1. Let Y be a positive random variable. Use Jensen's inequality to relate
   (a) $E[Y^p]^{1/p}$ to $E[Y^q]^{1/q}$ for $p > q \geq 1$;
   (b) $E[1/Y]$ to $1/E[Y]$;
   (c) $\log E[Y]$ to $E[\log Y]$.

2. Let $(w_1, w_2, w_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3)$.
   (a) Derive the expected value and variance of w_j for $j \in \{1, 2, 3\}$.
   (b) Derive the covariance of w_1 and w_2, and explain intuitively the sign of the result.
   (c) Derive the variance of w_1 and of $w_1 + w_2$.
   (d) Derive the distribution of w_1 and of $w_1 + w_2$.

3. Covariance:
   (a) Show that if X and Y are independent then $\text{Cov}[X, Y] = 0$.
   (b) Show that if $X = a + bY$ then $\text{Cor}[X, Y] = 1$ or -1.
   (c) Let X_1, X_2, X_3 be three potentially correlated random variables.
      i. Compute $\text{Cov}[a_1 + b_1 X_1, a_2 + b_2 X_2]$.
      ii. Compute $\text{E}[X_1 + X_2 + X_3]$ and $\text{Var}[X_1 + X_2 + X_3]$ using the definintion of expectation and variance, and check that your answer matches the formula from class.

4. For this exercise it is useful to know that if $Z \sim N(0, 1)$, then $E[Z^3] = 0$ and $E[Z^4] = 3$. Let $Y_1 = Z + X_1$, $Y_2 = Z + X_2$ and $Y_3 = Z^2 + X_3$, where Z, X_1, X_2 and X_3 are independent standard normal random variables.
   (a) Compute the expectation and variance of Y_1, Y_2 and Y_3.
   (b) Compute the variance-covariance matrix of Y_1, Y_2 and Y_3.
   (c) Are Y_1 and Y_3 independent?

5. For two jointly distributed random variables X and Y and functions f and g, show that

   (a) $\mathrm{E}[\mathrm{E}[f(Y)|X]] = \mathrm{E}[f(Y)]$;

   (b) $\mathrm{E}[f(X)g(X,Y)|X] = f(X)\mathrm{E}[g(X,Y)|X]$.