STA 732 Homework 1

Assigned 2025-01-14

Due 2025-01-21

1. Let L be a loss function for which $\mathcal{D} = \Theta$ and $L(\theta, d) = 0$ if $d = \theta$ and is strictly greater than zero otherwise. Obtain an example of a model for which there is a $\theta_0 \in \Theta$ such that $\delta(X) = \theta_0$ is not admissible.

2. Later we will show that X is an admissible estimator of θ in the model $X \sim N(\theta, 1),\ \theta \in \mathbb{R}$ under squared error loss. Using this fact, show the following:

   (a) If $X \sim N(\theta, \sigma_0^2)$, σ_0^2 known, then X is admissible for θ.

   (b) If $X \sim N(\theta, \sigma^2)$, σ^2 unknown, then X is admissible for θ.

   (c) If $X_1, \ldots, X_n \sim$ i.i.d. $N(\theta, \sigma_0^2)$, σ_0^2 known, then $\bar{X}$ is admissible for θ.

   (d) If $X_1, \ldots, X_n \sim$ i.i.d. $N(\theta, \sigma^2)$, σ^2 unknown, then $\bar{X}$ is admissible for θ.

3. Let $\theta \in (0, \infty)$ be an unknown parameter and X be a random variable such that $\mathrm{E}[X|\theta] = \theta$ and $\mathrm{Var}[X|\theta] = v(\theta)$ where $v(\theta)$ is specified. Consider estimation of θ by linear functions of the form

   $$\delta_a(X) = aX$$

   for $a \in (0, 1)$, with squared-error loss

   $$L(\delta(X), \theta) = [\delta(X) - \theta]^2.$$

   Let $\mathcal{A}$ be the set of all such estimators, indexed by $a \in (0, 1)$.

   (a) For $v(\theta) = \theta^2$, calculate the risk function of δ_a, and find a value of a that makes δ_a admissible within the class $\mathcal{A}$ (i.e., no member of $\mathcal{A}$ dominates it). Show that this estimator dominates the unbiased estimator $\delta_1(X) = X$.

   (b) For $v(\theta) = \theta$, prove that every member of $\mathcal{A}$ is admissible among the class $\mathcal{A}$ (i.e., no member dominates another), and that no member of $\mathcal{A}$ dominates δ_1.

   (c) Suppose $v(\theta) = \theta^k$ where k is a positive integer. Find a closed form expression for the Bayes estimator in the class $\mathcal{A}$ when θ has prior density $p(\theta) = e^{-\theta},\ \theta > 0$.

4. Let $X_1, \ldots, X_n$ be i.i.d. "failure times" with pdf $p(x|\theta) = e^{-x/\theta}/\theta$ on $0 < x < \infty$. Suppose we only see the censored variables $Y_1, \ldots, Y_n$ where $Y_i = \min(X_i, t)$ and t is a known censoring time.

   (a) Calculate $\Pr(Y = t|\theta)$ and the CDF $F(y)$ for a single observation $Y = \min(X, t)$.

   (b) Explain why the class of distributions $\mathcal{P}_Y = \{\Pr(Y \in \cdot|\theta) : 0 < \theta < \infty\}$ for the Y_i's is not dominated by Lebesgue measure μ. Define a new measure ν which is a simple modification of μ that dominates $\mathcal{P}_Y$.

   (c) Find the corresponding Radon-Nikodym derivatives, i.e. the probability densities $p(y|\theta)$ for Y, such that $\Pr(Y \in A|\theta) = \int_A p(y|\theta)\nu(dy)$.

   (d) Find a minimal sufficient statistic for θ based on the observed data $Y_1, \ldots, Y_n$ and explain why it is minimal.

5. Let $X \sim \text{binary}(\theta)$, $\theta \in \{\theta_l, \theta_h\}$ where $0 < \theta_l < \theta_h < 1$. Consider estimation of θ with 0-1 loss.

   (a) Characterize all non-randomized estimators of θ.

   (b) Characterize all estimators of θ, in terms of the randomized and non-randomized estimators.

   (c) The risk function $R(\theta, \delta)$ of each estimator δ can be expressed as a two dimensional vector, $r(\delta) = (R(\theta_l, \delta), R(\theta_h, \delta))$. Draw the risk set - the set of all risk functions.

   (d) Using your drawing, characterize all admissible estimators.