1. Let $p_1, \ldots, p_m \sim$ i.i.d. $(1-\gamma)P_0 + \gamma P_1$, where P_0 is the uniform distribution on $[0,1]$ and P_1 is some other distribution, with CDF F_1.

   (a) Write out the probability that $p_1 < \alpha/m$ in terms of α, m, F_1, γ.

   (b) Write out the probability that the Bonferroni procedure rejects the global null hypothesis $H_0 : \gamma = 0$ at level α, that is, the probability that the smallest p-value is less than α/m.

   (c) Approximate the above probability using the approximation that $\log(1-x) \approx -x$ for small x.

   (d) Based on this approximation, evaluate if the probability of rejection is increasing or decreasing in α and in γ. Explain why your answers make sense.

   (e) What are conditions on F_1 that suggest (based on the approximation) that the Bonferroni procedure will have good power as $m \to \infty$?

2. Let $Y_j \sim N(\theta_j, 1)$ independently for $i = j, \ldots, m$ with $m = 100$. Using a Monte Carlo approximation, compute the probability of rejecting the global null "$H_0 : \theta_1 = \ldots = \theta_m = 0$" at level $\alpha = .05$ using a χ^2 test based on the statistic $\sum Y_j^2$, and also using Bonferroni's procedure and Fisher's procedure both based on z-tests for each θ_j.

   (a) $\theta_1, \ldots, \theta_m \sim$ i.i.d. $N(0, K/100)$ for $K \in \{1, 4, 16, 64\}$.

   (b) $\theta_1 = K$ and $\theta_2 = \ldots = \theta_m = 0$, where $K \in \{1, 3, 5, 7\}$.

3. Let $Y_j \sim N(\theta_j, 1)$ independently for $j = 1, \ldots, m$.

   (a) Find the MLEs of $\theta_1, \ldots, \theta_m$.

   (b) Compute the -2 log likelihood ratio statistic for evaluating the global null hypothesis $H : \theta_1 = \cdots = \theta_m = 0$, and describe a level-$\alpha$ testing procedure for evaluating H based on this statistic.

(c) Suppose we are worried that the Y_j's might be correlated. As an alternative to the likelihood ratio test, suppose we get a p-value $p_j = 2 \times \Phi(-|Y_j|)$ for testing $H_j : \theta_j = 0$ for each j.

i. Show that p_j has a uniform distribution if $\theta_j = 0$.

ii. Suppose we will reject the global null hypothesis H if any of the H_j are rejected at level α/m. Show that this procedure will control the global error rate to be less than or equal to α, even if the Y_j's are correlated.

4. Consider a model for m p-values, $p_1, \ldots, p_m \sim$ i.i.d. from a mixture distribution $P = (1-\gamma)P_0 + \gamma P_1$, where P_0 is uniform on $[0,1]$ and P_1 is a beta$(1, b)$ distribution.

(a) Propose a modified Benjamini-Hochberg procedure to control the FDR at level α, in the case that γ and b are known.

(b) Compute the mean and variance of p_1 in terms of γ and b. Using these calculations, propose moment-based estimators of γ and b using the observed values of $p_1, \ldots, p_m$. Based on this, propose a modified BH procedure that can be used if γ and b are not known.

(*) Compare the FDR and the number of discoveries made by the BH and modified BH procedure in a simulation study, for the case that $b \in \{1, 2, 4, 8\}$ and some interesting values of α and γ.