# Social relations models for binary, ranked and ordinal relations

567 Statistical analysis of social networks

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# Adapting the normal SRM

Relational data are often valued, non-binary.

The standard SRM is appropriate for **normal** valued relational data.

**However,** relational data is often neither binary nor normal:

- Links can be **absent** or **continuous** within the same network.
  - meaning $Y_{i,j}$ is either 0 or some arbitrary real number;
  - communication networks (time spent communicating);
- Links can be **absent** or **ordinal** within the same network.
  - conflict networks (negative, positive or zero relation);
  - ranked nominations (friends are ranked, non-friends are not).

The SRM can be adapted via **ordinal probit models** to handle such data.

# Ordinal data

An **ordinal variable** has a meaningful ordering to the possible outcomes.

This is in contrast to a **categorical** (non-ordered) variable.

### Ordinal variable

- continuous: (all real numbers)
- discrete: (counts, ranks, etc.)

### Categorical variable

- non-orderable categories (religion, ethnicity, etc.)

# Binary variable as ordinal

The simplest ordinal variable is a binary random variable $Y \in \{0, 1\}$. Let

$$\Pr(Y = 1) = \theta$$
$$\Pr(Y = 0) = 1 - \theta$$

This model for $Y$ has the following **latent variable representation**:

$$\mu = \Phi^{-1}(\theta)$$
$$Z \sim N(\mu, 1)$$
$$Y = 1 \times (Z > 0)$$

Here, $\Phi^{-1}(\theta)$ is the $\theta$-quantile of the standard normal distribution.

## Probit representation

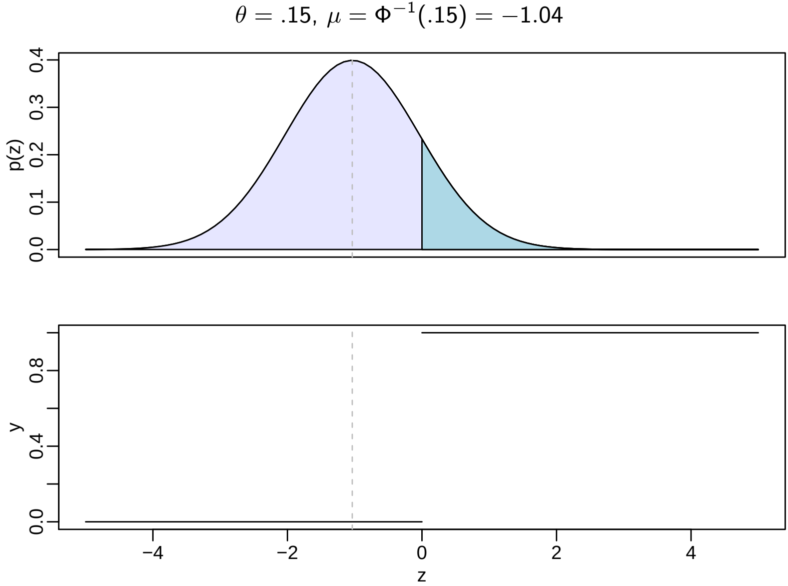
To confirm the representation, recall

- If $Z \sim N(\mu, 1)$ then $Z - \mu \sim N(0, 1)$;
- If $Z \sim N(\mu, 1)$ then $Z = \mu + \epsilon$, where $\epsilon \sim N(0, 1)$;
- If $\epsilon \sim N(0, 1)$ then $-\epsilon \sim N(0, 1)$.

Now do the calculation:

$$\begin{aligned} \Pr(Y = 1) &= \Pr(Z > 0) \\ &= \Pr(-Z < 0) \\ &= \Pr([-Z + \mu] < \mu) \\ &= \Pr(\epsilon < \mu) = \Phi(\mu) = \theta \end{aligned}$$

# Probit representation



# Probit regression

Now suppose we have

**binary data** $Y_1, \ldots, Y_n$

we want to relate to

**explanatory variables** $\mathbf{x}_1, \ldots, \mathbf{x}_n$

**Latent variable model:**

$$\epsilon_1, \ldots, \epsilon_n \sim \text{i.i.d. } N(0,1)$$
$$Z_i = \beta^T \mathbf{x}_i + \epsilon_i$$
$$Y_i = 1 \times (Z_i > 0)$$

Under this latent variable model, the $Y_i$'s are independent and

$$\begin{aligned}
\Pr(Y_i = 1) &= \Pr(Z_i > 0) \\
&= \Pr(\beta^T \mathbf{x}_i + \epsilon_i > 0) \\
&= \Pr(-\epsilon_i < \beta^T \mathbf{x}_i) \\
&= \Pr(\epsilon_i < \beta^T \mathbf{x}_i) = \Phi(\beta^T \mathbf{x}_i)
\end{aligned}$$

# Probit regression

This latent variable model is exactly the same as **probit regression**:

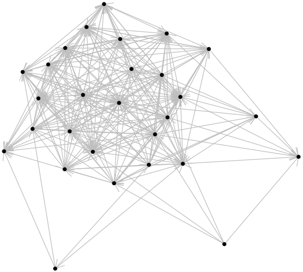
$$\Pr(Y_1 = y_1, \ldots, Y_n = y_n | \mathbf{x}_1, \ldots, \mathbf{x}_n) = \prod_{i=1}^{n} \Phi(\beta^T \mathbf{x}_i)^{y_i} [1 - \Phi(\beta^T \mathbf{x}_i)]^{1-y_i}$$

Compare to **logistic regression**:

$$\Pr(Y_1 = y_1, \ldots, Y_n = y_n | \mathbf{x}_1, \ldots, \mathbf{x}_n) = \prod_{i=1}^{n} \left(\frac{e^{\beta^T x_i}}{1+e^{\beta^T x_i}}\right)^{y_i} \left(\frac{1}{1+e^{\beta^T x_i}}\right)^{1-y_i}$$

In fact, logistic regression has a latent variable representation also:

$$\begin{aligned} \epsilon_1, \ldots, \epsilon_n &\sim \text{i.i.d. } L(0,1) \\ Z_i &= \beta^T \mathbf{x}_i + \epsilon_i \\ Y_i &= 1 \times (Z_i > 0) \end{aligned}$$



**Sheep dominance data**

- counts of dominance encounters between 28 male sheep;
- age of each sheep, in years.

Let's examine the effect of age difference on dominance.

# Sheep dominance data

Let

- `dom[i,j]` = indicator that $i$ has dominated $j$ at least once;
- `aged[i,j]` = $\text{age}_i$- $\text{age}_j$.

```
mean(aged[dom==1],na.rm=TRUE )

## [1] 2.184

mean(aged[dom==0],na.rm=TRUE )

## [1] -1.079051
```

```
summary(  glm(dom~aged,family=binomial)  )$coef

##                Estimate Std. Error   z value     Pr(>|z|)
## (Intercept) -0.8357694 0.08729001 -9.574628 1.022245e-21
## aged         0.2260270 0.02317097  9.754752 1.760303e-22

summary(  glm(dom~aged,family=binomial(link=probit))  )$coef

##                Estimate Std. Error   z value     Pr(>|z|)
## (Intercept) -0.5138332 0.05111585 -10.05232 8.972511e-24
## aged         0.1385751 0.01337034  10.36437 3.601223e-25
```

**Comparing logistic and probit regression output:**

- $\hat{\beta}$'s are on different scales;
- inference ($z$-scores) are typically similar.

Both models are **binary regression models** of the form:

$$\Pr(Y_1 = y_1, \ldots, Y_n = y_n | \mathbf{x}_1, \ldots, \mathbf{x}_n) = \prod_{i=1}^{n} g(\beta^T \mathbf{x}_i)^{y_i} [1 - g(\beta^T \mathbf{x}_i)]^{1-y_i},$$

where $g^{-1}$ is called the **inverse-link function**.

**Link functions:**

- **logistic regression:** $g^{-1}(\mu) = \exp(\mu)/[1 + \exp(\mu)]$;
- **probit regression:** $g^{-1}(\mu) = \Phi(\mu)$;
- **other binary regression:** $g^{-1}(\mu)$ is a strictly increasing function.

# SRM for binary relational data

Recall the latent variable representation of probit regression:

$$Z_{i,j} = \beta^T \mathbf{x}_{i,j} + \epsilon_{i,j}$$
$$Y_{i,j} = 1 \times (Z_{i,j} > 0)$$

We could estimate $\boldsymbol{\beta}$ with a probit regression analysis, but what about network dependence in the data?

Could there be

- across-row heterogeneity/within row correlation?
- across-column heterogeneity/within column correlation?
- within dyad correlation?

# SRM for binary relational data

Recall the latent variable representation of probit regression:

$$
\begin{aligned}
Z_{i,j} &= \beta^T \mathbf{x}_{i,j} + \epsilon_{i,j} \\
Y_{i,j} &= 1 \times (Z_{i,j} > 0)
\end{aligned}
$$

Dependence on the $Y$-scale can be induced by dependence on the $Z$-scale:

$$
\begin{aligned}
\epsilon_{i,j} &= a_i + b_j + e_{i,j} \\
\{(a_1, b_1), \dots, (a_n, b_n)\} &\sim \text{i.i.d. } N(0, \Sigma_{ab}) \\
\{(e_{i,j}, e_{j,i}) : i \neq j\} &\sim \text{i.i.d. } N(0, \Sigma_e)
\end{aligned}
$$

$$
\Sigma_{ab} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} \quad \Sigma_e = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}
$$

Note that the variance of $e_{i,j}$ is fixed at 1.

The scales of $e_{i,j}$ and $\boldsymbol{\beta}$ are not separately identifiable.

# SRM for binary relational data

**SRM probit model:**

$$
\begin{aligned}
Y_{i,j} &= 1 \times (Z_{i,j} > 0) \\
Z_{i,j} &= \beta^T \mathbf{x}_{i,j} + \epsilon_{i,j} \\
\epsilon_{i,j} &= a_i + b_j + e_{i,j},
\end{aligned}
$$

and $\{a_i, b_j, e_{i,j}\}$ are random effects as described previously.

**Parameter estimation**:

- The likelihood can't be expressed in closed form;
- Bayesian parameter estimates can be obtained via MCMC.

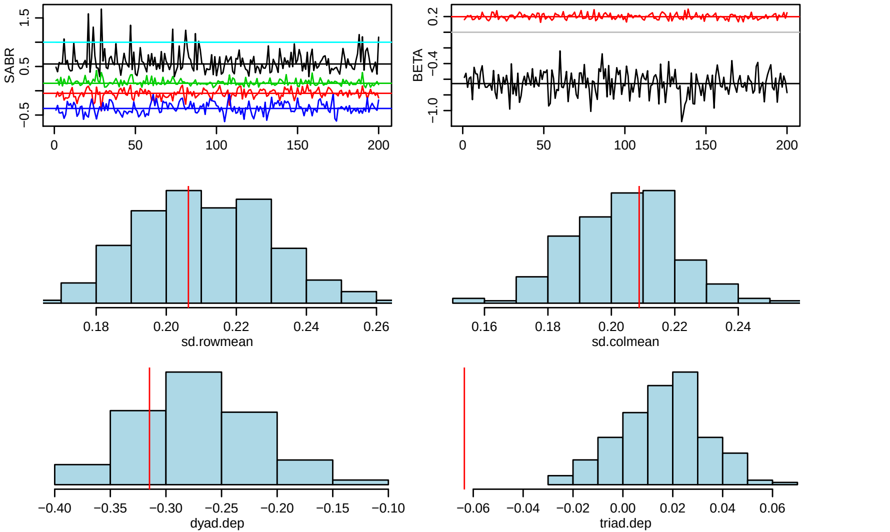
The latter is provided in the package `amen`.

# Binary SRM in amen

```
ame_bin                    package:amen                    R Documentation

AME fit for binary relational data

Description:

     An MCMC routine providing a fit to an additive and multiplicative
     effects (AME) regression model for binary relational data

Usage:

     ame_bin(Y, X, rvar = TRUE, cvar = TRUE, dcor = TRUE, R = 0, seed = 1, nscan = 50000, b

Arguments:

       Y: an n x n square relational matrix

       X: an n x n x p array of covariates

    rvar: logical: fit row random effects?

    cvar: logical: fit column random effects?

    dcor: logical: fit a dyadic correlation?
```

# SRM probit in amen

```
XD<-outer(X,X,"-")

fit_ame_bin<-ame(YB,XD,model="bin")
```



# Posterior analysis

The `summary` command provides a canned summary of the fitted model:

```
summary(fit_ame_bin)

##
## beta:
##              pmean   psd z-stat p-val
## intercept -0.654 0.149 -4.374     0
## .dyad      0.200 0.034  5.835     0
##
## Sigma_ab pmean:
##        a      b
## a  0.603 -0.057
## b -0.057  0.172
##
## rho pmean:
##  -0.359
```

# Posterior analysis

Confidence intervals can be obtained via the `quantile` command:

```
apply( fit_ame_bin$BETA , 2 , quantile, prob=c(.025,.5, .975))

##        intercept     .dyad
## 2.5%  -0.9423416 0.1398794
## 50%   -0.6554213 0.1985873
## 97.5% -0.4028070 0.2637526

apply( fit_ame_bin$SABR , 2 , quantile, prob=c(.025,.5, .975))

##              va         cab         vb        rho ve
## 2.5%  0.3111828 -0.26359446 0.07500074 -0.5818362  1
## 50%   0.5511227 -0.05253038 0.15423887 -0.3646930  1
## 97.5% 1.2468605  0.09875416 0.33123579 -0.1495458  1
```

# Posterior inference

**Summary of results:**

Strong evidence of an age effect:

- Older sheep dominate younger ones.

Row and column heterogeneity is not substantial:

- Compare $\hat{\sigma}_a^2 = 0.55$ and $\hat{\sigma}_b^2 = 0.15$ to the error variance $\sigma_e^2 = 1$.

There is evidence that dominance tends to go one-way in a dyad:

- The 95% CI for $\rho$ is $(-0.58, -0.15)$.

# Goodness of fit

The default `amen` plot provides four goodness of fit plots:

Outdegree distribution: Comparing outdegree distribution to simulated;

Indegree distribution: Comparing indegree distribution to simulated;

Reciprocity: Comparing fraction reciprocated ties to simulated fraction;

Transitivity: Comparing number of triangles to simulated number.

These statistics are computed with the commands

```
t_degree
t_recip
t_trans
```

# Model comparison

Let's use these statistics to evaluate the fit of three models:

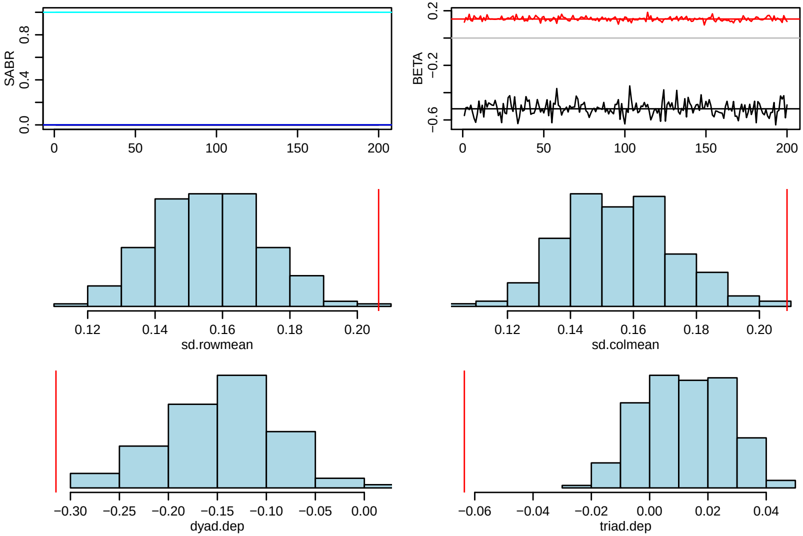
fit_bin_000: the probit model, fix $\sigma_a^2 = \sigma_b^2 = \rho = 0$.

fit_bin_001: dyadic correlation $\rho$ estimated, $\sigma_a^2 = \sigma_b^2 = 0$.

fit_bin_111: full SRM covariance model.

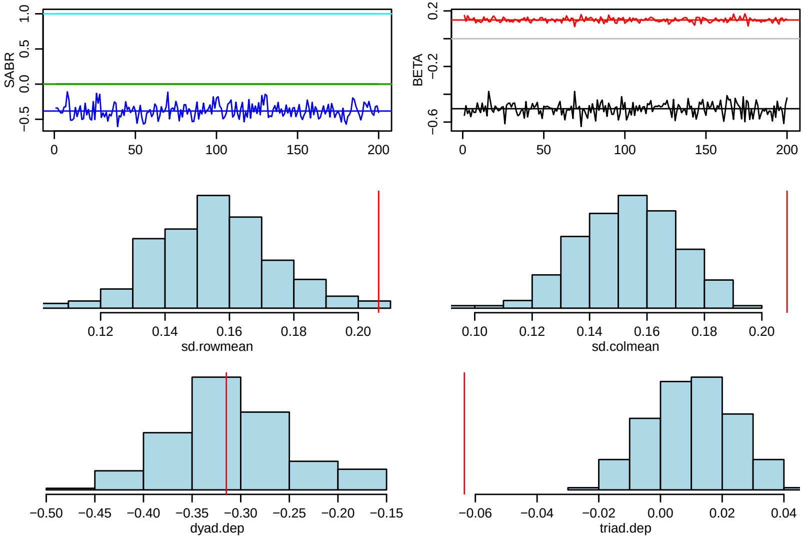
# Model comparison

```
fit_bin_000<-ame(YB,XD,model="bin",rvar=FALSE,cvar=FALSE,dcor=FALSE)
```



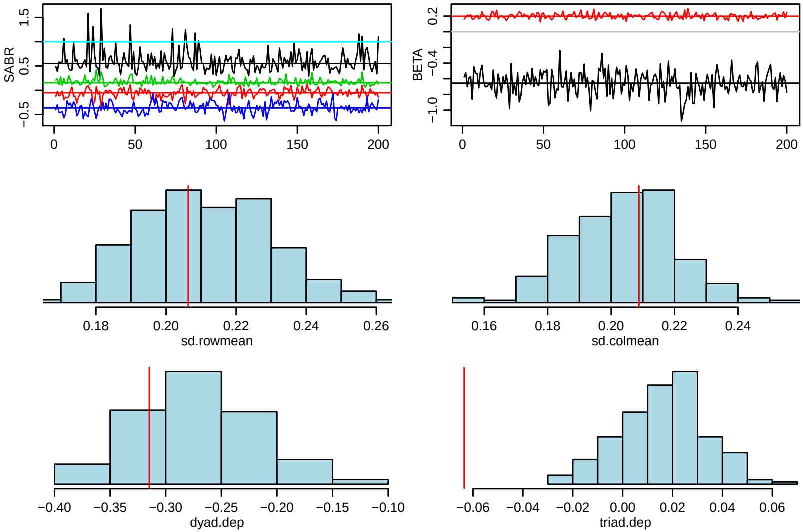
# Model comparison

```
fit_bin_001<-ame(YB,XD,model="bin",rvar=FALSE,cvar=FALSE,dcor=TRUE)
```



# Model comparison

```
fit_bin_111<-ame(YB,XD,model="bin")
```



# Model comparison

The SRM model `fit_bin_111` looks best based on these statistics, except possibly for the transitivity statistic (more on that soon).

# Ordered probit models for ordinal data

The original sheep data consists of counts:

```
YV[1:8,1:8]

##      V1 V2 V3 V4 V5 V6 V7 V8
## [1,] NA  0  0  0  0  0  0  1
## [2,]  0 NA  0  0  5  2  1  0
## [3,]  0  0 NA  0  7  4  0  0
## [4,]  0  0  8 NA  0  0  0  0
## [5,]  0  0  0  0 NA  1  0  0
## [6,]  0  0  0  7  0 NA  0  0
## [7,]  0  0  1  0  0  0 NA  0
## [8,]  0  1  1  4  5  3  0 NA
```

Ideally, this information should be taken into consideration.

- In principle, throwing away information is not efficient.
- Effects of some covariates might distinguish high values of the relation.

**Latent variable model:**

$$Z_{i,j} = \beta^T x_{i,j} + \epsilon_{i,j}$$
$$\epsilon_{i,j} = a_i + b_j + e_{i,j}$$

**Binary probit:**

$$Y_{i,j} = \begin{cases} 0 & \text{if } Z_{i,j} < 0 \\ 1 & \text{if } Z_{i,j} > 0 \end{cases}$$

What if $Y_{i,j} \in \{y_1, \ldots, y_K\} = \mathcal{Y}$?

Here, $\mathcal{Y}$ is an ordered, countable set of possible values of $Y_{i,j}$.

# Ordered probit models for ordinal data

**Latent variable model:**

$$Z_{i,j} = \beta^T x_{i,j} + \epsilon_{i,j}$$
$$\epsilon_{i,j} = a_i + b_j + e_{i,j}$$

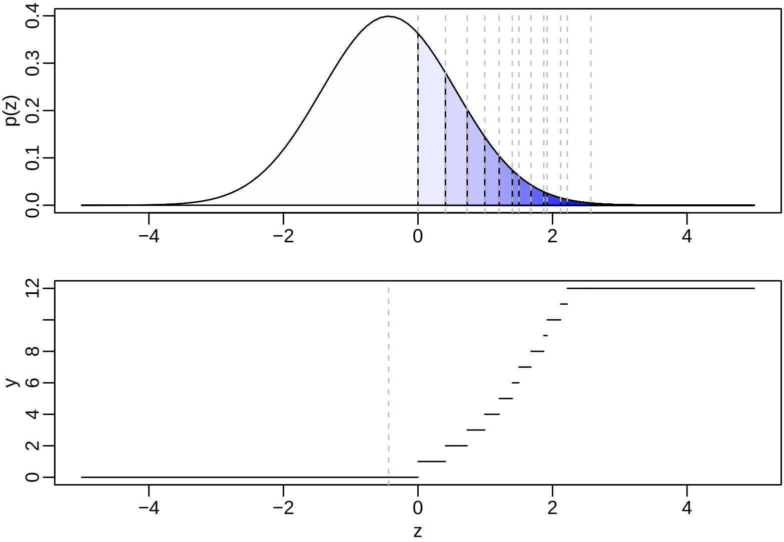
**Ordered probit:**

$$Y_{i,j} = \begin{cases} y_1 & \text{if } Z_{i,j} \in & (-\infty, c_1) \\ y_2 & \text{if } Z_{i,j} \in & (c_1, c_2) \\ & \vdots & \\ y_{K-1} & \text{if } Z_{i,j} \in & (c_{K-2}, c_{K-1}) \\ y_K & \text{if } Z_{i,j} \in & (c_{K-1}, \infty) \end{cases}$$

# Ordered probit models for ordinal data

```
table(c(YB))

##
##   0   1
## 506 250

table(c(YV))

##
##   0   1   2   3   4   5   6   7   8   9  10  11  12
## 506 100  59  34  20  13   5   7   5   1   3   1   2

round( table(c(YV))/sum(table(c(YV)))  ,3 )

##
##     0     1     2     3     4     5     6     7     8     9    10    11
## 0.669 0.132 0.078 0.045 0.026 0.017 0.007 0.009 0.007 0.001 0.004 0.001
##    12
## 0.003
```

## Link function for ordered probit

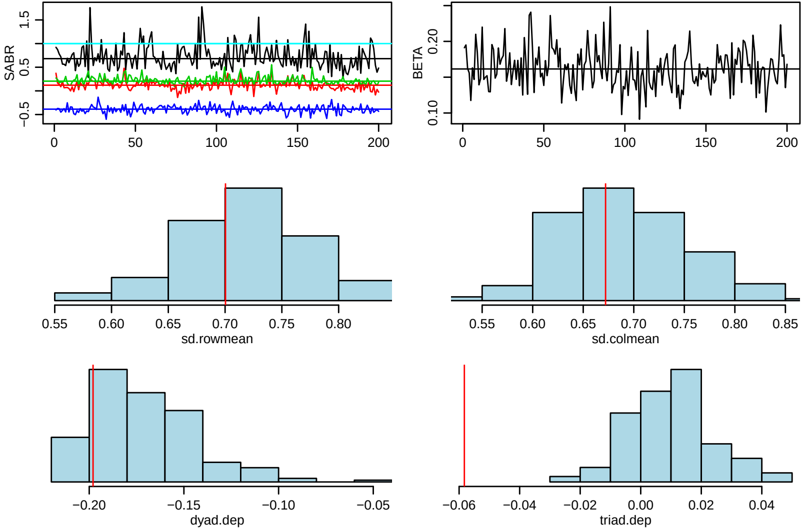


# Fitting ordered probit models in amen

```
ame_ord                  package:amen                  R Documentation

AME fit for ordinal relational data

Description:

     An MCMC routine providing a fit to an additive and multiplicative
     effects (AME) regression model for ordinal relational data

Usage:

     ame_ord(Y, X, rvar = TRUE, cvar = TRUE, dcor = TRUE, R = 0, seed = 1, nscan
 = 50000, burn = 500, odens = 25, plot = TRUE, print = TRUE)

Arguments:

       Y: an n x n square relational matrix

       X: an n x n x p array of covariates

    rvar: logical: fit row random effects?

    cvar: logical: fit column random effects?

    dcor: logical: fit a dyadic correlation?
```

# SRM probit in amen

```r
fit_ame_ord<-ame(YV,XD,model="ord")
```



**Summary:**

```
summary(fit_ame_ord)

##
## beta:
##        pmean   psd z-stat p-val
## .dyad 0.162 0.028   5.72     0
##
## Sigma_ab pmean:
##       a     b
## a 0.731 0.125
## b 0.125 0.224
##
## rho pmean:
##  -0.388
```

# Posterior analysis

## Confidence intervals

```
apply( fit_ame_ord$BETA , 2 , quantile, prob=c(.025,.5, .975))

##           .dyad
## 2.5%  0.1138031
## 50%   0.1615003
## 97.5% 0.2231967

apply( fit_ame_ord$SABR , 2 , quantile, prob=c(.025,.5, .975))

##              va         cab        vb        rho ve
## 2.5%  0.3728079 -0.06367146 0.1060518 -0.5601718  1
## 50%   0.6811580  0.12026448 0.2057822 -0.3863772  1
## 97.5% 1.4126096  0.37453941 0.4239469 -0.2343322  1
```

These results are very similar to those obtained from the binary probit analysis using the dichotomized data.

## The role of covariance

**Regression modeling:**

$$
\begin{aligned}
Y_{i,j} &= \boldsymbol{\beta}^T \mathbf{x}_{i,j} + \epsilon_{ij} \\
\mathbf{Y} &= \langle \mathbf{X}, \boldsymbol{\beta} \rangle + \mathbf{E}
\end{aligned}
$$

**OLS estimation:**

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

**Precision of OLS estimators:**

Let $\mathbf{C} = \text{Cov}[\mathbf{E}]$

$$
\begin{aligned}
\text{Cov}[\hat{\boldsymbol{\beta}}] &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{C}\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1} \\
&= (\mathbf{X}^T\mathbf{X})^{-1}\sigma^2 \text{ if } \mathbf{C} = \sigma^2\mathbf{I}
\end{aligned}
$$

For networks and relational data, typically $\mathbf{C} \neq \sigma^2\mathbf{I}$.

Accurate standard errors can't be obtained unless we know/estimate $\text{Cov}[\mathbf{E}]$.

# Justifying the SRM

The social relations covariance model can by a symmetry principle.

**Exchangeability:**

1. randomly sample $n$ individuals from a population;
2. observe $\epsilon_{i,j}$ = directed relation between $i$th and $j$th person.

Consider a probability model $\Pr(\mathbf{E})$ for the possible outcomes of $\mathbf{E} = \{\epsilon_{i,j}\}$

$$\Pr\left(\begin{pmatrix} NA & -0.94 & 0.15 \\ -0.7 & NA & 0.63 \\ 0.63 & -0.42 & NA \end{pmatrix}\right) = ?$$

$$\Pr\left(\begin{pmatrix} NA & 0.63 & -0.42 \\ 0.15 & NA & -0.94 \\ 0.63 & -0.7 & NA \end{pmatrix}\right) = ?$$

Note the second matrix is the same as the first with (1,2,3) relabled as (2,3,1).

# Justifying the SRM

Let $\pi$ be some permutation of $\{1, \ldots, n\}$.

$$\mathbf{E} = \{\epsilon_{i,j} : i \neq j\}$$
$$\mathbf{E}_\pi = \{\epsilon_{\pi_i,\pi_j} : i \neq j\}$$

**Exchangeability:** A probability distribution $\Pr(\mathbf{E})$ is **exchangeable** if

$$\Pr(\mathbf{E}) = \Pr(\mathbf{E}_\pi)$$

for all $\mathbf{E}$ and permutations $\pi$.

Exchangeability can be justified by

- random sampling of nodes from a population;
- symmetry in the uncertainty in the $\epsilon_{i,j}$'s.

**Interesting result:**
Suppose

1. $\epsilon_{i,j}$'s are normal and mean zero;
2. our probability for $\mathbf{E} = \{\epsilon_{i,j}\}$ is exchangeable.

Then

$$\begin{aligned} \epsilon_{i,j} &= a_i + b_j + e_{i,j} \\ \{(a_1, b_1), \ldots, (a_n, b_n)\} &\sim \text{i.i.d. } N(0, \Sigma_{ab}) \\ \{(e_{i,j}, e_{j,i}) : i \neq j\} &\sim \text{i.i.d. } N(0, \Sigma_e) \end{aligned}$$

for some $\Sigma_{ab}$ and $\Sigma_e$ (Li and Loken, 2002).

**Interpretation:**

$$\text{normality} + \text{exchnageability} \Rightarrow \text{SRM}$$

# Justifying the SRM

There is a stronger result than this: If **E** is exchangeable, then

$$\begin{aligned} \epsilon_{i,j} &= a_i + b_j + e_{i,j} \\ \mathrm{Cov}[(a_i, b_i)] &= \Sigma_{ab} \\ \mathrm{Cov}[(e_{i,j}, e_{j,i})] &= \Sigma_e \end{aligned}$$

for some $\Sigma_{ab}, \Sigma_e$.

**Interpretation:**

exchangeability $\Rightarrow$ SRM covariance structure

**Implication:**

$$\mathrm{Cov}[\hat{\boldsymbol{\beta}}] = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{C}\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1},$$

where $\mathbf{C} = \mathrm{Cov}[(\epsilon_{i,j}, \epsilon_{j,i})]$.

Under exchangability, the SRM provides appropriate SEs and CIs for $\boldsymbol{\beta}$.