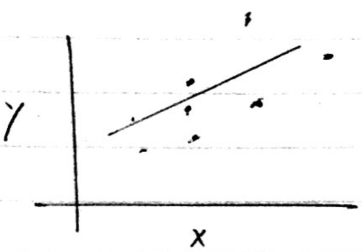
Tests, CI's and PI's



Questions: Given data $(\underline{y}, \underline{x})$, how can we

1) Evaluate evidence that $\beta_1 \neq 0$ (Test)
2) Obtain a plausible range for β_1 (CI)
3) Obtain a plausible range for y^* given X^* (PI)

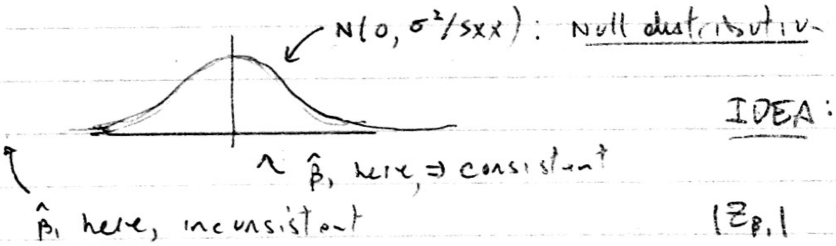
(see Demo)

Hypothesis Test: Assuming $E[Y|X=x] = \beta_0 + \beta_1 x$, evaluate $H: \beta_1 = 0$ versus $K: \beta_1 \neq 0$

Recall, if A1: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$
A23: $\epsilon_1 \ldots \epsilon_n \sim \text{IID } N(0, \sigma^2)$

then $\hat{\beta}_1 \sim N(\beta_1, \sigma^2/sxx)$
$\sim N(0, \sigma^2/sxx)$ if $\beta_1 = 0$.

Testing Question: Is observed value of $\hat{\beta}_1$ "consistent" with $H: \beta_1 = 0$?
with $\hat{\beta}_1 \sim N(0, \sigma^2/sxx)$?



IDEA: $z_{\beta_1} = \hat{\beta}_1 / \sqrt{\sigma^2/sxx} \sim N(0,1)$ if $\beta_1 = 0$

$|z_{\beta_1}|$ big compared to $N(0,1) \Rightarrow$ reject H
$|z_{\beta_1}|$ small " $\Rightarrow$ accept H.

Problem: don't know σ $\rightarrow$ can't compute z_{β_1}

Remedy: Plug in $\hat{\sigma}^2 = RSS/n-2$ for σ^2.

$t_{\beta_1} = \hat{\beta}_1 / \sqrt{\hat{\sigma}^2/sxx}$. What is dist of t_{β_1} under $H: \beta_1 = 0$?

Recall from math stat:
$$\left.\begin{array}{l} Z \sim N(0,1) \\ X \sim \chi^2_k \\ Z, X \text{ indep} \end{array}\right\} \rightarrow Z/\sqrt{X/k} \sim t_k$$
, t-dist w/ k deg. of freedom

Now look at t_{β_1}: $$t_{\beta_1} = \frac{\hat{\beta}_1}{\sqrt{\frac{\hat{\sigma}^2}{S_{xx}}}} = \frac{\hat{\beta}_1}{\sqrt{\frac{\sigma^2}{S_{xx}}}} \times \frac{\sqrt{\sigma^2/S_{xx}}}{\sqrt{\hat{\sigma}^2/S_{xx}}} = \frac{z_{\beta_1}}{\sqrt{\frac{\hat{\sigma}^2}{\sigma^2}}}$$

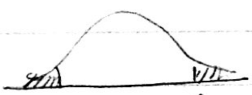
Now $z_{\beta_1} \sim N(0,1)$ under $\beta_1 = 0$.
Also, $$\hat{\sigma}^2/\sigma^2 = \frac{\Sigma(y_i - \hat{y}_i)^2/\sigma^2}{n-2} \sim \chi^2_{n-2}, \text{ indep. of } \hat{\beta}_1$$

$$\Rightarrow \quad t_{\beta_1} = \frac{\hat{\beta}_1}{\sqrt{\hat{\sigma}^2/S_{xx}}} = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)} \sim t_{n-2} \quad \text{if } \beta_1 = 0$$

Similarly, $$\frac{\hat{\beta}_0}{SE(\hat{\beta}_0)} \sim N(0,1) \quad \text{if } \beta_0 = 0$$

Hypothesis test/p-values

Procedure: reject $H: \beta_1 = 0$ if $\left|\hat{\beta}_1/SE(\hat{\beta}_1)\right| > t_{1-\alpha/2}$
$\quad$ (qt(1-α/2, n-2)



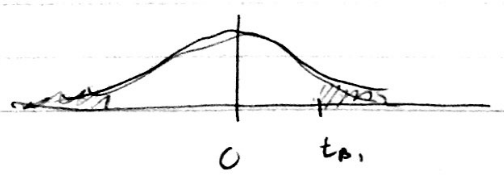
Note: $\alpha = 0.05 \rightarrow t_{1-\alpha/2} \approx 2$.
Rule of thumb: Reject $H: \beta_1 = 0$ if $|\hat{\beta}_1| > 2 \times SE(\hat{\beta}_1)$

Property: $\Pr(\text{Reject } H \mid \beta_1 = 0) = \Pr(|t_{\beta_1}| > t_{1-\alpha/2} \mid \beta_1)$
$\quad = \Pr(|t| > t_{1-\alpha/2}) = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$

P-value: p-value = lowest value of α s.t. H is rejected
= Probability a replicate experiment will produce a result as extreme as current one, given $\beta_1 = 0$

$= \Pr(|t| > |t_{\beta_1}|)$
↑ new experiment
fixed at observed value, not random.

$= 2 \times (1 - \text{pt}(\text{abs}(t_{\beta_1}), n-2))$



(3)

Confidence Intervals: given $(\underset{\sim}{x}, \underset{\sim}{y})$, what are plausible values of β_1?

Is $\tilde{\beta}_1$ a plausible value? Evaluate with a test: $H: \beta_1 = \tilde{\beta}_1$
$K: \beta_1 \neq \tilde{\beta}_1$

If H is rejected then $\tilde{\beta}_1$ not plausible.

Testing $H: \beta_1 = \tilde{\beta}_1$:

If $\beta_1 = \tilde{\beta}_1$, then $t_{\beta_1} = \dfrac{\hat{\beta}_1 - \tilde{\beta}}{SE(\hat{\beta}_1)} \sim t_{n-2}$ (Homework or exercise!)

$\tilde{\beta}_1$ not rejected if

$$|t_{\beta_1}| < t_{1-\alpha/2} \iff$$
$$-t_{1-\alpha/2} < \frac{\hat{\beta}_1 - \tilde{\beta}_1}{SE(\hat{\beta}_1)} < t_{1-\alpha/2} \iff$$
$$\hat{\beta}_1 - SE \times t_{1-\alpha/2} < \tilde{\beta}_1 < \hat{\beta}_1 + SE \times t_{1-\alpha/2}$$

$(1-\alpha)$% CI: The $(1-\alpha)$% CI = values of β_1 not rejected at level α

$= \hat{\beta}_1 \pm SE(\hat{\beta}_1) \times t_{1-\alpha/2}$

$\approx \hat{\beta}_1 \pm 2 \times SE(\hat{\beta}_1)$ if $\alpha = 0.05$ (95% interval)

Properties of CI:

$$\Pr(\beta_1 \in \hat{\beta}_1 \pm SE(\hat{\beta}_1) \times t_{1-\alpha/2} \mid \beta_1) = \Pr\left(\left| \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} \right| < t_{1-\alpha/2} \mid \beta \right)$$
$$= \Pr(|t_{n-2}| < t_{1-\alpha/2})$$
$$= 1-\alpha$$

Interpretation: Pre-experiment/survey, data are random
CI is random
$\Pr(\beta_1 \in CI) = 0.95$

Post-experiment/survey CI is fixed
$\Pr(\beta_1 \in CI) = 0$ or 1.

$\Rightarrow$ This CI procedure is a procedure that gives correct results on 95% of datasets. May or may not give correct result on yours.

(see demo again)

Prediction Intervals:

Task: given $(\underset{\sim}{x}, \underset{\sim}{y})$, predict what y^* will be for a given x^*.

Idea: $y^* = \beta_0 + \beta_1 x^* + \epsilon^*$

$$\approx \hat{\beta}_0 + \hat{\beta}_1 x^* \equiv \hat{y}^* = \text{predicted value.}$$

How accurate is the prediction? (Note: error in textbook)

$$\hat{\epsilon}^* = y^* - \hat{y}^* = (\beta_0 + \beta_1 x^* + \epsilon^*) - (\hat{\beta}_0 + \hat{\beta}_1 x^*)$$
$$= (\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1) x^* + \epsilon^*$$

$$E[\hat{\epsilon}^*] = E[\beta_0 - \hat{\beta}_0] + E(\beta_1 - \hat{\beta}_1) + E(\epsilon^*)$$
$$= 0 + 0 + 0$$ Great!

what about variance?

$$V(\hat{\epsilon}^*) = \text{Var}([\beta_0 - \hat{\beta}_0] + [\beta_1 - \hat{\beta}_1]x^* + \epsilon^*)$$ $\{\epsilon^* \text{ indep of } \underset{\sim}{x}, \underset{\sim}{y}\}$

$$= \text{Var}(\quad) + \text{Var}(\epsilon^*)$$

$$= \left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{Sxx}\right]\sigma^2 + \sigma^2 = \left[1 + \frac{1}{n} + \frac{x^* - \bar{x}}{Sxx}\right]\sigma^2$$

Discuss: what makes $V(y^* - \hat{y}^*)$ big or small?
for fixed σ^2, how small can $V(y^* - \hat{y}^*)$ be?

Prediction interval: Let $SE(\hat{\epsilon}^*) = \left[1 + \frac{1}{n} + \frac{x^* - \bar{x}}{Sxx}\right]\hat{\sigma}$

Then $\hat{\epsilon}^* / SE(\hat{\epsilon}^*) \sim t_{n-2}$
$\hat{\epsilon}^* \pm SE(\hat{\epsilon}^*) \times t_{1-\alpha/2}$ is a $1-\alpha$ PI for $\hat{\epsilon}^*$

$y^* = \hat{y}^* + \hat{\epsilon}^*$, so $\hat{y}^* \pm SE(\hat{\epsilon}^*) \times t_{1-\alpha/2}$ is a $(1-\alpha)$ PI for y^*.