STA 732 Homework 3

Assigned 2025-01-28

Due 2025-02-04

1. Consider estimation of $\theta \in \mathbb{R}^p$ based on $X \sim N_p(\theta, \sigma^2)$ with σ^2 known, using "compound loss" $L(\theta, d) = \sum(\theta_j - d_j)^2/p$.

   (a) Obtain the Bayes estimator under the prior $\theta \sim N_p(0, \tau^2 I_p)$, and compute the Bayes risk.

   (b) Show that the unbiased estimator X is a pointwise limit of Bayes estimators, and has a Bayes risk that is the limiting Bayes risk of the corresponding Bayes estimators.

   (c) Attempt to show admissibility of X using Blyth's method. In particular, compute the limiting ratio of $(R(\pi_n, X) - R(\pi_n, \delta_{\pi_n}))/\pi_n(B)$ for a bounded subset $B \in \mathbb{R}^p$. Describe what happens for $p = 1$, $p = 2$ and $p > 2$.

2. Based on $\bar{X} \sim N(\theta, 1/n)$, suppose you need to decide among three actions: 1) stating nothing, 2) stating $\theta < 0$ or 3) stating $\theta > 0$. Refer to these decisions numerically as $d = 0$, $d = -1$ and $d = 1$, respectively, and let the loss be $L(\theta, 0) = 1$, $L(\theta, -\text{sign}(\theta)) = c$ for some $c > 2$, and zero otherwise.

   (a) Find the Bayes rule under a $N(0, \tau^2)$ prior for θ, and find a formula for the frequentist risk function.

   (b) Find the frequentist risk function of the following "z-test" procedure: If the standard z-test of $\theta = 0$ is rejected at level $\alpha = 0.05$ then take d to be the sign of $\bar{X}$. If the test doesn't reject, then take $d = 0$.

   (c) Compare the risk functions graphically for $n \in \{5, 10, 100, 1000\}$ and $\tau^2 \in \{1, 10, 100\}$. Comment similarities and differences in the frequentist risk functions as n changes.

   (*) Describe the Bayes procedures and admissible procedures for the case $d = 2$.

3. Consider Bayesian inference using a posterior density $\pi(\theta|x)$:

   (a) Find the form of the Bayes estimator under absolute loss $L(\theta, d) = |\theta - d|$, and prove your result.

(b) Find the form of the Bayes estimator under zero-one loss $L(\theta, d) = 1(\theta \neq d)$ for the case that $\pi(\theta|x)$ is discrete.

4. Let $X \sim P_\eta$ for some unknown value of $\eta \in \mathbb{R}^p$, where P_η has density $p(x|\eta) = \exp(\eta \cdot x - A(\eta))h(x)$. Show that the Bayes estimator under prior $\pi(\eta)$ and sum of squared error loss is given by

$$\hat{\eta}_j = \tfrac{\partial}{\partial x_j} \log \tfrac{p_\pi(x)}{h(x)}.$$