Course Overview

594 Multiway Data Analysis

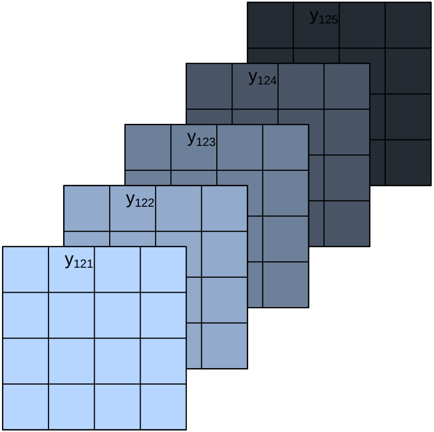
Peter Hoff

Statistics, University of Washington

Array-valued data

$y_{i,j,k} =$

- jth measurement on ith subject under condition k (psychometrics)
- sample mean of variable i for group j in state k (cross-classified data)
- type-k relationship between i and j (multivariate relational data)
- time-k relationship between i and j (dynamic relational data)



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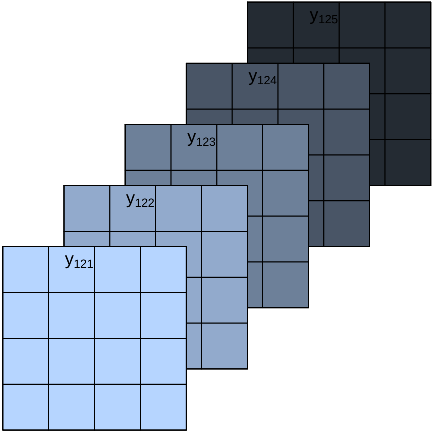

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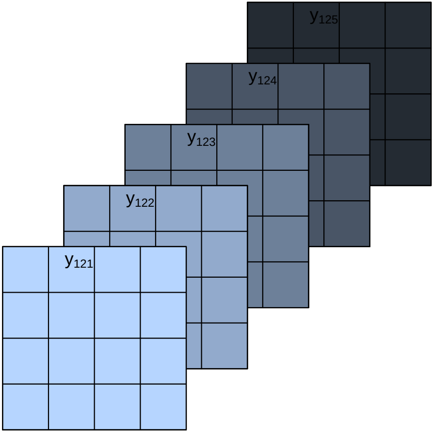

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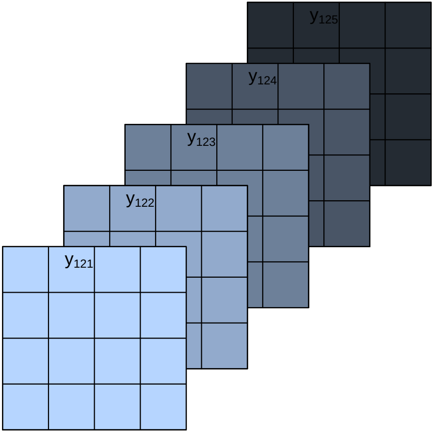

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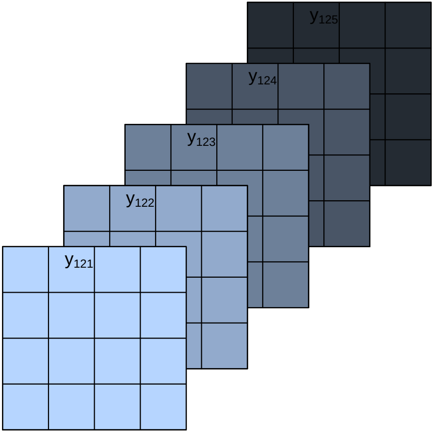

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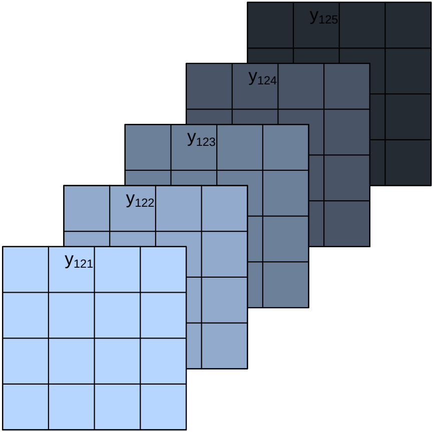
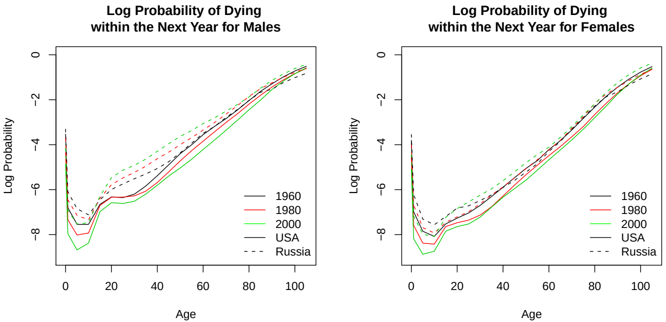


Mortality tables

(Joint work with Bailey Fosdick)



Human Mortality Database: (log) probability of dying in the next year

- 38 countries
- 23 age levels (0, 1 and then every 5 years)
- 9 times periods (1960 to 2000 every 5 years)
- 2 sexes

A $39 \times 23 \times 9 \times 2$-dimensional table.

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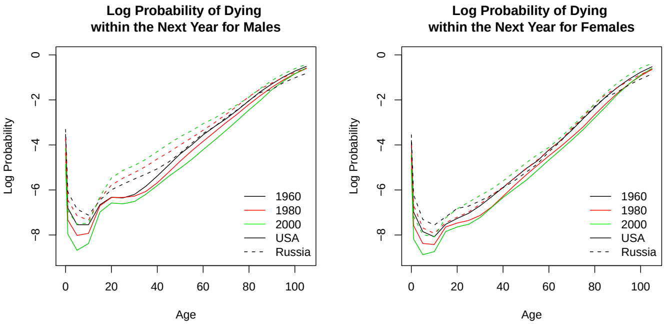

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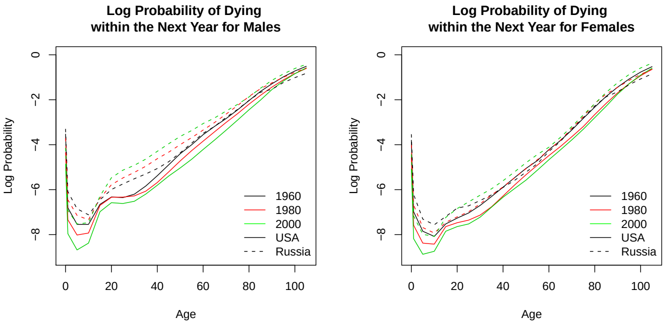

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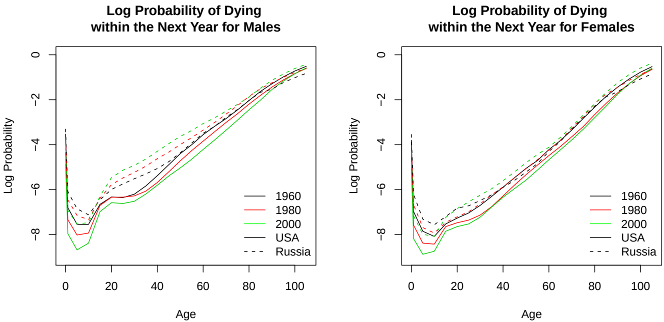

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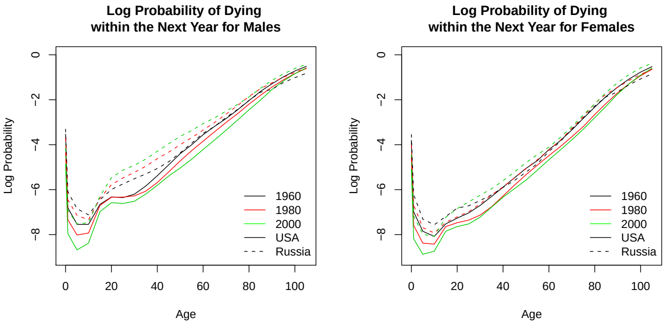

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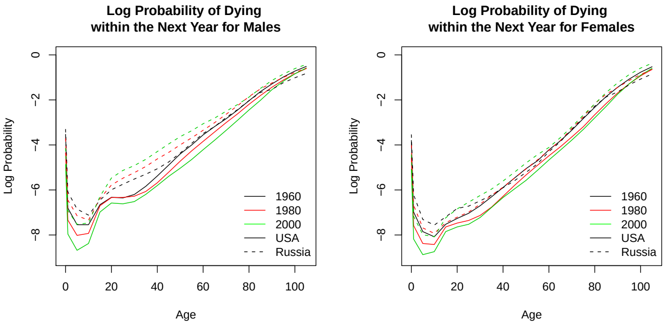

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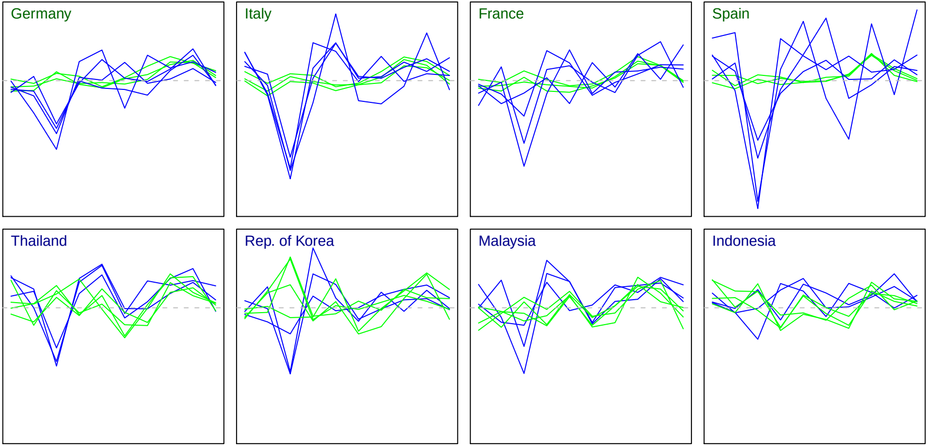
Longitudinal trade

Yearly change in log exports (2000 dollars) : $\mathbf{Y} = \{y_{i,j,k,l}\} \in \mathbb{R}^{30\times 30\times 6\times 10}$

- $i \in \{1,\ldots,30\}$ indexes exporting nation
- $j \in \{1,\ldots,30\}$ indexes importing nation
- $k \in \{1,\ldots,6\}$ indexes commodity
- $l \in \{1,\ldots,10\}$ indexes year



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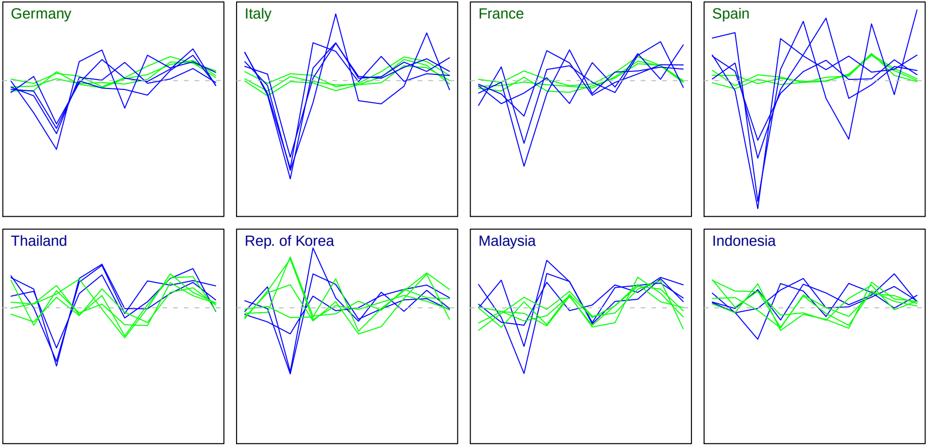

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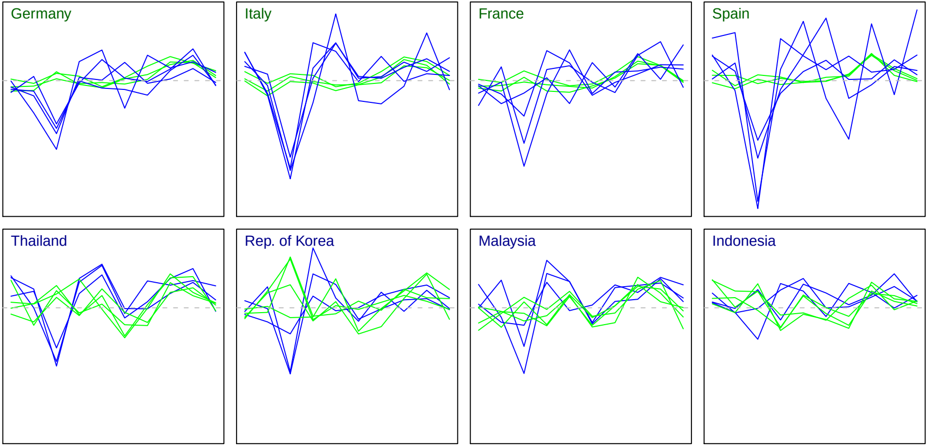

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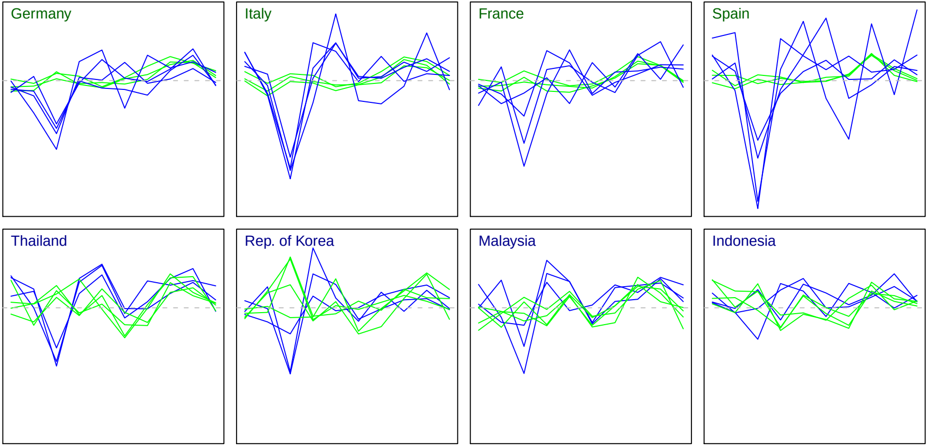

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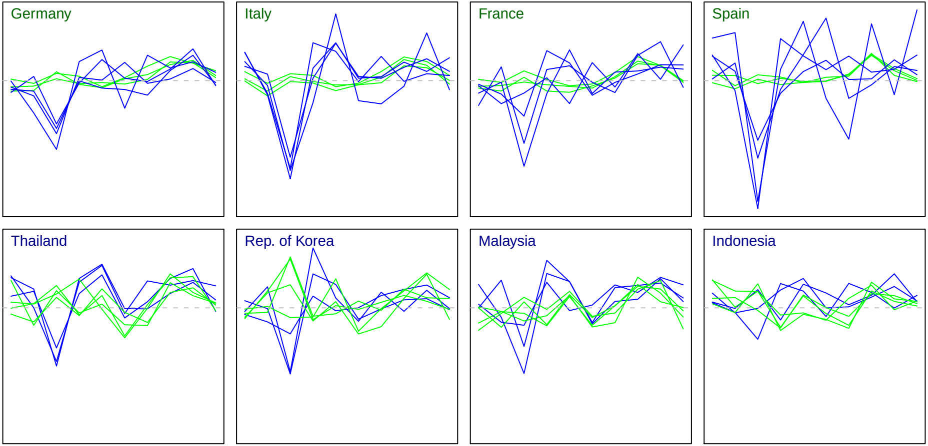
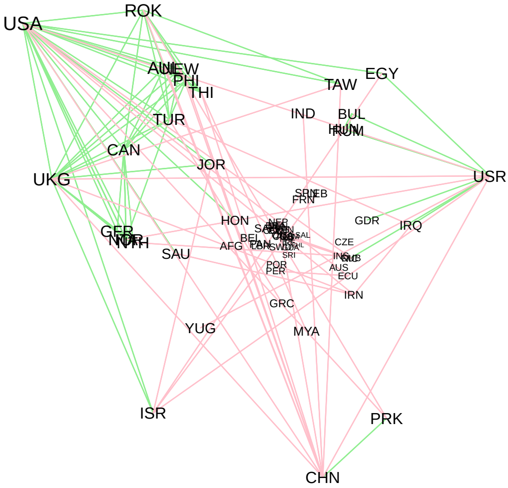


Longitudinal network example

Cold war cooperation and conflict

- 66 countries
- 8 years (1950,1955,...,1980,1985)
- $y_{i,j,t}$ =relation between i, j in year t
- also have data on gdp and polity



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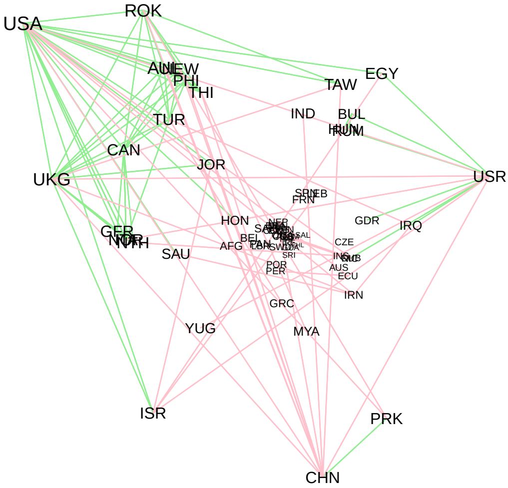

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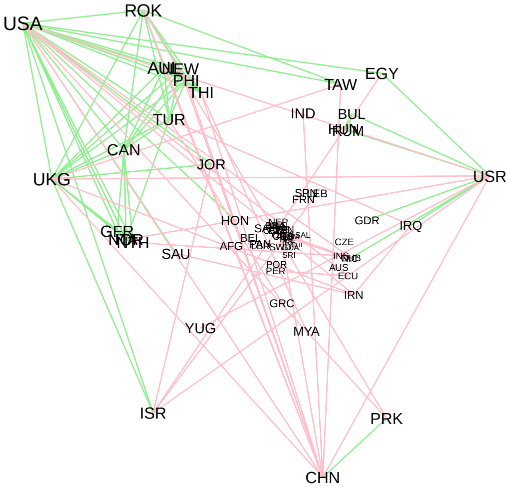

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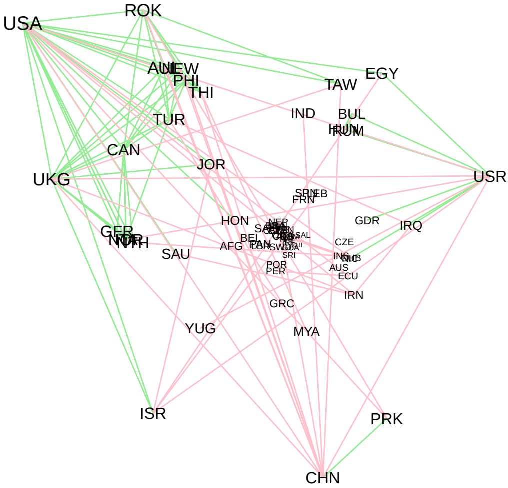

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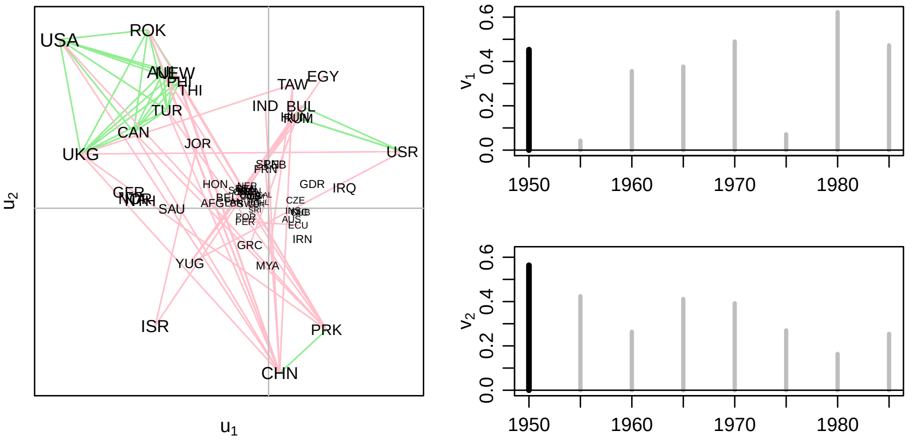

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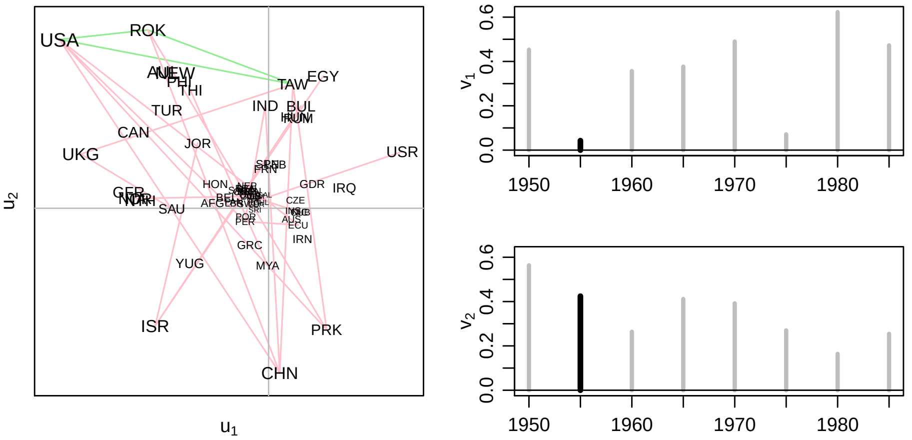

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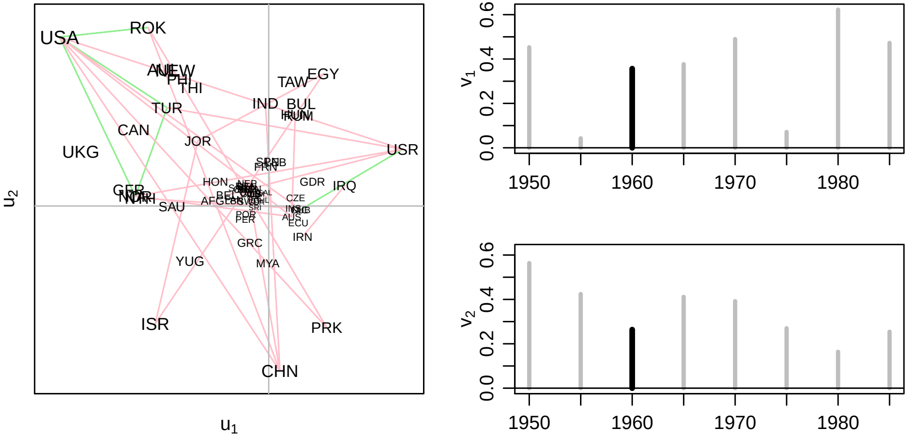

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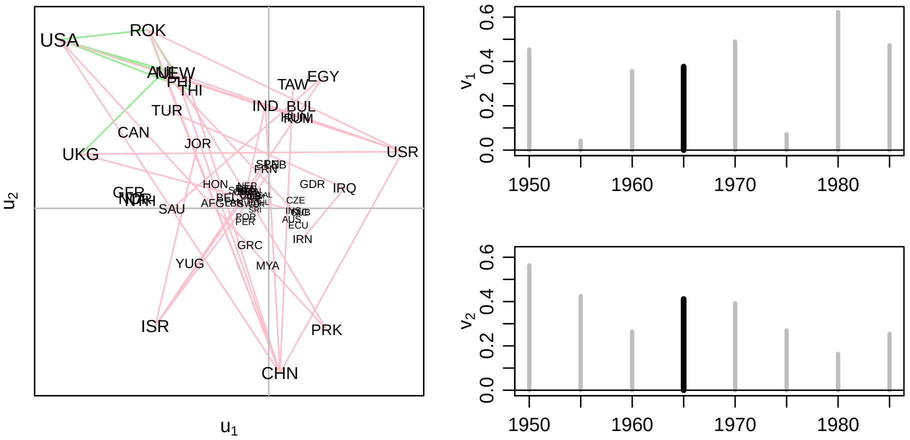

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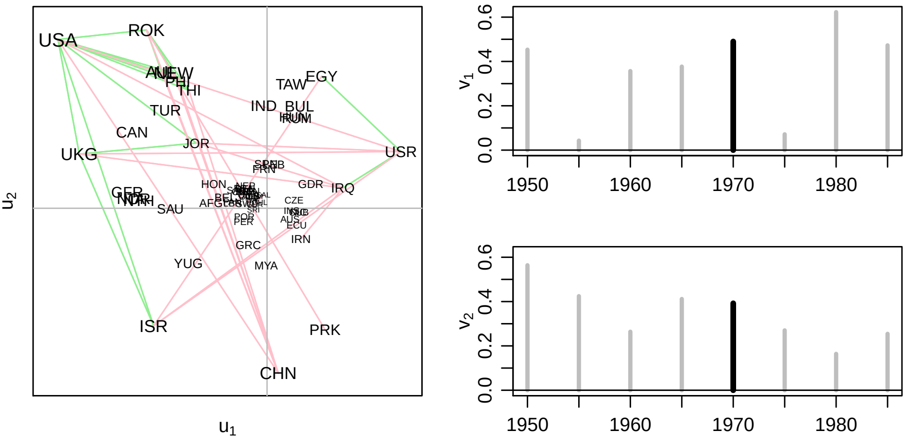

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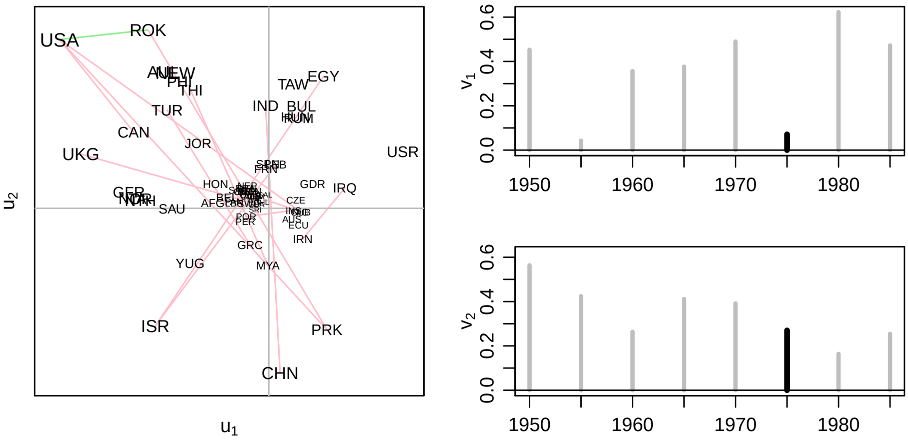

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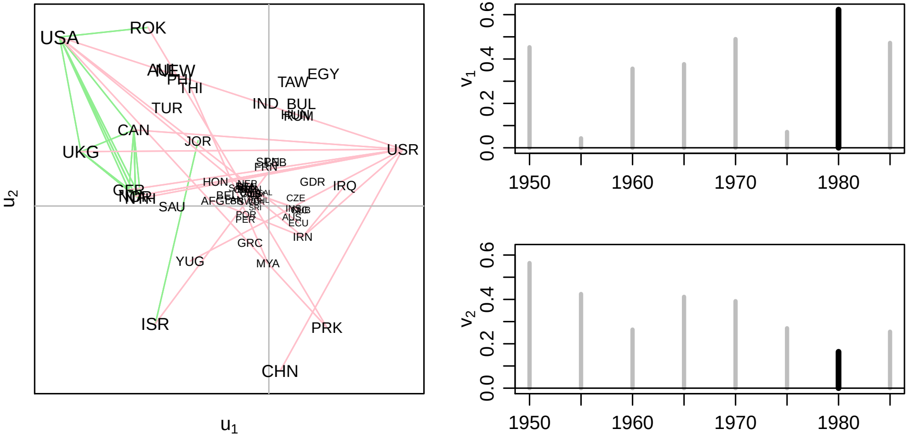

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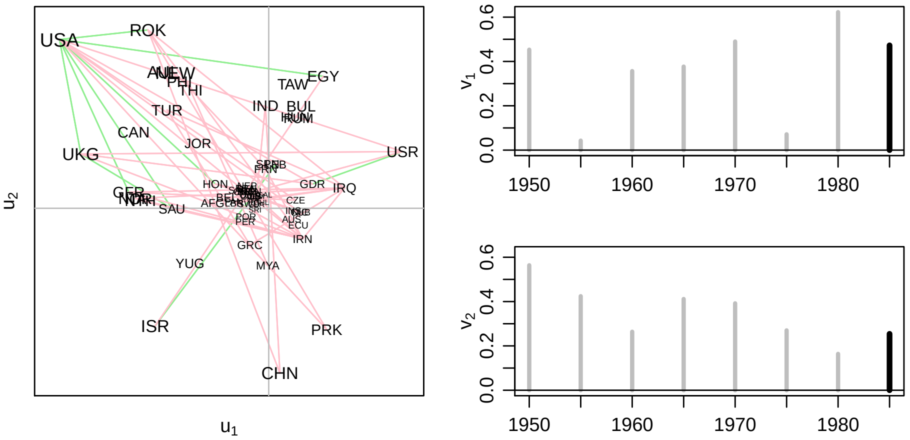

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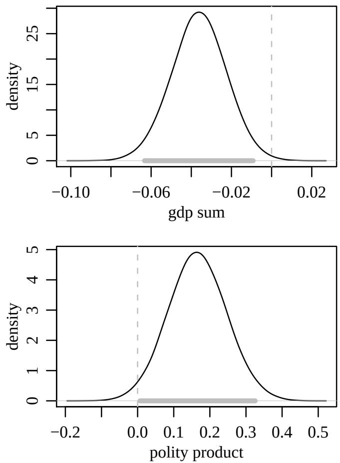

Deep interactions

(Joint work with Alex Volfovsky)

Consider the usual three-factor "ANOVA decomposition" model:

$$\begin{aligned} y_{i,j,k,l} &= \mu_{j,k,l} + \epsilon_{i,j,k,l} \\ &= \mu + \Big[a_j + b_k + c_l\Big] + \Big[(ab)_{j,k} + (ac)_{j,l} + (bc)_{k,l}\Big] + \Big[(abc)_{j,k,l}\Big] + \epsilon_{i,j,k,l} \end{aligned}$$

Parameters are vectors, matrices and arrays based on three index sets.

Estimation methods:

- OLS estimation
- OLS with reduced model
- Bayes/penalized estimation

Parameters in models for contingency table data are similar.

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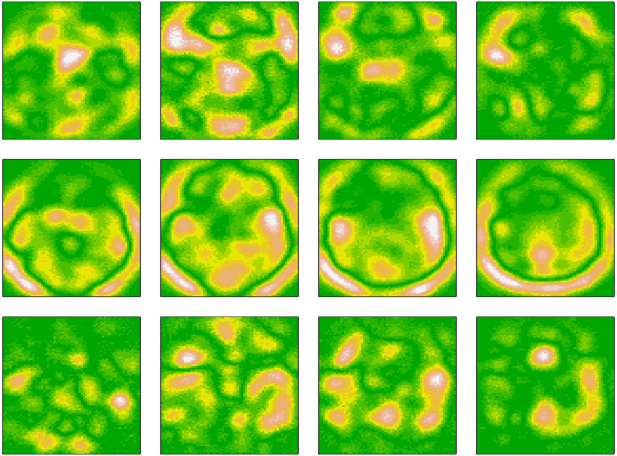
Neuropsychology imaging data

Blood flow in the brain: $\mathbf{Y} = \{y_{i,j,k,l,m,n}\}$

- (i, j, k) index spatial location
- l indexes time
- m indexes treatment/stimulus
- n indexes subject



Data models and Probability models

$$\mathbf{Y} = \mathbf{\Theta} + \mathbf{E}$$

$\mathbf{\Theta}$ contains the "main features" we hope to recover,

$\mathbf{E}$ is "patternless."

Data model:

- $\mathbf{\Theta}$ represents main features of the data
- $\mathbf{E}$ represents "residual" features
- Goal is to compactly represent/summarize/describe the data

Probability model:

- $\mathbf{\Theta}$ represents a fixed process or population parameter
- $\mathbf{E}$ represents measurement error or sample-to-sample variation
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Mean models

"$\boldsymbol{\Theta}$ = main features" means "$\boldsymbol{\Theta}$ = low dimensional"

$$\text{dimension}(\boldsymbol{\Theta}) < \text{ambient dimension}(\boldsymbol{\Theta}) = \text{dimension}(\mathbf{Y})$$

Modeling possibilities:

- regression: $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{B}, \mathbf{X})$, $\mathbf{X}$ known.
- replication: $\boldsymbol{\Theta} = \boldsymbol{\mu} \circ \mathbf{1}$
- rank reduction/factor model: $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{A}, \mathbf{B})$

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PC decomposition

$$\mathbf{Y} = \mathbf{\Theta} + \mathbf{E}$$

Matrix decomposition: If $\mathbf{\Theta}$ is a rank-R matrix, then

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Tucker decomposition and HOSVD

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Decompose $\mathbf{\Theta}$ using the Tucker decomposition (Tucker 1964,1966):

$$\begin{aligned} \theta_{i,j,k} &= \sum_{r=1}^{R} \sum_{s=1}^{S} \sum_{t=1}^{T} z_{r,s,t} a_{i,r} b_{j,r} c_{k,r} \\ \mathbf{\Theta} &= \mathbf{Z} \times \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \end{aligned}$$

- $\mathbf{Z}$ is the $R \times S \times T$ *core array*
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Covariance modeling

Matrix-variate data:

Additive random effects

$$y_{i,j} = \theta_{i,j} + a_i + b_j + \epsilon_{i,j}$$
$$\{a_i\} \sim \text{i.i.d.} N(0, \sigma_a^2)$$
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$$\{\epsilon_{i,j}\} \sim \text{i.i.d.} N(0, \sigma_\epsilon^2)$$

Multiplicative factors

$$y_{i,j} = \theta_{i,j} + \mathbf{u}_i^T \mathbf{v}_j + \epsilon_{i,j}$$

Multilinear transformations

$$\mathbf{Y} = \mathbf{\Theta} + \mathbf{AEB}^T$$

In this case, we have $\text{Cov}[\mathbf{Y}] = \mathbf{AA}^T \circ \mathbf{BB}^T \equiv \Sigma_1 \circ \Sigma_2$.

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Statistical analysis of matrix data

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   1.1 singular value decomposition
   1.2 ANOVA decomposition
2. Error models
   2.1 additive random effects
   2.2 multiplicative effects and the matrix normal model

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2. CANDECOMP/PARAFAC
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1. Alternating least squares
2. Penalized likelihoods
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Covariance models and estimation

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