STA 732 Homework 9

Assigned 2025-03-27

Due 2025-04-03

1. Let $X \sim N(\mu, \sigma^2)$, with μ and σ^2 both unknown. Show that σ^2 is not U-estimable.

2. For the exponential family models below, write the joint density of the data in minimal form, show that it is a full rank model, and identify a complete sufficient statistic. Comment on some estimands that are U-estimable, and some that are not.

   (a) The one-way ANOVA model $X_{i,j} = \mu + a_j + \epsilon_{i,j}$, $\{\epsilon_{i,j} : i = 1, \ldots, n, j = 1, \ldots, m\} \sim$ i.i.d. $N(0, \sigma^2)$, where $\mu \in \mathbb{R}$, $(a_1, \ldots, a_m) \in \mathbb{R}^m$, and $\sigma^2 > 0$ are all unknown.

   (b) The Poisson regression model $X_i \sim \text{Poisson}(e^{\beta^\top z_i})$, where $\beta \in \mathbb{R}^p$ is unknown.

3. Let $X_1, \ldots, X_n \sim$ i.i.d. uniform$(0, \theta)$, $\theta > 0$.

   (a) Identify a complete sufficient statistic, and prove that it is complete.

   (b) Find the UMVUE of $h(\theta) = \theta^k$ for any $k > -n$.

4. Let $(X_1, Y_1), \ldots, (X_n, Y_n) \sim$ i.i.d. $P \in \mathcal{P}$ where $\mathcal{P}$ is a nonparametric family for which the $\hat{P}_n$ is a complete sufficient statistic.

   (a) For the case that $(X_i, Y_i) \in \mathbb{R}^2$ and $\mathcal{P}$ includes only distributions with finite second moments, find the UMVUE of $\text{Cov}(X, Y)$.

   (b) Find the UMVUE of $\Pr(X < Y)$; where $(X, Y) \sim P$.

   (c) Find the UMVUE of $\Pr(\{X \leq X'\} \cap \{Y \leq Y'\})$ where $(X, Y) \sim P$ and $(X', Y') \sim P$ are independent.

*. Recall the model $X_1, \ldots, X_n \sim$ i.i.d. uniform$(\theta - 1/2, \theta + 1/2)$ from the previous homework.

   (a) Show that the minimum variance location equivariant estimator (the UMREE) is unbiased.

   (b) Is the UMREE the UMVUE? If so, explain why. If not, try to come up with an unbiased estimator that beats the UMREE for some θ values.