STA 732 Homework 2
Assigned 2025-01-21
Due 2025-01-28

1. Consider an exponential family $\mathcal{P}$ having parameter space $\mathcal{H} \subset \mathbb{R}^p$ and densities $\{p(x|\eta) = \exp(t(x) \cdot \eta - A(\eta)) : \eta \in \mathcal{H}\}$ with respect to a dominating measure ν. Here, $\mathcal{H}$ is the natural parameter space, $\mathcal{H} = \{\eta : \int \exp(t(x) \cdot \eta)\, \nu(dx) < \infty\}$, and $A(\eta) = \log \int \exp(t(x) \cdot \eta)\, \nu(dx)$ is the normalizing constant. Derive expressions for $\mathrm{E}[t(x)|\eta]$ and $\mathrm{Var}[t(x)|\eta]$ as a function of the derivatives of $A(\eta)$. Apply this result to obtain $\mathrm{E}[t(x)|\eta]$ and $\mathrm{Var}[t(x)|\eta]$ for the following families:

   (a) the normal(μ, σ^2) family;

   (b) the gamma(a, b) family, having mean ab and variance ab^2.

   (c) the multinomial(θ) family, where $\theta = (\theta_1, \ldots, \theta_K)$ with $\theta_k \geq 0$ and $\sum \theta_k = 1$.

2. Consider the Kullback-Leibler loss:

$$L(\theta, d) = \int \log \frac{p(x|\theta)}{p(x|d)} p(x|\theta)\ \mu(dx),$$

   which measures the predictive accuracy of $p(x|d)$ against the truth $p(x|\theta)$.

   (a) Show that $L(\theta, d) \geq 0$ unless $p(x|\theta) = p(x|d)$ a.e. μ.

   (b) Obtain an expression for this loss function for estimating the natural parameter in a (multiparameter) exponential family model in the natural parameter space, and show that this loss function is convex.

   (c) Obtain expressions for the loss function when the model is

      i. Poisson with unknown mean;

      ii. normal with unknown mean and variance;

      iii. gamma with unknown shape and scale.

3. Let $X \sim N(\theta, 1)$ and $L(\theta, d) = (\theta - d)^2$ for some $\theta \in \mathbb{R}$.

   (a) Show formally that $\delta(X) = \theta_0$ is an admissible estimator.

(b) Consider a randomized estimator of the form $\delta(X, U) = \theta_0 \times 1(U < c) + \theta_1 \times 1(U > c)$, where $U \sim$ uniform(0,1). Decide whether or not δ is admissible, and prove your result.

4. Use a limiting Bayes argument to show that x/a is an admissible estimator of b based on $X \sim \text{gamma}(a, b),\ b > 0$ and a known, under loss $L(b, d) = (b - d)^2/b^2$.