying convergents of irrational numbers:

$$\frac{-1}{m_e q^2} < \alpha - \frac{p}{q} < \frac{1}{m_o q^2}, \tag{8}$$

where a rational is a convergent of α if and only if it satisfies inequality (8). This interval seems to make sense, seeing as how theorem 1 and the results of Mobius and Komatsu all result as specific cases. More importantly, by the above arguments, there is not yet enough knowledge about convergents to create a smaller interval (not dependent on n) that contains all convergents of α. Thus if inequality (8) fails to classify convergents, our hopes of creating an if and only if theorem for convergents may simply remain as wishes. We should point out that in the case when $\alpha = [\ \overline{a, b}\ ]$, this interval did not classify convergents for three specific values, hence we already know that a theorem based on this interval will not hold for all irrational numbers. We hope to find all irrationals for which such a theorem does hold.

Proving that all convergents of α satisfy inequality (8) is not hard at all, the proof in fact lies in the very reasoning we employed in creating the interval. However, as in the proof of theorem 1, we will have a much harder time showing that every rational satisfying inequality (8) is in fact a convergent. Due to Legendre's result, if m_e and m_o are at least 2 then we know every rational satisfying inequality (8) is a convergent. Whereas the proof follows rather trivially, it is encouraging to note that our conjecture holds for *every* irrational number with no ones in its continued fraction expansion! This in itself classifies convergents, and accomplishes our goal for a huge class of irrational numbers! We should note that in this case we would have already been able to determine (using Legendre) that a rational satisfying inequality (8) is a convergent, but this inequality goes beyond that. It places much narrower bounds (how much narrower depending precisely on how much greater than 2 m_e and m_o are) on the accuracy of best approximates for α, but more importantly, we know that *all* convergents satisfy inequality (8), so we may state such a result as if and only if.

Unfortunately, 1 is the most common coefficient in continued fraction expansions, and it would be nice if the conjecture holds as well for irrationals containing ones in their continued fraction expansion. Luckily, if m_e and/or m_o is 1, we may still employ our knowledge that every rational satisfying inequality (8) must be either a convergent or a secondary convergent. In order for such a theorem (a rational is a convergent if and only if it satisfies inequality (8)) to hold for an irrational α we again wish to show that every secondary convergent of α either is itself a convergent or does not satisfy inequality (8) (in which case every rational satisfying inequality (8) must be a convergent).

Since we only need to turn to an examination of secondary convergents when m_e or m_o is 1, we are really interested is when a secondary convergent satisfies

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}, \tag{9}$$

and so satisfies inequality (8). Turning once again to fact 3, this is true precisely when λ_n and λ'_n are less than 1. We analyze exactly when this is true with a series of three lemmas. The first concerns type 1 secondary convergents, and tells us when $\lambda_n < 1$.

Lemma 1. *Let* $\alpha_n = [a_n, a_{n+1}, a_{n+2}, ...]$, $\gamma_n = [a_n, a_{n-1}, a_{n-2}, ..., a_1]$, *and*

$$\lambda_n = 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1}.$$

If $a_{n+1} \geq a_n + 2$, *and whenever* $n > 1$ *and* $a_{n+1} = a_n + 2$ *it follows that* $a_{n-1} \geq a_{n+2} + 1$, *then* $\lambda_n < 1$.

Proof. We claim that $\alpha_{n+1} > \gamma_n + 2$. To verify this claim, first suppose $a_{n+1} > a_n + 2$, so $a_{n+1} \geq a_n + 3$. We now suppose that $n > 1$, (so γ_{n-1} exists) and produce the following string of inequalities:

$$\begin{aligned} \alpha_{n+1} &> a_{n+1} \\ &\geq a_n + 3 \\ &= \gamma_n - \frac{1}{\gamma_{n-1}} + 3 \end{aligned}$$

$$> \gamma_n + 2,$$

which establishes our claim in the case when $a_{n+1} > a_n + 2$ and $n > 1$. We next suppose that $n = 1$, so we have $a_2 \geq a_1 + 2$. It then follows that

$$\begin{aligned} \alpha_{n+1} &= \alpha_2 \\ &> a_2 \\ &\geq a_1 + 2 \\ &= \gamma_1 + 2 \\ &= \gamma_n + 2. \end{aligned}$$

Lastly, we suppose $n > 1$, $a_{n+1} = a_n + 2$, and $a_{n-1} \geq a_{n+2} + 1$, and observe that

$$\begin{aligned} \alpha_{n+1} &= a_{n+1} + \frac{1}{\alpha_{n+2}} \\ &= a_n + 2 + \frac{1}{\alpha_{n+2}}. \end{aligned}$$

Since $n > 1$, $a_n = \gamma_n - \frac{1}{\gamma_{n-1}}$, and hence we see

$$\alpha_{n+1} = a_n + 2 + \frac{1}{\alpha_{n+2}} = \gamma_n - \frac{1}{\gamma_{n-1}} + \frac{1}{\alpha_{n+2}} + 2.$$

Next, we note that

$$\begin{aligned} \gamma_{n-1} &\geq a_{n-1} \\ &\geq a_{n+2} + 1 \\ &> a_{n+2} + \frac{1}{\alpha_{n+3}} \\ &= \alpha_{n+2}, \end{aligned}$$

and combining the above two results, we have

$$\alpha_{n+1} = \gamma_n - \frac{1}{\gamma_{n-1}} + \frac{1}{\alpha_{n+2}} + 2 > \gamma_n + 2,$$

which establishes our claim that $\alpha_{n+1} > \gamma_n + 2$. It follows directly that $\alpha_{n+1} - 1 > \gamma_n + 1$, yielding

$$\frac{1}{\alpha_{n+1} - 1} < \frac{1}{\gamma_n + 1},$$

which leads us to conclude

$$\lambda_n = 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1} < 1,$$

and completes our proof. □

The astute reader may notice that the above lemma is not if and only if, meaning that there may be situations in which the conditions are not upheld yet λ_n is still less than one. This is true, but only for certain very specific cases when $n > 1$, $a_{n+1} = a_n + 2$, and $a_{n-1} = a_{n+2}$. If all of these conditions hold then depending on the relationship between γ_{n-2} and α_{n+3} the value of λ_n may be either less than or greater than 1. However, this is a rare case and its consideration would significantly complicate the lemma. So although with the above lemma we cannot write if and only if, we do note that in cases excepting the one just mentioned, if the conditions of lemma 1 are not upheld than $\lambda_n > 1$. We verify this by assuming $\lambda_n < 1$, hence

$$1 > 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1}.$$

From this we observe

$$\alpha_{n+1} > \gamma_n + 2,$$

and so

$$a_{n+1} > a_n + \frac{1}{\gamma_{n-1}} + 2 - \frac{1}{\alpha_{n+2}}.$$

Thus clearly we must have $a_{n+1} > a_n + 1$, therefore $a_{n+1} \geq a_n + 2$. Moreover, if $a_{n+1} = a_n + 2$, we require

$$\alpha_{n+2} > \gamma_{n-1},$$

yielding

$$a_{n+2} > a_{n-1} + \frac{1}{\gamma_{n-2}} - \frac{1}{\alpha_{n+3}},$$

so we must have $a_{n+2} > a_{n-1} - 1$ and hence $a_{n+2} \geq a_{n-1}$. Thus indeed if $a_{n+1} < a_n + 2$ or if

$a_{n+1} = a_n + 2$ and $a_{n+2} < a_{n-1}$ we observe $\lambda_n > 1$. Therefore, if the conditions of lemma 1 are not satisfied, excepting the specific case given above, we can be sure that $\lambda_n > 1$.

We will consider a similar lemma concerning type 2 secondary convergents in a moment, but first we pause to examine what lemma actually means in the context we care about. Recall that we wish to show for an irrational $\alpha = [a_0, a_1, a_2, ...]$ that every secondary convergent is either a convergent or does not satisfy inequality (8). Also recall that we are particularly concerned with when m_e or m_o is 1, thus a type 1 secondary convergent satisfies inequality (8) when inequality (9) is upheld, which occurs precisely when $\lambda_n > 1$. Lemma 1 tells us that if there exists an n for the irrational we are considering such that the conditions of lemma 1 are met, then this type 1 secondary convergent will not satisfy inequality (8), and hence we do not need to worry about it contradicting our conjecture. On the other hand, if there exists an n that does not satisfy the conditions of lemma 2 we have to be wary, we must check these secondary convergents... if they themselves are convergents then we are fine, however if they are not convergents then this signals that our conjecture probably does not hold for this irrational.

We next present the analogous lemma, now concerning type 2 secondary convergents. The lemma and proof are very similar to lemma 1, yet are presented for completeness.

Lemma 2. *Let* $\alpha_n = [a_n, a_{n+1}, a_{n+2}, ...]$, $\gamma_n = [a_n, a_{n-1}, a_{n-2}, ..., a_1]$, *and*

$$\lambda'_n = 1 + \frac{1}{\gamma_n - 1} - \frac{1}{\alpha_{n+1} + 1}.$$

If $a_n \geq a_{n+1} + 3$ *and whenever* $n > 1$ *and* $a_n = a_{n+1} + 2$*, it follows that* $a_{n+2} \geq a_{n-1} + 1$*, then* $\lambda'_n < 1$.

Proof. Here we claim $\gamma_n > \alpha_{n+1} + 2$. To verify this claim, we first suppose $a_n \geq a_{n+1} + 3$, which yields

$$\begin{aligned} \gamma_n &\geq a_n \\ &\geq a_{n+1} + 3 \end{aligned}$$

$$
\begin{aligned}
&= \alpha_{n+1} - \frac{1}{\alpha_{n+2}} + 3 \\
&> \alpha_{n+1} + 2,
\end{aligned}
$$

which establishes our claim in the case when $a_n \geq a_{n+1} + 3$. We next suppose that $n > 1$, $a_n = a_{n+1} + 2$ and $a_{n+2} \geq a_{n-1} + 1$, and observe

$$
\begin{aligned}
\alpha_{n+2} &= a_{n+2} + \frac{1}{\alpha_{n+3}} \\
&\geq a_{n-1} + 1 + \frac{1}{\alpha_{n+3}} \\
&> \gamma_{n-1} - \frac{1}{\gamma_{n-2}} + 1 \\
&> \gamma_{n-1}.
\end{aligned}
$$

Thus we may deduce

$$
\begin{aligned}
\gamma_n &= a_n + \frac{1}{\gamma_{n-1}} \\
&= a_{n+1} + 2 + \frac{1}{\gamma_{n-1}} \\
&> a_{n+1} + 2 + \frac{1}{\alpha_{n+2}} \\
&= \alpha_{n+1} + 2,
\end{aligned}
$$

establishing our claim.

Thus we have $\gamma_n > \alpha_{n+1} + 2$, yielding $\gamma_n - 1 > \alpha_{n+1} + 1$. This implies

$$
\frac{1}{\gamma_n - 1} < \frac{1}{\alpha_{n+1} + 1},
$$

and hence we conclude

$$
\lambda'_n = 1 + \frac{1}{\gamma_n - 1} - \frac{1}{\alpha_{n+1} + 1} < 1,
$$

concluding the proof. □

As in the above case, except in the specific case when $n > 1$, $a_n = a_{n+1} + 2$ and $a_{n-1} = a_{n+2}$, if the conditions of lemma 2 are not upheld then $\lambda'_n > 1$.

Lastly, we require one more lemma before we state our main result (which hopefully you are beginning to anticipate!). The above lemmas are concerned with one type of secondary convergent at a time. However, if you know in advance that neither secondary convergent is a convergent, then you have the added criteria that both $lambda_n < 1$ and $\lambda'_n < 1$. It is a certain instance of this case that we examine in more detail here.

Lemma 3. *Suppose that α is a real irrational number with $\alpha = [a_0, a_1, a_2, ...]$. Let $\alpha_n = [a_n, a_{n+1}, a_{n+2}, ...]$, $\gamma_n = [a_n, a_{n-1}, a_{n-2}, ..., a_1]$,*

$$\lambda_n = 1 + \frac{1}{\alpha_{n+1} - 1} - \frac{1}{\gamma_n + 1},$$

and

$$\lambda'_n = 1 + \frac{1}{\gamma_n - 1} - \frac{1}{\alpha_{n+1} + 1}.$$

If $a_n = 3$, $a_{n+1} = a_{n-1} = 1$, and there exists an integer $1 \leq k < \frac{n-1}{2}$ such that for all integers $1 \leq i < 2k$, $a_{n\pm i} = 1$ and one of the following three conditions holds:

(i) $a_{n\pm 2k} > 2$;

(ii) $a_{n+2k} > 2, a_{n-2k} = a_{n-2k-1} = 1$;

(iii) $a_{n-2k} > 2, a_{n+2k} = a_{n+2k+1} = 1$,

then $\lambda_{n-1} < 1$ and $\lambda'_n < 1$.

Before embarking on the proof, this lemma merits a little explanation. First of all, it is only concerned with the special case when a coefficient in the continued fraction expansion is 3 and the coefficients on either side are both 1. Thus, it should be immediately obvious that this lemma is only relevant to a small class of irrationals. The rest of the conditions are best explained with a few examples. Consider $\alpha = [5, 1, 1, 1, 3, 1, 1, 1, 4, ...]$, $\beta = [5, 1, 3, 1, 1, 1, ...]$, and $\delta = [2, 1, 3, 1, ...]$. Clearly in our examination of α, in applying lemma 3, $n = 4$ since $a_4 = 3$. We have that $a_3 = a_5 = 1$, so that condition is upheld. Letting $k = 2$ we observe $a_{4\pm 2} = a_{4\pm 3} = 1$, as required. Lastly, condition

(1) is met since $a_{4\pm 4} > 2$. Hence for the irrational number α, $\lambda_4 < 1$ and $\lambda_4' < 1$ so neither of the secondary convergents using $n = 4$ would satisfy inequality (9). Applying lemma 3 to β we have $n = 2$ and $k = 1$, and observe that the the conditions are met, with β falling under condition (iii). For δ however, there exists no k such that the conditions may be met, thus we may not apply lemma 3. Although lemma 3 is technical and applies only when the continued fraction expansion of an irrational number satisfies very specific conditions, it will be necessary in the proof of our main result, therefore we commence with the proof.

Proof. Let $a_n = 3$, $a_{n+1} = a_{n-1} = 1$, and assume there exists an integer $k \geq 1$ that satisfies the above conditions. Thus we have for all i, $1 \leq i < 2k$ that $a_{n\pm i} = 1$ and that one of cases (i), (ii), or (iii) holds. We first claim that $\alpha_{n+2k}\gamma_{n-2k} > \alpha_{n+2k} + 1$. To establish this claim we will begin by examining cases (i) and (iii), in which we have $a_{n-2k} > 2$. Note that $\gamma_{n-2k} \geq a_{n-2k} > 2$, and $\alpha_{n+2k} > a_{n+2k} \geq 1$. Thus we may deduce

$$\begin{aligned}
\alpha_{n+2k}\gamma_{n-2k} &> 2\alpha_{n+2k} \\
&= \alpha_{n+2k} + \alpha_{n+2k} \\
&> \alpha_{n+2k} + 1,
\end{aligned}$$

establishing our small claim if we are in cases (i) or (iii). Now let's suppose we are in case (ii), so we have $a_{n+2k} > 2$ and $a_{n-2k} = a_{n-2k-1} = 1$. We have that $k < \frac{n-1}{2}$, so we must have $n - 2k > 1$. We first consider $n - 2k = 2$, hence

$$\gamma_{n-2k} = \gamma_2 = a_2 + \frac{1}{a_1} = 2.$$

Then again we see $\alpha_{n-2k}\gamma_{n-2k} = 2\alpha_{n+2k} > \alpha_{n+2k} + 1$. We next consider $n - 2k > 2$, so γ_{n-2k-2} exists. We thus observe

$$\gamma_{n-2k} = 1 + \frac{1}{\gamma_{n-2k-1}} = 1 + \frac{1}{1 + \frac{1}{\gamma_{n-2k-2}}} \geq 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2},$$

yielding

$$\begin{aligned} \alpha_{n+2k}\gamma_{n-2k} &\geq \frac{3}{2}\alpha_{n+2k} \\ &= \alpha_{n+2k} + \frac{\alpha_{n+2k}}{2}. \end{aligned}$$

In view of the fact that $a_{n+2k} > 2$, we must have $\alpha_{n+2k} > 2$. Thus we conclude

$$\begin{aligned} \alpha_{n+2k}\gamma_{n-2k} &\geq \alpha_{n+2k} + \frac{\alpha_{n+2k}}{2} \\ &> \alpha_{n+2k} + 1, \end{aligned}$$

establishing our small claim in all cases.

We will now use induction to show that that for all integers $1 \leq r \leq k$, $\alpha_{n+2r} + 1 < \alpha_{n+2r}\gamma_{n-2r}$ implies $\alpha_n > \gamma_{n-1} + 2$. We first note that since k is an integer and $1 \leq k < \frac{n-1}{2}$, we must have $n \geq 4$, hence γ_{n-1} and γ_{n-2} both exist.

Base Case: $r = 1$, so assume $\alpha_{n+2} + 1 < \alpha_{n+2}\gamma_{n-2}$. Dividing both sides by α_{n+2} we see

$$1 + \frac{1}{\alpha_{n+2}} < \gamma_{n-2}.$$

Combining this with the fact that $a_{n+1} = a_{n-1} = 1$, we observe

$$\alpha_{n+1} = a_{n+1} + \frac{1}{\alpha_{n+2}} = 1 + \frac{1}{\alpha_{n+2}} < \gamma_{n-2},$$

from which it follows immediately that

$$\frac{1}{\alpha_{n+1}} + 1 > \frac{1}{\gamma_{n-2}} + 1.$$

Since $a_{n-1} = 1$, note that $1 + \frac{1}{\gamma_{n-2}} = \gamma_{n-1}$. Hence we see

$$a_n + \frac{1}{\alpha_{n+1}} - (a_n - 1) > \gamma_{n-1},$$

which can be rewritten as follows:

$$\alpha_n > \gamma_{n-1} + (a_n - 1),$$

and recalling that $a_n = 3$, we get our desired result:

$$\alpha_n > \gamma_{n-1} + 2,$$

verifying the base case.

Now suppose for all $r < k$ that $\alpha_{n+2r} + 1 < \alpha_{n+2r}\gamma_{n-2r}$ implies $\alpha_n > \gamma_{n-1} + 2$, and we assume that $\alpha_{n+2k} + 1 < \alpha_{n+2k}\gamma_{n-2k}$. We wish to show that this implies $\alpha_n > \gamma_{n-1} + 2$. So we have

$$\alpha_{n+2k} + 1 < \alpha_{n+2k}\gamma_{n-2k}.$$

Adding $3\alpha_{n+2k}\gamma_{n-2k} + 2\gamma_{n-2k} + 2\alpha_{n+2k} + 1$ to both sides, we observe

$$3\alpha_{n+2k} + 3\alpha_{n+2k}\gamma_{n-2k} + 2\gamma_{n-2k} + 2 < 4\alpha_{n+2k}\gamma_{n-2k} + 2\gamma_{n-2k} + 2\alpha_{n+2k} + 1,$$

which can be factored as follows:

$$(3\alpha_{n+2k} + 2)(\gamma_{n-2k} + 1) < (2\alpha_{n+2k} + 1)(2\gamma_{n-2k} + 1).$$

This yields

$$\frac{(3\alpha_{n+2k} + 2)}{(2\alpha_{n+2k} + 1)} < \frac{(2\gamma_{n-2k} + 1)}{(\gamma_{n-2k} + 1)},$$

which equals

$$1 + \frac{(\alpha_{n+2k} + 1)}{(2\alpha_{n+2k} + 1)} < 1 + \frac{(\gamma_{n-2k})}{(\gamma_{n-2k} + 1)}.$$

Thus we observe

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\alpha_{n+2k}}}} < 1 + \frac{1}{1 + \frac{1}{\gamma_{n-2k}}}.$$

Recall that for all $1 \leq i < 2k$ we have $a_{n \pm i} = 1$. Thus we see that $a_{n+(2k-1)} = a_{n+(2k-2)} =$

$a_{n-(2k-1)} = a_{n-(2k-2)} = 1$. We therefore may deduce

$$1 + \frac{1}{a_{n+(2k-2)} + \frac{1}{a_{n+(2k-1)} + \frac{1}{\alpha_{n+2k}}}} < a_{n-(2k-2)} + \frac{1}{a_{n-(2k-1)} + \frac{1}{\gamma_{n-2k}}},$$

from which we see

$$1 + \frac{1}{\alpha_{n+(2k-2)}} < \gamma_{n-(2k-2)}.$$

Multiplying through by $\alpha_{n+(2k-2)}$, we conclude

$$\alpha_{n+(2k-2)} + 1 < \alpha_{n+(2k-2)}\gamma_{n-(2k-2)}.$$

Thus for $r = k-1 < k$, we have $\alpha_{n+2r} + 1 < \alpha_{n+2r}\gamma_{n-2r}$, which by our induction hypothesis yields $\alpha_n > \gamma_{n-1} + 2$, as desired. Therefore, by induction, $\alpha_{n+2k} + 1 < \alpha_{n+2k}\gamma_{n-2k}$ implies $\alpha_n > \gamma_{n-1} + 2$.

Since we have already verified our claim that $\alpha_{n+2k} + 1 < \alpha_{n+2k}\gamma_{n-2k}$, we now see $\alpha_n > \gamma_{n-1} + 2$, and so $\alpha_n - 1 > \gamma_{n-1} + 1$. This establishes our desired result:

$$\lambda_{n-1} = 1 + \frac{1}{\alpha_n - 1} - \frac{1}{\gamma_{n-1} + 1} < 1.$$

The proof that $\lambda'_n < 1$ follows identically to the proof that $\lambda_{n-1} < 1$, simply reverse α and γ, and reverse $n + j$ and $n - j$ wherever they appear in the proof. Also, in proving $\lambda'_n < 1$, we first consider cases (i) and (ii) and then case (iii), and in case (iii) we do not need to consider the special case when $n - 2k = 2$, but since α_0 exists we may show right away that for $n - 2k \geq 2$ that $\alpha_{n+2k} > \frac{3}{2}$.

□

Utilizing the above three lemmas, we are ready to state our main result. The following theorem takes the same form as our original conjecture, but with conditions specifying for which irrationals the theorem holds.

Theorem 2. *Let* $\alpha = [a_0, a_1, a_2, ...]$ *be a real, irrational number, and for any integer* $N \geq 0$ *let*

$$m_e = \min_{2n \geq N}\{a_{2n}\} \text{ and } m_o = \min_{2n+1 \geq N}\{a_{2n+1}\}.$$

Suppose there exists an $N \geq 0$ *such that one of the following four cases holds.*

(i) $m_e > 1$ *and* $m_o > 1$.

(ii) If $m_e = 1$ *and* $m_o > 1$, *then*[1]

(a) $a_n \geq a_{n+1} + 2$ *for all odd* $n \geq N$

(b) $a_n \geq a_{n-1} + 2$ *for all odd* $n > N$.

(iii) If $m_e > 1$ *and* $m_o = 1$, *then the conclusions of (ii) hold, replacing "odd" with "even".*

(iv) If $m_e = m_o = 1$, *then for all* $n \geq N$

(a) $a_n \neq 2$;

(b) If $a_n \neq 1$, *then* $a_{n-1} = a_{n+1} = 1$;

(c) If $a_n = 3$, *then there exists an integer* $1 \leq k < \frac{n-1}{2}$ *such that for all* $1 \leq i < 2k$, *we have* $a_{n \pm i} = 1$ *and either* $a_{n \pm 2k} \neq 1$, *or* $a_{n+2k} \neq 1$ *and* $a_{n-2k} = a_{n-2k-1} = 1$, *or* $a_{n-2k} \neq 1$ *and* $a_{n+2k} = a_{n+2k+1} = 1$.

Then, a rational number $\frac{p}{q}$ *expressed in lowest terms with* $q \geq \max\{2, q_N\}$ *(where* q_N *is the denominator of the* N^{th} *convergent of* α*), is a convergent (and hence, best approximate) of* α *if and only if*

$$\frac{-1}{m_e q^2} < \alpha - \frac{p}{q} < \frac{1}{m_o q^2}. \tag{10}$$

This theorem may at first appear daunting, as the conditions on α are rather complex. However, notice that one one has verified that an irrational number satisfies the above conditions, the theorem is straightforward and easy to use. It may be complicated to understand the conditions and check for them, yet if the conditions hold the result is a complete classification of all

[1] For a few very small classes of α's, the following additional conditions must be met or else N must be increased: If N is even and $a_{N+1} = a_N + 2$ then we require $a_{N-1} \leq a_N + 1$ and if N is odd and $a_N = a_{N+1} + 2$ then we require $a_N \leq a_{N-1} + 1$. Moreover, in case (ii) if $N \leq 1$, we cannot have $a_1 = a_2 + 2$.

best approximates for that irrational number... exactly what we set out to find! For example, although we will consider the following case more thoroughly after the proof of the above theorem, we demonstrate the usefulness of this theorem with a corollary below:

Corollary. *If $\frac{p}{q}$ is a rational number expressed in lowest terms with $q > 2$, then $\frac{p}{q}$ is a convergent to e if and only if*

$$\left| e - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Before proving this corollary however, or even yet proving our main result, we will spend a little time in gaining a better understanding of the conditions.

For which irrationals does this theorem apply? In order to check the conditions, one must know the entire continued fraction expansion. Unfortunately, for many numbers, such as π, this is not possible. The coefficients of the continued fraction expansion of π (also know as the *partial quotients* of π) go on infinitely with no apparent pattern. Since we require that our conditions hold for ALL a_n with $n \geq N$, we must exclude numbers such as π where checking each partial quotient is impossible. However, there are a large class of numbers for which we *can* easily find all of their partial quotients, namely those irrational numbers with periodic continued fraction expansions. It is not hard to show (see Theorem 7.6 in [1]) that an irrational number has a periodic continued fraction expansion if and only if that number is quadratic. Thus for any quadratic irrational, one can find the period of its continued fraction expansion, and thus if the period satisfies our above conditions, we know the entire continued fraction expansion will. Finding this continued fraction expansion may be tedious by hand, but with the help of computer software packages such as Pari GP, one has merely to type in the irrational in consideration and the continued fraction expansion will be produced. Thus for quadratic irrationals, checking the conditions is not only feasible, but relatively painless. Although theorem 2 applies mainly to quadratics (due to their periodic structure), other irrational numbers display a pattern to their partial quotients. For example, the number $e = [2,1,2,1,1,4,1,1,6,1,1,8,...] = [2,1,2,\overline{1,1,2k}]$, where $k : 2 \rightarrow \infty$. Similarly, $e^{\frac{1}{a}} = [0,\overline{a(2k-1)-1,1,1}]$ for $a \geq 2$, and $tan1 = [1,\overline{(2k-1),1}]$, both with $k : 1 \rightarrow \infty$. These three examples do in fact meet our conditions, and thus our theorem classifies their best approximates. Clearly however, the theorem does not apply to every irrational whose partial quotients are either periodic or exhibit a pattern; we need to make sense of the conditions enough to apply them.

Case (i) is fairly straightforward, it simply says that whenever none of the partial quotients beyond a certain point are 1, then the theorem may be applied. Consider the irrational number $\alpha_1 = [2,3,\overline{2,5,6,8}]$. Since here $m_e = 2 > 1$ and $m_o = 3 > 1$, case (i) holds and we may conclude that $\frac{p}{q}$ is a best approximate of α_1 if and only if

$$\frac{-1}{3q^2} < \alpha_1 - \frac{p}{q} < \frac{1}{2q^2}.$$

Cases (ii) and (iii) are concerned with irrationals where after a certain point, either none of the even partial quotients are 1, or none of the odd partial quotients are 1. For example, let $\alpha_2 = [1,1,3,5,\overline{1,4,2,8,4,6}]$. Using $N = 2$, $m_e = 1$ and $m_o = 3$. We check and observe that every odd partial quotient is at least 2 above its adjacent partial quotients on either side, and hence case (ii) is upheld. The requirement of each partial quotient being at least 2 above its adjacent partial quotients stems from the criteria of lemmas 1 and 2. If this criteria is not met, there will exist secondary convergents that satisfy inequality (8) (and which are not themselves convergents). If $\alpha_3 = [1,1,3,5,\overline{1,4,2,5,4,6}]$, we have identical m_e and m_o, and so are again in case (ii). However this time, since $a_7 = 5$ and $a_8 = 4$, there exists an odd n with $a_n < a_{n+1} + 2$ and hence α_3 does not meet our criteria. Notice that for α_2 to work, we required $N = 2$ (at least). If we had chosen $N = 1$ then m_o would have been 1, and we would have been in case (iv) (with much stricter conditions, which α_2 would not satisfy. The integer N is chosen as the point (if such a point exists) from which there onward the conditions are met.

Case (iv) is a little trickier to get a grasp of. It has the strictest conditions, and unfortunately, due to the fact that 1 is the most common partial quotient, is probably the most likely case. Case (iv) deals with all the irrationals that don't fall in to one of the above three cases, namely any irrational for which there exists no point where beyond that point either all of the even or all of the odd partial quotients are greater than 1. If this is the case, then for every partial quotient not equal to 1, its adjacent partial quotients must both be 1. In some sense, if you are going to have 1's, you need a lot of them. This will be more obvious in the proof, but it results from the fact that secondary convergents are likely to satisfy inequality (10) if $m_e = m_o = 1$, and are only themselves convergents when $a_{n+1} = 1$ or $a_n = 1$. In addition to the "many ones" criteria, no partial quotient may be 2. If $a_i = 2$, the secondary convergent formed with a_i will satisfy inequality (8) but not be a convergent. Lastly, the third condition in case (iv) applies if a partial quotient is 3, and is a direct result of lemma 3. To demonstrate case (iv), we examine $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$. Letting $N = 3$, we observe that all integers greater than 1 have 1's adjacent to them and none of the partial quotients are 2 or 3, hence e satisfies our conditions (resulting in the above corollary). The number $[\overline{4, 1, 2, 1}]$ does not however because it contains a 2, and neither does $[\overline{4, 5, 1}]$ because it contains two adjacent integers both greater than 1.

The footnote specific to cases (ii) and (iii) involves some extra specifications. In the very specific cases described in the footnote, N may need to be increased for α to meet the conditions. This follows directly from the conditions of lemmas 1 and 2. The criteria added in the footnote will not prevent any α from satisfying the conditions of the theorem, it just may require N to be increased by 1 or 2. However the footnote will very rarely come into play, and requires inclusion simply for completeness.

Now that we have a little more understanding of the theorem, we commence with the proof. Several components of the proof are virtually identical to the proof of theorem 1, in which case we just refer the reader to that proof to avoid repetition.

Proof. Let $\frac{p_n}{q_n}$ be a convergent of α with $q_n \geq 2, q_N$ (so $n \geq N$). If n is even, $a_{n+1} \geq m_o$, thus by inequality (4) we see

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{m_o q_n^2}.$$

Similarly, if n is odd, $a_{n+1} \geq m_e$, hence

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{m_e q_n^2}.$$

From here this direction of the proof follows identically to the forward direction of the proof of theorem 1, therefore we will not repeat it but jump straight to proving the other direction.

Let $\frac{p}{q}$ be a rational number expressed in lowest terms with $q \geq \max\{2, q_N\}$ that satisfies inequality (10), and we wish to show that $\frac{p}{q}$ is a convergent.

Case 1: $m_e > 1$ and $m_o > 1$.
See Case 1 of the proof of theorem 1.

In each of the remaining three cases, either $m_e = 1$ or $m_o = 1$. In either case, inequality (10) yields

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}, \tag{11}$$

and hence $\frac{p}{q}$ is either a convergent or a secondary convergent of α. If $\frac{p}{q}$ is a convergent of α then we have established our desired result. Therefore we may now assume that $\frac{p}{q}$ is a secondary convergent of α.

We first remark that if $q = q_n + q_{n-1}$ then by our assumption that $q \geq max\{2, q_N\}$, we observe $q_n + q_{n-1} \geq q_N \geq q_{N-1} + q_{N-2}$. Thus we must have $n \geq N - 1$. Note that if $n = N - 1$, then $q_{N-1} + q_{N-2} \geq q_N$, which requires $q_{N-1} + q_{N-2} = q_N$ and hence requires $a_N = 1$ (recall $q_n = a_n q_{n-1} + q_{n-2}$). In this case (when $q = q_{N-1} + q_{N-2}$) we observe

$$\frac{p}{q} = \frac{p_{N-1} + p_{N-2}}{q_{N-1} + q_{N-2}} = \frac{p_N}{q_N},$$

which is a convergent of α. Thus if $\frac{p}{q}$ is a type 1 secondary convergent we may assume $n > N - 1$, which implies $n \geq N$.

If $\frac{p}{q}$ is a type 2 secondary convergent, then we have $q = q_n - q_{n-1}$ and so require $q_n - q_{n-1} \geq q_N$. Thus we have $q_n \geq q_N + q_{n-1} > q_N$, and hence must have $n > N$. Also note that since $q \geq 2$, we require $n > 0$. Thus when $\frac{p}{q}$ takes the form of either type of secondary convergent, we may assume $n \geq \max\{1, N\}$.

One can verify (see [8]) that for type 1 secondary convergents

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} = [a_0, a_1, a_2, ..., a_{n-1}, a_n + 1]$$

and for type 2 secondary convergents

$$\frac{p_n - p_{n-1}}{q_n - q_{n-1}} = [a_0, a_1, a_2, ..., a_{n-1}, a_n - 1].$$

Since $\alpha = [a_0, a_1, a_2, ..., a_{n-1}, \alpha_n]$ and since $a_n - 1 < \alpha_n < a_n + 1$, we can deduce the following depending on the parity of n:
If n is even:

$$\frac{p_n - p_{n-1}}{q_n - q_{n-1}} < \alpha < \frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \tag{12}$$

and if n is odd:

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} < \alpha < \frac{p_n - p_{n-1}}{q_n - q_{n-1}}. \tag{13}$$

With these preliminary observations established, we proceed to the remaining three cases of our proof.

<u>Case 2</u>: $m_e = 1$ and $m_o > 1$.
We observe by inequality (10) that

$$\frac{-1}{q^2} < \alpha - \frac{p}{q} < \frac{1}{2q^2}.$$

Hence by Legendre's Theorem, we see that if $\frac{p}{q} < \alpha$, then $\frac{p}{q}$ is a convergent of α, as desired. Thus by inequalities (12) and (13), we observe that when n is even, $\frac{p_n - p_{n-1}}{q_n - q_{n-1}}$ is a convergent, and when n is odd, $\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ is a convergent. Hence we only need to consider type 1 secondary convergents when n is even, and type 2 secondary convergents when n is odd.

We will first suppose there exists a positive even integer $n \geq N$ such that

$$\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

Then recall by fact 3 we have

$$\left|\alpha - \frac{p}{q}\right| = \left|\alpha - \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right| = \frac{1}{\lambda_n q^2}.$$

Since n is even, we observe by condition (ii)a of our hypothesis that $a_{n+1} \geq a_{n+2} + 2$ and by condition (ii)b that $a_{n+1} \geq a_n + 2$. Also, by condition (ii)a and our specific footnoted assumption, we notice that if $a_{n+1} = a_n + 2$, then $a_{n-1} \geq a_n + 1$. So, if $a_{n+1} = a_n + 2$, we see

$$\begin{aligned} a_{n-1} &\geq a_n + 1 \\ &= a_{n+1} - 1 \\ &\geq a_{n+2} + 1. \end{aligned}$$

Therefore we may apply Lemma 1 and conclude that $\lambda_n < 1$. However, this yields

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{\lambda_n q^2} > \frac{1}{q^2},$$

contradicting inequality (11). Thus when n is even, $\frac{p}{q}$ is not a type 1 secondary convergent (unless it is also a convergent).

Now suppose there exists an odd $n \geq N$ such that

$$\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}}.$$

Given that n is odd, it follows from condition (ii) that $a_n \geq a_{n+1} + 2$, $a_{n+2} \geq a_{n+1} + 2$, and $a_n \geq a_{n-1} + 2$. From the last two inequalities, we find that when $a_n = a_{n+1} + 2$,

$$\begin{aligned} a_{n+2} &\geq a_{n+1} + 2 \\ &= a_n \\ &\geq a_{n-1} + 2. \end{aligned}$$

Applying Lemma 2, we find $\lambda'_n < 1$. Utilizing fact 3, we again have a contradiction to inequality (11). Therefore, when n is odd, $\frac{p}{q}$ is not a type 2 secondary convergent (again, unless it is also a convergent). Since these were the two remaining cases with which we were concerned, we may now conclude that $\frac{p}{q}$ must be a convergent of α, as desired.

<u>Case 3</u>: $m_e > 1$ and $m_o = 1$.

Inequality (1) yields J J J

$$\frac{-1}{2q^2} < \alpha - \frac{p}{q} < \frac{1}{q^2}.$$

This follows analogously to case 2, and hence does not deserve repetition.

<u>Case 4</u>: $m_e = m_o = 1$.

We first suppose $\frac{p}{q}$ is a type 1 secondary convergent, so there exists a positive $n \geq N$ such that $\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}$. By condition (iv)a, we must have either that $a_n = 1$ or $a_{n+1} = 1$. If $a_{n+1} = 1$, then

$$\frac{p}{q} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}},$$

so $\frac{p}{q}$ is a convergent as desired. Thus we may now assume $a_{n+1} \neq 1$, so by condition (iv)a and (iv)b, we must have $a_n = 1$ and $a_{n+1} \geq 3$. Thus, $a_{n+1} \geq a_n + 2$. If $a_{n+1} > a_n + 2$, then by Lemma 1 we conclude $\lambda_n < 1$. If $a_{n+1} = a_n + 2 = 3$, then by condition (iv)c we may apply Lemma 3 and again observe that $\lambda_n < 1$. This (together with fact 3) contradicts inequality (11), hence unless $\frac{p}{q}$ is a convergent, it cannot be a type 1 secondary convergent.

We now suppose $\frac{p}{q}$ is a type 2 secondary convergent, so there exists a positive $n \geq N$ such that $\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}}$. Again by condition (iv)a we see that either $a_n = 1$ or $a_{n+1} = 1$. Here if $a_n = 1$ then we have

$$\frac{p}{q} = \frac{p_n - p_{n-1}}{q_n - q_{n-1}} = \frac{p_{n-2}}{q_{n-2}},$$

so we must have $n > 1$ (or else this fraction is undefined), and $\frac{p}{q}$ is a convergent. Thus we may now assume $a_n \neq 1$. Then by conditions (iv)a and (iv)b we must have $a_{n+1} = 1$ and $a_n \geq 3 = a_{n+1} + 2$. If $a_n > a_{n+1} + 2$ then it follows from Lemma 2 that $\lambda'_n < 1$ and if $a_n = a_{n+1} + 2 = 3$ then by condition (iv)c and Lemma 3 again we see $\lambda'_n < 1$. This again contradicts inequality (11), so $\frac{p}{q}$ can only be a type 2 secondary convergent if it is a convergent. This is exactly what we want, for we may now conclude that indeed $\frac{p}{q}$ is a convergent of α.

Thus in all cases we have shown that if a rational $\frac{p}{q}$ expressed in lowest terms with $q \geq \max\{2, q_N\}$ satisfies inequality (10), $\frac{p}{q}$ must be a convergent of α, establishing our claim.

□

We remark that this theorem cannot be expanded to include anymore irrational numbers, without losing the simplicity of the result. Granted, the theorem itself is anything but simple, but once the conditions are met the result is simple. As the theorem stands now, any irrational not satisfying the conditions will have a secondary convergent satisfying inequality (10) which is not itself a convergent. (We should note that this is not quite true; recall our examination of lemmas 1 and 2 in which we noted a few very specific cases that did not satisfy the conditions for these lemmas, yet still produced $\lambda_n < 1$ and/or $\lambda'_n < 1$. If the continued fraction expansion for α contains one of these cases then the hypothesis of our theorem will not be met but it still may possess no secondary convergents satisfying inequality (10). However, for the most part, for every irrational not satisfying the conditions, the conclusion of the theorem would not hold). Excepting these few cases, recall that each of the lemmas could not be expanded to include any more irrationals. Since the conditions of Theorem 2 are almost completely reliant on these lemmas, then inherently these conditions may not be expanded to include more irrationals. Also, recall in our creation of the interval that in order to create a narrower interval, we would have to resort to considering each value of n individually, which would render the whole theorem irrelevant. Therefore similar techniques may not be used to classify the best approximates of the rest of the irrationals. So in that sense, our quest is completed. We conclude this segment with a proof of the aforementioned corollary:

Corollary 1. *If $\frac{p}{q}$ is a rational number expressed in lowest terms with $q > 2$, then $\frac{p}{q}$ is a convergent to e if and only if*

$$\left| e - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{14}$$

Proof. Recall that $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...] = [2, 1, 2, \overline{1, 1, 2k_i}]$, $k : 2 \to \infty$. Using $N = 3$ and defining m_e and m_o as in Theorem 2, we have $m_e = m_o = 1$. Thus, e lies in case (iv) and satisfies the conditions placed on α in Theorem 2. Since $q_3 = 4$, it follows from Theorem 2 that for all rationals $\frac{p}{q}$ expressed in lowest terms with $q \geq 4$, $\frac{p}{q}$ is a convergent to e if and only if

$$\left| e - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Thus it only remains to consider $q = 3$.

Since $\frac{p}{q}$ satisfies inequality (14), $\frac{p}{q}$ must be either a convergent or a secondary convergent of e. If $\frac{p}{q}$ is a convergent then we have established our desired result, thus we may now assume $\frac{p}{q}$ is a secondary convergent.

First consider $3 = q = q_n + q_{n-1}$. This would necessitate $q_{n-1} = 1$ and $q_n = 2$ or $q_{n-1} = 0$ and $q_n = 3$. However, there does not exist an n such that $q_n = 2$, and while $q_2 = 3$, $q_{(2-1)} = q_1 = 1 \neq 0$. Thus, we cannot have $3 = q_n + q_{n-1}$. Now consider $3 = q = q_n - q_{n-1}$. So we have $3 = a_n q_{n-1} + q_{n-2} - q_{n-1} = (a_n - 1)q_{n-1} + q_{n-2}$. This equation can be satisfied when $q_{n-2} \in \{0, 1, 2, 3\}$. We will consider each of these cases. If $q_{n-2} = 0$, we must have $n = 1$. However, when $n = 1, q_n - q_{n-1} = 1 - 1 = 0 \neq 3$. If $q_{n-2} = 1$ we must have $n \in \{2, 3\}$. However,

$q_2 - q_1 = 3 - 1 \neq 3$ and $q_3 - q_2 = 4 - 3 \neq 3$. There exists no n such that $q_{n-2} = 2$. Lastly, if $q_{n-2} = 3$, we must have $n = 4$, in which case we see

$$\frac{p}{q} = \frac{p_4 - p_3}{q_4 - q_3} = \frac{19 - 11}{7 - 4} = \frac{8}{3} = \frac{p_2}{q_2}$$

which is a convergent of e. Thus if $q = 3$, the only rational number satisfying inequality (14) is a convergent of e.

Thus if $\frac{p}{q}$ is a rational number expressed in lowest terms with $q > 2$ that satisfies inequality (14), then $\frac{p}{q}$ is a convergent of e.

□

In producing Theorem 2 we have improved on the famous result of Legendre for certain irrational numbers, creating a criteria which a rational satisfies if and only if it is a best approximate. Moreover, once an irrational has been shown to satisfy the conditions of the hypothesis, whether or not a rational satisfies the criteria may be easily checked and more importantly, requires no knowledge of the continued fraction expansion or of any other convergents. For the irrational numbers for which our theorem applies, this accomplishes the goal with which we began. Since the ideas used in the above work may not be extended to other irrationals, our journey here takes us no farther and we are forced to turn to other endeavors.

V. SIMULTANEOUS APPROXIMATION

Recall the initial questions questions we began the paper with. We presented the number e, and the rational number $\frac{1650}{607}$. Using the above corollary we may now easily verify that this is a best approximate, however here I want to jump back to the original question we asked: How do you come up with the number $\frac{1650}{607}$, and once you have it, how do you determine if it really is a best approximate? We said the first question was answered in the 18th century and thus up until this point we have dwelt on the latter question. At this point however, we return to the original question, yet in a slightly different context, one in which this question has not been fully answered. Here we move into the field of simultaneous approximation, where we consider the first question, and watch the answer to the second easily emerge. So, in continuing our quest for best approximates, we extend our search into the simultaneous approximation of a finite number of irrationals. In doing so we also acquire a deeper understanding for a certain Diophantine equation. However, before we make this jump to simultaneous approximation, we make a few additional necessary comments on finding best approximates of just one irrational.

When determining the sequence of best approximates (or equivalently, convergents), giving the sequence of rational numbers is nice, but it is superfluous information. If we have the sequence of denominators of the convergents of an irrational α, the numerator that goes with each denominator is simply whatever integer brings the rational number closest to α. Hence, our search for the sequence of best approximates may have been reduced not only to a search for convergents, but even farther, to a search simply for the denominators of these convergents, known as *continuants* of α.

Another way of thinking about best approximates is using the nearest integer function $|| \ ||$, where $||x|| = \min\{|x-k| \mid k \in \mathbf{Z}\}$. If you want to find the sequence of best approximates to α, the sequence of integer q's minimizing $||\alpha q||$ will give you the denominators of the best approximates and the k's from the definition above will be precisely the numerators. To see this, we first observe $||\alpha q|| = |\alpha q - p|$, where p is the integer αq is closest to. Taking the q out of the absolute value, we observe

$$||\alpha q|| = q\left|\alpha - \frac{p}{q}\right|.$$

Thus the sequence of q's minimizing $||\alpha q||$ are also the sequence of denominators of rationals that get closer and closer to α, taking into account the size of the denominator. Hence, they are indeed the sequence of denominators of the best approximates of α.

Now, suppose we want to approximate two irrationals at once, or simultaneously approximate any finite number of irrationals, using the same denominator for each. We thus want to find the sequence of integer q's minimizing

$$\max\{||\alpha_0 q||, ||\alpha_1 q||, ||\alpha_2 q||, ..., ||\alpha_l q||\},$$

where the α_i's are the irrational numbers we would like to simultaneously approximate. In other words, we would like to find the complete list of denominators that successively minimize the maximum distance between any irrational you are approximating and its rational approximate with that denominator. This is known as *simultaneous approximation.*

In general, classifying these sequences of denominators for finite lists of irrational numbers is a very hard task. Here we consider a special case, and look at the problem for certain irrational numbers. Specifically, we consider the golden ratio, $\varphi = \frac{1+\sqrt(5)}{2} = [\overline{1}\ ]$, and the generalized golden ratios, $\varphi_a = \frac{a+\sqrt{a^2+4}}{2} = [\ \overline{a}\ ]$. We will look at simultaneously approximating $\varphi_{a_0}, \varphi_{a_1}, \varphi_{a_2}, ..., \varphi_{a_l}$, where each φ_{a_i} is an element of the same vector space, $\mathbf{Q} + \mathbf{Q}\alpha$. Before jumping to our conclusion however, we first require an examination of generalized golden ratios belonging to the same vector space.

Generalized golden ratios are quadratic, hence every φ_a belongs to the vector space $\mathbf{Q}+\mathbf{Q}\sqrt{d}$ for some positive square-free integer d. If we are to deal with simultaneously approximating generalized golden ratios belonging to the same vector space, we require insight into exactly when φ_a belongs to the vector space $\mathbf{Q} + \mathbf{Q}\sqrt{d}$. Moreover, it would be nice to have some knowledge regarding the sequence of a's for which their corresponding φ_a's belong to the same vector space. The following two lemmas shed light onto both of these matters. As will be established with the following small lemma, we utilize a particular Diophantine equation to determine exactly when $\varphi_a \in \mathbf{Q} + \mathbf{Q}\sqrt{d}$.

Lemma 4. *For a positive integer a, let $\varphi_a = [\ \overline{a}\ ] = \frac{a+\sqrt{a^2+4}}{2}$. Then $\varphi_a \in \mathbf{Q}+\mathbf{Q}\sqrt{d}$ for a square-free positive integer d if and only if there exists an integer solution to*

$$a^2 - dt^2 = -4.$$

Proof. We note that

$$\varphi_a = \frac{a + \sqrt{a^2+4}}{2},$$

and assume $\varphi_a \in \mathbf{Q} + \mathbf{Q}\sqrt{d}$. Hence we must have $\sqrt{a^2+4} = k\sqrt{d}$, for some integer k, and squaring both sides we observe an integral solution to $a^2 - dt^2 = -4$.

We now assume there exists an integer, call it t_0, satisfying

$$a^2 - dt_0^2 = -4,$$

so $a^2 + 4 = dt_0^2$. Then

$$\varphi_a = \frac{a + \sqrt{a^2+4}}{2} = \frac{a + t_0\sqrt{d}}{2} \in \mathbf{Q} + \mathbf{Q}\sqrt{d}.$$

□

Although this lemma may seem somewhat trivial, it will be important in the simultaneous approximation of generalized golden ratios in the same vector space, our ultimate goal. Keeping in mind that we would like to find the sequence of a's for which φ_a is in the same vector space, the above lemma serves as motivation for a deeper understanding of the integral solutions to the diophantine equation $a^2 - dt^2 = -4$.

Lemma 5. *Let $(a_0, t_0), (a_1, t_1), (a_2, t_2), ...$ be the complete increasing sequence of positive integral solutions to*

$$a^2 - dt^2 = -4. \tag{15}$$

This sequence is nonempty (and infinite) if and only if the continued fraction expansion for $\sqrt{d}$ has odd period length, and moreover, if (a_0, t_0) is the minimum positive solution to (2), then all other solutions are defined recursively by

$$(a_n, t_n) = J\frac{1}{2}\left((a_0^2+2)a_{n-1} + da_0t_0t_{n-1}, (a_0^2+2)t_{n-1} + a_0t_0a_{n-1}\right). \tag{16}$$

Proof. We will first assume that there exists a solution to (15), and show that this implies infinitely many solutions to (15), defined recursively as above. Then we will show that such a solution exists if and only if the continued fraction expansion for $\sqrt{d}$ has odd period length.

Assume that there exists a solution to (15), and denote the minimum positive solution by (a_0, t_0). By factoring, (15) can be rewritten as

$$(a + t\sqrt{d})(a - t\sqrt{d}) = -4,$$

and for positive integers a and t, we call the first term, $a + t\sqrt{d}$, a *solution in* $\mathbf{Z}[\sqrt{d}]$. Thus there exists a solution in $\mathbf{Z}[\sqrt{d}]$ precisely when there exists an integral solution.

By our assumption, the minimal solution in $\mathbf{Z}[\sqrt{d}]$ to (15) is $a_0 + t_0\sqrt{d}$. Theorem 8.8 in [3], the minimal solution to $a^2 - dt^2 = 4$ is then given by (see for example Theorem 8.8 in [6])

$$\frac{(a_0 + t_0\sqrt{d})^2}{2}.$$

Also (see again [6]) we have that all solutions (with positive components) in $\mathbf{Z}[\sqrt{d}]$ to (15) are given by

$$\alpha_n = (a_0 + t_0\sqrt{d})\left(\frac{(a_0 + t_0\sqrt{d})^2}{4}\right)^n,$$

where $n \geq 0$ is an integer. Thus letting $\alpha_n = a_n + t_n\sqrt{d}$, we observe infinitely many integral solutions (a_n, t_n) to (15).

We now let k be a positive integer, and consider α_k. We have

$$\begin{aligned}
\alpha_k &= (a_0 + t_0\sqrt{d})\left(\frac{(a_0 + t_0\sqrt{d})^2}{4}\right)^k \\
&= (a_0 + t_0\sqrt{d})\left(\frac{(a_0 + t_0\sqrt{d})^2}{4}\right)^{k-1}\left(\frac{(a_0 + t_0\sqrt{d})^2}{4}\right) \\
&= \alpha_{k-1}\left(\frac{(a_0 + t_0\sqrt{d})^2}{4}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left[(a_{k-1}+t_{k-1}\sqrt{d})(a_0^2+2a_0t_0\sqrt{d}+dt_0^2)\right]\\
&= \frac{1}{4}\left[(a_{k-1}a_0^2+da_{k-1}t_0^2+2da_0t_0t_{k-1})+(2a_{k-1}a_0t_0+a_0^2t_{k-1}+dt_0^2t_{k-1})\sqrt{d}\right]\\
&= \frac{1}{4}\left[[a_{k-1}(a_0^2+dt_0^2)+2da_0t_0t_{k-1}]+[2a_0t_0a_{k-1}+t_{k-1}(a_0^2+dt_0^2)]\sqrt{d}\right]\\
&= \frac{1}{4}\left[[a_{k-1}(a_0^2+a_0^2+4)+2da_0t_0t_{k-1}]+[2a_0t_0a_{k-1}+t_{k-1}(a_0^2+a_0^2+4)]\sqrt{d}\right]\\
&= \frac{1}{2}\left[[(a_0^2+2)a_{k-1}+da_0t_0t_{k-1}]+[(a_0^2+2)t_{k-1}+a_0t_0a_{k-1}]\sqrt{d}\right]
\end{aligned}
$$

Recalling that $\alpha_k = a_k + t_k\sqrt{d}$, we observe that a_k is defined recursively by

$$a_k = \frac{1}{2}\left[(a_0^2+2)a_{k-1}+da_0t_0t_{k-1}\right],$$

and t_k is defined recursively by

$$t_k = \frac{1}{2}\left[(a_0^2+2)t_{k-1}+a_0t_0a_{k-1}\right],$$

establishing our result.

Lastly, we will show that there exists a solution to (15) if and only if the continued fraction expansion for $\sqrt{d}$ has odd period length. It is known that there exists a solution to

$$a^2 - dt^2 = -1 \tag{17}$$

if and only if the continued fraction expansion for $\sqrt{d}$ has odd period length (see ???). We will show that there exists a solution to (15) if and only if there exists a solution to (17). Trivially, we observe that if there exists a solution (a_*, t_*) satisfying (17), multiplying through by 4 we find $4a_*^2 - 4dt_*^2 = -4$, hence $(2a_*, 2t_*)$ is a solution to (15). Now suppose there exists a solution to (15), and we continue to denote the smallest such positive solution (a_0, t_0). We consider two cases, depending on the parity of a_0.

Case 1: Suppose a_0 is even, so $a_0 = 2r$ for some integer r, giving us

$$(2r)^2 - dt_0^2 = -4.$$

This is equivalent to

$$dt_0^2 = 4(r^2 + 1),$$

from which it follows immediately that $4|dt_0^2$. We know that d is square-free, hence at most only one factor of 2 may divide into d, forcing t_0^2 to be even, hence t_0 must be even. Therefore, there exists an integer s such that $t_0 = 2s$, giving us

$$(2r)^2 - d(2s)^2 = -4.$$

Dividing through by 4 we observe

$$r^2 - ds^2 = -1,$$

giving us an integral solution to (17), as desired.

Case 2: Suppose a_0 is odd. Notice that if d is even, to satisfy equation (15) we require a_0^2 and thus a_0 to be even. Hence here we may assume that d is odd. Using our recurrence relation established above, we see

$$\begin{aligned} a_1 &= \frac{1}{2}\left[(a_0^2 + 2)a_0 + da_0t_0t_0\right] \\ &= \frac{1}{2}\left[a_0(a_0^2 + 2 + dt_0^2)\right] \\ &= \frac{1}{2}\left[a_0(2a_0^2 + 6)\right] \\ &= a_0^3 + 3a_0, \end{aligned}$$

and we observe a_1 is even. Thus using the above argument for when a_0 is even and replacing a_0 with a_1, a solution to (17) may be deduced.

Thus we may conclude there exists a solution to (15) if and only if there exists a solution to (17), and hence (15) is solvable if and only if the continued fraction expansion for $\sqrt{d}$ has odd period length.

□

Of further interest to the curious reader, many various recursive definitions where found to give the solutions to (15). As a taste, when the vector space we are considering is $\mathbf{Q} + \mathbf{Q}\sqrt{5}$, the space whose minimum solution to (15) is $(1, 1)$, generating the golden ratio, a_n can also be (independently) expressed recursively by

$$a_n = \sqrt{9a_{n-1}^2 + a_{n-2}^2 + 36 - 6\sqrt{(4 + a_{n-1}^2)(4 + a_{n-2}^2)}},$$

and by

$$a_n = \frac{1}{2^n}\left[\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 3^{2k} 5^{\lfloor \frac{n-2k+1}{2} \rfloor} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 3^{2k+1} 5^{\lfloor \frac{n-2k}{2} \rfloor}\right].$$

Even more fascinating, t_n may be expressed by the simple recurrence

$$\begin{aligned} t_n &= t_{n-1} + \sum_{k=0}^{n-1} t_k \\ &= 3t_{n-1} - t_{n-2}. \end{aligned}$$

However, here these facts are purely for interest, as they serve no practical purpose (as yet) in our search for best approximates. Hence, I omit the proof, and leave it to the skeptical reader to verify. Of more concern to our goal, in finding these different methods of expressing the integral solutions to (15) recursively (the a_n's and t_n's), we sought to generate a closed form classifying integral solutions to (15), or perhaps a simpler recurrence for the a_n's. However, such an equation was elusive to us, and may be a topic for further consideration in the future. On the other hand, probably even more likely, this might be a search where there is nothing to be found.

Although examination of equation (15) is interesting and rewarding in its own right, in sticking to our purpose we return to our search for best approximates for irrationals in the same vector space. In ??? E. Burger examined precisely this question (see [2]). If $\alpha, \alpha_1, \alpha_2, ..., \alpha_l$ are irrational belong to the same vector space, then α_i may be expressed as $\frac{A_i+B_i\alpha}{C_i}$, where A_i, B_i, and C_i are integers. Using properties and relations among these coefficients, Burger presents an algorithm for producing an auxiliary irrational number, which he denotes by α^*. The part of his result relevant to our work here is presented in the following theorem:

Theorem (Burger, ???). *Let $\alpha, \alpha_1, ..., \alpha_L \in \mathbf{Q} + \mathbf{Q}\alpha$ be $L+1$ irrational numbers. Then the sequence $Dq_j^*, Dq_{j+1}^*, D_{j+2}^*, ...$ is the complete list of values for integers $n > \tilde{C}^2 \varepsilon M$ that successively minimize the quantity*

$$\max \{\|\alpha q\|, \|\alpha_1 q\|, ..., \|\alpha_L q\|\},$$

where q_j^* denotes the j^{th} continuant of the auxiliary number α^*.

Do not be concerned with the constants $D, \tilde{C}, \varepsilon$, and M, they are merely constants computed by examination of the above mentioned integers, A_i, B_i and C_i. The computation of these constants is too involved to include here, but we refer the curious reader to [2] for more information. The important thing to note about these constants is that when $C_i = 1$ for all i, then the constant $D = 1$, and hence the best approximates (after a certain point) are simply the continuants of α^*.

As we saw in the motivation for Mobius's and Komatsu's theorems, the continuants of φ are precisely the fibonacci numbers and the continuants of φ_a are exactly the generalized fibonacci numbers ($q_0 = 1, q_1 = a$ and for $n > 1$, $q_n = aq_{n-1} + q_{n-2}$). We proceed to show that each generalized golden ratio $\varphi_{a_0}, \varphi_{a_1}, ..., \varphi_{a_l}$ belonging to the same vector space may be written as a linear combination of φ_{a_0}, where φ_{a_0} is the minimum such generalized golden ratio belonging to that vector space. In writing each φ_{a_i} as a linear combination of φ_{a_0}, the C_i's in Burger's algorithm are all 1, and hence the best approximates are simply the continuants of our auxiliary number (which nicely turns out to be precisely the reciprocal of φ_{a_0} itself). Hence, the best approximates of the sequence of generalized golden ratios belonging to the same vector space are exactly the generalized fibonacci numbers! We present this result more formally with the following theorem.

Theorem 3. *Let $\varphi_a = [\ \overline{a}\ ] = \frac{a+\sqrt{a^2+4}}{2}$ (a generalized golden ratio), and let $\varphi_{a_0}, \varphi_{a_1}, \varphi_{a_2}, ...\varphi_{a_l} \in$*

$\mathbf{Q} + \mathbf{Q}\sqrt{d}$, *where* d *is a positive, square-free integer and* φ_{a_0} *is the minimum generalized golden ratio belonging to this vector space. Let* $m = \sqrt{\frac{a_l^2+4}{a_0^2+4}}$. *Then the generalized fibonacci numbers for* a_0 *(*q_n *given recursively by* $q_0 = 1$, $q_0 = a_0$ *and for* $n > 1$, $q_n = a_0 q_{n-1} + q_{n-2}$*) greater than* $m + m^2$ *give the complete sequence of integers* $q > m + m^2$ *that successively minimize*

$$\max \{\|\varphi_{a_0} q\|, \|\varphi_{a_1} q\|, \|\varphi_{a_2} q\|, ..., \|\varphi_{a_l} q\|\}.$$

In simpler language, when simultaneously approximating generalized golden ratios belonging to the same vector space, the denominators of the best approximates are precisely generalized fibonacci numbers! More specifically, if φ_{a_0} is the smallest generalized golden ratio in the vector space $\mathbf{Q} + \mathbf{Q}\sqrt{d}$, then the denominators of the best simultaneous approximates to φ_{a_0} and any other generalized golden ratios in that vector space are the generalized fibonacci numbers using a_0. We now have only to establish this claim.

Proof. We first claim that for each i, we can write $\varphi_{a_i} = A_i + B_i \varphi_{a_0}$, where A_i and B_i are relatively prime integers. Without loss of generality, we will assume $\varphi_{a_0} < \varphi_{a_1} < ... < \varphi_{a_l}$. Since $\varphi_{a_i} \in \mathbf{Q} + \mathbf{Q}\sqrt{d}$ for each i, by Lemma 4 for each i there exists a positive integer t_i satisfying

$$a_i^2 - dt_i^2 = -4. \tag{18}$$

Defining $A_i = \frac{a_i t_0 - t_i a_0}{2t_0}$ and $B_i = \frac{t_i}{t_0}$, we will verify $A_i + B_i \varphi_{a_0} = \varphi_{a_i}$, and then show A_i and B_i to be relatively prime integers. We observe the following:

$$\begin{aligned} A_i + B_i \varphi_{a_0} &= \left(\frac{a_i t_0 - t_i a_0}{2t_0}\right) + \frac{t_i}{t_0}\left(\frac{a_0 + \sqrt{a_0^2+4}}{2}\right) \\ &= \frac{a_i t_0 + t_i\sqrt{a_0^2+4}}{2t_0}. \end{aligned}$$

In light of equation (18), we observe that $\sqrt{a_0^2+4} = t_0\sqrt{d}$, and consequently,

$$A_i + B_i \varphi_{a_0} = \frac{a_i t_0 + t_i t_0 \sqrt{d}}{2t_0}$$

$$
\begin{aligned}
&= \frac{a_i + t_i\sqrt{d}}{2} \\
&= \frac{a_i + \sqrt{a_i^2 + 4}}{2} \\
&= \varphi_{a_i},
\end{aligned}
$$

as desired.

We will first show B_i to be an integer, then A_i to be an integer, and lastly we will show them to be relatively prime. We show by induction that $B_i = \frac{t_i}{t_0} \in \mathbf{Z}$. $B_0 = \frac{t_0}{t_0} = 1$ is clearly an integer, establishing the base case. Let n be a positive integer, and suppose that $B_k \in \mathbf{Z}$ for all $0 \leq k < n$. By Lemma 5, $t_n = \frac{1}{2}\left[(a_0^2 + 2)t_{n-1} + a_0 a_{n-1} t_0\right]$, which can be rewritten using equation (18) to yield

$$
t_n = \frac{1}{2}\left[(dt_0^2 - 2)t_{n-1} + t_0 a_0 a_{n-1}\right].
$$

Therefore we see

$$
\begin{aligned}
B_n &= \frac{t_n}{t_0} \\
&= \frac{(dt_0^2 - 2)t_{n-1} + t_0 a_0 a_{n-1}}{2t_0} \\
&= \frac{1}{2}\left(dt_0 t_{n-1} - 2B_{n-1} + a_0 a_{n-1}\right).
\end{aligned}
$$

If d is odd, then by equation (18) a_i and t_i must be the same parity, forcing $a_0 a_i$ and $t_0 t_i$ to be the same parity. Thus if d is odd $dt_0 t_{n-1}$ and $a_0 a_{n-1}$ have the same parity, hence their sum is even. Also, as seen by equation (18), if d is even, a_0 must be even, so clearly $dt_0 t_{n-1} + a_0 a_{n-1}$ is even. Therefore we observe B_n to be an integer, and hence by induction $B_i \in \mathbf{Z}$ for all i.

We now consider A_i, and observe the following:

$$
\begin{aligned}
A_i &= \frac{a_i t_0 - t_i a_0}{2t_0} \\
&= \frac{a_i - a_0 B_i}{2}.
\end{aligned}
$$

First suppose a_0 is even, and we will show by induction that a_i must be even for all i. The base case (a_0 is even) is already established by assumption. Let n be a positive integer, suppose a_k is even for $0 \leq k < n$, and consider

$$a_n = \frac{1}{2}\left[(a_0^2 + 2)a_{n-1} + da_0t_0t_{n-1}\right].$$

Clearly $a_0^2 + 2$ is even, and by our induction hypothesis a_{n-1} is even, hence $(a_0^2 + 2)a_{n-1}$ must be divisible by 4. Since a_0 is even, $a_0^2 + 4 = dt_0^2$ is divisible by 4, and since d is square-free it follows that t_0^2 must be divisible by 2. This forces t_0 to be even, hence $da_0t_0t_{n-1}$ must also be divisible by 4. As a result, $(a_0^2 + 2)a_{n-1} + da_0t_0t_{n-1}$ is divisible by 4, and thus a_n is even. So by induction, if a_0 is even, a_i is also even for all i. Given this information, and recalling that $B_i \in \mathbf{Z}$, it follows immediately that $a_i - a_0B_i$ is even, and hence A_i is an integer.

We now suppose a_0 is odd. In order to satisfy equation (18), both d and t_0 must also be odd. This implies that $B_i = \frac{t_i}{t_0}$ has parity equivalent to t_i. Since d is odd, t_i and a_i must have the same parity, hence B_i and a_i have the same parity. Since a_0 is odd, it follows that $a_i - a_0B_i$ is even, and consequently, A_i is an integer. Therefore, we have shown both A_i and B_i to be integers for all i. Our next step is to show that they are relatively prime.

Let $g = (A_i, B_i)$. Then there exist integers r and s such that

$$A_i = \frac{a_it_0 - t_ia_0}{2t_0} = gr, \tag{19}$$

and

$$B_i = \frac{t_i}{t_0} = gs.$$

From this it is evident that $t_i = t_0gs$ and substituting into equation (19) we find

$$a_it_0 - (t_0gs)a_0 = 2t_0gr,$$

from which we observe

$$a_i = g(2r + sa_0).$$

We may now rewrite equation (18) in terms of r and s, yielding

$$[g(2r + sa_0)]^2 - d(gst_0)^2 = -4,$$

and factoring out a g^2 we see

$$g^2[(2r + sa_0)^2 - d(st_0)^2] = -4.$$

Squaring the terms and writing t_0 in terms of a_0 gives us

$$g^2 \left[4r^2 + 4rsa_0 + s^2a_0^2 - ds^2 \left(\frac{a_0^2 + 4}{d} \right) \right] = -4,$$

and hence

$$4g^2(r^2 + rsa_0 - s^2) = -4.$$

Dividing by 4 it becomes evident that $g^2 | -1$, which forces $g = 1$, as desired. Hence A_i and B_i are indeed relatively prime integers.

Now for any $l \geq 0$ we have $l + 1$ irrational numbers, $\varphi_{a_0}, \varphi_{a_1}, \varphi_{a_2}, ..., \varphi_{a_l}$, all elements of $\mathbf{Q} + \mathbf{Q}\sqrt{d}$, which can be expressed as $\varphi_{a_0}, (A_1 + B_1\varphi_{a_0}), (A_2 + B_2\varphi_{a_0}), ..., (A_l + B_l\varphi_{a_0})$. We may then apply Burger's theorem, and use his algorithm to computer the auxiliary number. Using his algorithm (see [2]), we create the auxiliary number associated with our $l + 1$ irrational numbers to be

$$\alpha^* = -a_0 + \varphi_{a_0} = [0, \overline{a_0}].$$

Hence, the continuants of α^* are exactly the continuants of φ_{a_0}, which in turn are precisely the generalized fibonacci numbers for a_0! In this case we find the constants defined in Burger's theorem to be $\tilde{C} = 1$, $\varepsilon = \max\{B_i\}$, and $M = \max\{B_i\} + 1$. We know $B_i = \frac{t_i}{t_0}$ and by equation (18), $t_i = \sqrt{\frac{a_i^2+4}{d}}$. Since the sequence of a_i's is increasing, the sequence of t_i's must also be increasing, hence

$$\max\{B_i\} = \frac{t_l}{t_0} = \frac{\sqrt{\frac{a_l^2+4}{d}}}{\sqrt{\frac{a_0^2+4}{d}}} = \sqrt{\frac{a_l^2+4}{a_0^2+4}} = m.$$

Thus, $\tilde{C}^2 \varepsilon M = m(m+1) = m^2+1$. Therefore, we have that the generalized fibonacci numbers greater than m^2+m are the complete list of values for integers $q > m+m^2$ that successively minimize

$$\max\{\|\varphi_{a_0}q\|, \|\varphi_{a_1}q\|, \|\varphi_{a_2}q\|, ..., \|\varphi_{a_l}q\|\},$$

verifying the rest of our claim. □

We support this result with an example for clarity. We shall consider generalized golden ratios of the vector space $\mathbf{Q}+\mathbf{Q}\sqrt{5}$. It is easy to verify that the smallest such generalized golden ratio is the golden ratio itself, $\frac{1+\sqrt{5}}{2}$. Combining Lemmas 4 and 5 we observe the sequence of a_i's for which $\varphi_{a_i} \in \mathbf{Q}+\mathbf{Q}\sqrt{5}$ to be $1, 4, 11, 29, 76, ...$ and so on. What our result tells us is that if you want to simultaneously approximate say φ, φ_4, and φ_{76}, then the denominators of these best approximates are precisely the fibonacci numbers.

When you actually think about this theorem, it is rather fascinating. It says that whether you are approximating solely φ_{a_0} or simultaneously approximating φ_{a_0} and a bunch of other irrationals, you will find the same best approximates! So in some sense, considering the whole issue of simultaneous approximation in this case is rendered unnecessary, as one can simply consider the (much easier) case of finding best approximates of φ_{a_0}. Therefore, we may apply Komatsu's theorem (or equivalently, theorem 2), and very quickly and easily determine whether or not a rational is a best approximate. Thus in answering our first question (how do you find best approximates), the answer to the second (how do you know if a number is best approximate) follows as a direct result of previous work.

The astute reader will notice that the theorem requires that the minimum generalized golden ratio of the vector space in which the irrationals lie must be one of the irrationals being simultaneously approximated for the theorem to hold. If φ_{a_0} is the minimum generalized golden ratio of a vector space, we relied heavily on the fact that all other generalized golden ratios in that vector space may be written as linear combinations of φ_{a_0}. However, it is not necessarily the case that they may all be written as linear combinations of, say, φ_{a_1}. Hence if for example we are approximating φ_{a_1} and φ_{a_2}, if one may be written as a linear combination of the other then the best approximates will be the generalized fibonacci numbers by a_1. If however, one cannot write φ_{a_2} as a linear combination of φ_{a_2} then we can not make such a claim. This leads us to examine when φ_{a_j} may be written as a linear combination of φ_{a_i}. Recall that both have been shown to be expressible as linear combinations of φ_{a_0}, with $\varphi_{a_i} = A_i + B_i\varphi_{a_0}$ and $\varphi_{a_j} = A_j + B_j\varphi_{a_0}$. Thus we may write

$$\begin{aligned}\varphi_{a_j} &= A_j + B_j\varphi_{a_0} \\ &= A_j + B_j\left(\frac{\varphi_{a_i} - A_i}{B_i}\right)\end{aligned}$$

Recalling that $\frac{B_j}{B_i} = \frac{t_j}{t_i}$, we see that φ_{a_j} can be written as a linear combination of φ_{a_i} if and only if $t_i \mid t_j$.

Thus if you are simultaneously approximating $\varphi_{a_1}, \varphi_{a_2}, ...\varphi_{a_l} \in \mathbf{Q} + \mathbf{Q}\sqrt{d}$, if $t_1 \mid t_i$ for all i, then the best simultaneous approximates are the generalized golden ratios for a_1. However, simply given the generalized golden ratios, we do not wish to compute each t_i. Although this is at the brink of knowledge, we present the following conjecture which would rectify this situation:

Conjecture 1. *If (a_m, t_m) and (a_n, t_n) are both solutions to $a_i^2 - dt_i^2 = -4$, then $t_m \mid t_n$ if and only if $a_m \mid a_n$.*

We also present the following general conjecture concerning the case when φ_{a_0} is not one of the irrationals being simultaneously approximated.

Conjecture 2. *Let $\varphi_{a_1}, \varphi_{a_2}, ...\varphi_{a_l} \in \mathbf{Q} + \mathbf{Q}\sqrt{d}$, and φ_{a_0} be the minimum generalized golden ratio belonging to this vector space. Let $m = \sqrt{\frac{a_l^2+4}{a_0^2+4}}$. If there exists an k such that a_1 does not divide a_k, then the generalized fibonacci numbers for a_0 give the complete sequence of integers $q > m + m^2$*

that successively minimize

$$\max \{\|\varphi_{a_1} q\|, \|\varphi_{a_2} q\|, ..., \|\varphi_{a_l} q\|\}.$$

Otherwise, the generalized fibonacci numbers for a_1 *give the complete sequence of integers* $q > m + m^2$ *that successively minimize*

$$\max \{\|\varphi_{a_1} q\|, \|\varphi_{a_2} q\|, ..., \|\varphi_{a_l} q\|\}.$$

It is our hope that both of these conjectures will be established as theorems before the final draft of this dissertation is completed.

While these conjectures represent the boundary of our knowledge, there is obviously still a lot beyond this boundary. One natural extension being the question, what about simultaneously approximating generalized golden ratios not belonging to the same vector space? Unfortunately, this deceptively simple extension is by no means easy. However, propelled by curiosity and hope, we spent a good deal of time examining this case (without result). Thus to save repetition of these efforts, I will briefly summarize this excursion.

We considered the simultaneous approximation of two generalized golden ratios, not belonging to the same vector space. We created and ran a computer program generating the sequence of best approximates for a number of these pairs, generating the sequence out to $100,000$, or in some cases, $1,000,000$. We then analyzed these patterns on a mission to find something (anything!).

We first checked to see if these best approximates were the continuants of some auxiliary irrational number, looking for q_i expressed as $a_i q_{i-1} + q_{i-2}$ for some a_i. Although we observed with $q_i = q_{i-1} + q_{i-2}$ with great frequency, most often if $a_i \neq 1$, then such an a_i did not exist, immediately dashing our hopes of finding an "auxiliary number". We checked the quotients of consecutive terms, which in a few cases appeared to be converging to some specific number, but not often enough to generate any really interest. We look for any sort of patterns we could find within each sequence, finding nothing significant. We also looked for similarities between similar pairs. For example, we analyzed the sequence of best approximates to φ_2, φ_3 in comparison to the sequence for φ_2, φ_4 and in comparison to many other sequences where one of the irrationals was φ_2. We did this for many different irrationals, hoping to find some numbers appearing with significant frequency, but to no avail. We especially looked at comparing pairs such as φ, φ_2 and φ_2, φ_4, where one in each pair stays constant and the other in each pair belong to the same vector space. Although here many of the same numbers appeared in these sequences, the sequences were not identical. It may be that you need to go out farther in the sequences before a pattern is discernable, it may be that a pattern exists that we failed to discover, yet probably the most likely possibility of all is that there is simply nothing there. Nonetheless, it provided an interesting source of investigation, and remains such a source for people in the future.

VI. CONCLUSION

We began with a goal... to discover exactly when a rational is a best approximate of an irrational. For a large class of irrationals numbers, we accomplished this goal. Namely, for any irrational satisfying the conditions set forth in Theorem 2, we now know exactly when a rational is a best approximate. This makes a large leap forward beyond what was previously known on the topic. In spite of this, the task is far from complete. What about quadratic irrationals who don't meet the conditions of Theorem 2? What about the countless irrationals that are not quadratic (almost all of whom cannot be shown to satisfy Theorem 2). For all of these numbers, the field remains entirely open.

For all quadratic irrationals, perhaps something along the lines of the above results may be achievable. We have shown that the above methods cannot be expanded to work for more irrationals, but conceivably they may serve as a springboard for further ideas. Regarding irrational numbers in general, this is likely to be a much more challenging issue. For irrationals where the complete continued fraction expansion is not known (which includes the majority of irrationals), the techniques demonstrated in this paper are of little use. For the great thinker with ambition, completely classifying best approximates for every irrational number is a worthy goal to pursue.

On the subject of simultaneous approximation, much less is known. We have found the best simultaneous approximates to very specific sets of irrational numbers (generalized golden ratios in the same vector space). In the specific case we considered, by finding the best simultaneous approximates, we enjoyed the added benefit of completely classifying them through our previous result. Needless to say, within the field of simultaneous approximation, there remain opportunities galore for making discoveries. Simply producing an algorithm (reminiscent of convergents perhaps) for finding the best simultaneous approximates for irrationals not in the same vector space would be a huge advance. Once this is done, as I am confident it some day will be, there always remains the next goal, discovering exactly when a rational is a best simultaneous approximate of a sequence of irrationals.

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