By signing below I pledge that I have not communicated with anyone about this exam other than the instructor.

Signature: ______________________________

1. Let W and Z be continuous random variables with joint density $p(w, z)$, conditional density $p(z|w)$ and marginal density $p(w)$.

   (a) Show that $\mathrm{E}[f(W)g(Z)|W = w] = f(w)\mathrm{E}[g(Z)|W = w]$.

   (b) Show that $\mathrm{E}[Z] = \mathrm{E}[\mathrm{E}[Z|W]]$.

   (c) Show that $\mathrm{Var}[Z] = \mathrm{Var}[\mathrm{E}[Z|W]] + \mathrm{E}[\mathrm{Var}[Z|W]]$.

2. Let X be a real-valued random variable and let a be a number.

   (a) Show that $\Pr(|X-a| \geq \epsilon) \leq \mathrm{E}[(X-a)^2]/\epsilon^2$.

   (b) Show that $\mathrm{E}[(X-a)^2] = \mathrm{Var}[X] + (\mathrm{E}[X]-a)^2$.

   (c) Let $\hat{\theta}$ be an estimator of θ. Show that $\Pr(|\hat{\theta}-\theta| \geq \epsilon) \leq (\mathrm{Var}[\hat{\theta}] + \mathrm{Bias}[\hat{\theta}]^2)/\epsilon^2$.

   (d) Now suppose for each natural number $n \in \mathbb{N}$, $\hat{\theta}_n$ is an estimator of θ. Find conditions on $\mathrm{Var}[\hat{\theta}_n]$ and $\mathrm{Bias}[\hat{\theta}_n]$ so that $\hat{\theta}_n$ is consistent for θ.

3. Let $(X_1, Y_1), \ldots, (X_n, Y_n) \sim$ i.i.d., where each (X_i, Y_i) are positive random variables with $\mathrm{E}[Y_i|X_i] = \mathrm{Var}[Y_i|X_i] = \theta X_i$, and $\mathrm{E}[|X_i|^r] < \infty$ for all values of r.

   (a) Let $\hat{\theta} = \frac{1}{n} \sum Y_i/X_i$. Find the expectation and variance of $\hat{\theta}$.

   (b) Show that $\hat{\theta}$ is a consistent estimator of θ.

   (c) Show that $\hat{\theta}$ is CAN and find its asymptotic variance, that is, the value V such that $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, V)$. State any theorems that you use.

4. Now consider the same setup as in the previous problem but assume that the conditional distribution of Y_i given X_i is Poisson(θX_i).

   (a) Write out the likelihood and log-likelihood for θ, and find the MLE $\hat{\theta}_{MLE}$.

   (b) Compute the expectation and variance of $\hat{\theta}_{MLE}$.

   (c) Make an argument that the variance of the MLE should converge to zero as $n \to \infty$ (you don't have to prove this), and thus that the MLE is consistent for estimating θ (without appealing to any general results about MLEs).

   (d) Find the value V_{MLE} such that $\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, V_{MLE})$ and compare it to the asymptotic variance V of the estimator obtained in the previous problem, $\hat{\theta} = \frac{1}{n} \sum Y_i / X_i$.

5. Write a 1-2 page essay on your favorite result or concept from the course.

   (a) Describe the concept and explain why you think it is useful or interesting.

   (b) State mathematically what the concept is, and provide an interpretation.

   (c) Give a specific example in which the concept is used.

   (d) Explain how the concept is connected to other concepts from the course.