1. Consider a bivariate Gibbs sampler with target (posterior) \( p(x) \equiv p(x_1, x_2) \) having complete conditionals \( p_1(x_1|x_2) \) and \( p_2(x_2|x_1) \). Refer to Section 3.3 of the Gibbs Sampling lecture notes that discusses how the Gibbs sampler defines a Markov chain with transition density \( f(x|x') \) for all \( x = (x_1, x_2) \) and \( x' = (x'_1, x'_2) \).

   - Verify that the joint target density \( p(x) = p(x_1, x_2) \) is indeed the stationary distribution of the resulting Markov chain, i.e., that it satisfies \( p(x) = \int f(x|x')p(x')dx' \) where \( f(x|x') \) is the transition density.

   The result can be proven similarly for the general case of a \( q \)-component Gibbs sampler; this simple special case of \( q = 2 \) is instructive as it makes the result transparent.

2. (a) If \( x \sim N(m, s) \), then \( y = \exp(x) \) is log-normally distributed. Show that \( E(y) = \exp(m + s/2) \).
   (b) In the AR(1) model \( x_t \leftarrow AR(1|\theta) \) with \( \theta = (\phi, v) \), set \( y_t = \exp(x_t) \) for each \( t \). Show that \( \{y_t\} \) is a first-order Markov process, and give an expression for \( E(y_t|y_{t-1}) \) as a function of \( y_{t-1} \).

   Note: The model in (b) is an example of a non-linear, non-normal reversible AR(1) process - related to those arising in stochastic volatility models.

3. In the AR(1) HMM SV model with the normal mixture error distribution, we have

   \[
   (y_t|x_t) \sim \sum_{j=1}^{J} q_j N(b_j + \mu + x_t, w_j)
   \]

   where \( x_t \leftarrow AR(1|(\phi, v)) \).

   Suppose that \( |\phi| < 1 \) and that we are in the stationary distribution of the volatility process \( x_t \), so that \( (x_t|\phi, v) \sim N(0, s) \) with \( s = v/(1 - \phi^2) \).

   - What is the implied distribution \( p(y_t|\mu, \phi, v) \)?
4. Suppose $x^{(t)}$ is a reversible, stationary Markov process with autocovariances $\gamma(k)$ ($k = 0, 1, 2, \ldots$), and that the stationary distribution $\pi(x)$ has mean $\mu$ and variance $\sigma^2 = \gamma(0)$.

For a fixed $n > 0$, suppose that $\bar{x}_n = n^{-1} \sum_{t=1}^{n} x^{(t)}$.

It can be shown that

$$V(\bar{x}_n) = \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{k=1}^{n-1} (1 - k/n)\gamma(k).$$

(a) Show that, for large $n$ and assuming the sum in the following expression converges, $V(\bar{x}_n) \approx \tau^2/n$ where

$$\tau^2 = \sigma^2 (1 + 2 \sum_{k=1}^{\infty} \rho(k))$$

where $\rho(k)$ is the autocorrelation at lag $k$.

(b) In running a MCMC analysis, $x$ is a parameter of interest and we compute a long MCMC series $x^{(t)}$ ($t = 1, \ldots, n$), assuming the MCMC has converged. The above theory indicates that we can estimate the MC sampling variability in $\bar{x}_n$ as the MC estimate of $E(x) = \mu$ using the formula for $\tau$ with the sample autocorrelations substituted for the $\rho(k)$. This is the standard Monte Carlo standard error for $\bar{x}_n$ from the MCMC.

i. Comment on how the autocorrelations in the MC sequence influence the MC accuracy measured this way, with particular comment on cases of large, positive autocorrelations.

ii. In a specific example, one Gibbs sampler generates a sequence that has sample autocorrelations that are apparently like those of a linear AR(1) process with $\phi \approx 0.9$. An alternatively designed MCMC sampler has a similar outcome but with $\phi \approx 0.4$. To achieve roughly similar Monte Carlo standard errors, how much longer should you run the first sampler than the second?

**Bonus:** Can you prove (‡)? Remember the formula for the variance of a sum of correlated random variables, and that $\gamma(k) = \gamma(-k)$ here.