Tensor SVD

Matrix Rank (First definition), and extension

$M \in \mathbb{R}^{m_1 \times m_2}$ $\quad m_1 \geq m_2$

Def: $M \in \mathbb{R}^{m_1 \times m_2}$ is rank-1 if $M = ab^T = a \circ b$,

some $a \in \mathbb{R}^{m_1} / \{0\}$ $b \in \mathbb{R}^{m_2} / \{0\}$.

Def: M is rank-R if R is the smallest number for which

$$M = \sum_{k=1}^{R} a_k b_k^T.$$

The SVD recovers the rank:

$$M = UDV^T = \sum_{k=1}^{m_2} d_k u_k v_k^T, \quad d_1 \geq d_2 \geq \cdots \geq d_{m_2} \geq 0,$$

so $\operatorname{rank}(M) = \#$ nonzero d_k's

Extension to Tensors:

Def: $M \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ is rank-1 if $\exists\, a \in \mathbb{R}^{m_1}/\{0\}$, $b \in \mathbb{R}^{m_2}/\{0\}$, $c \in \mathbb{R}^{m_3}/\{0\}$, s.t.

$$M = a \circ b \circ c$$
$$\operatorname{vec}(M) = (c \otimes b \otimes a) \quad (\in \mathbb{R}^{m_1 m_2 m_3})$$
$$m_{ijk} = a_i b_j c_k$$

Def: M is rank-R if R is the smallest number for which

$$M = \sum_{k=1}^{R} a_r \circ b_r \circ c_r, \quad m_{ijk} = \sum a_{ir} b_{jr} c_{kr}, \quad \operatorname{vec}(M) = \sum (c_r \otimes b_r \otimes a_r)$$

One notion of a tensor SVD is based on such a decomposition

$$M = \sum_{r=1}^{R} d_r \, \underset{\sim}{u}_r \circ \underset{\sim}{v}_r \circ \underset{\sim}{w}_r \qquad (u_r^T u_r = v_r^T v_r = w_r^T w_r = 1)$$

This representation is called the "CP" decomposition

(PARAFAC : Harshman (1970)
CANDECOMP : Carrol & Chang (1970))

Matrix Rank (2nd def) and Extension

<u>Def</u>: The column rank of a matrix M is $\dim(\mathrm{span}(M_{[:,1]}, \ldots M_{[:,m_2]}))$

<u>Def</u>: The row rank " is $\dim(\mathrm{span}(M_{[1,:]}, \ldots M_{[m_1,:]}))$

<u>Thm</u>: row rank = col rank.

SVD recovers bases of row/col spaces, and ranks.

$M = UDV^T$: $M[i,j] = U[i,1] d_1 v_{1j} + \cdots + U[i,m_2] d_{m_2} v_{m_2 j}$

$\Rightarrow$ columns of U form orthog. basis for $M_1, \ldots, M_{m_2}$.

Similarly, columns of V form orthog. basis for $M[1,] \ldots M[m_1,]$

Derive SVD from bases:

① obtain U = orthog basis for $\text{span}(M[,1] \ldots, M[,m_2])$
V = " $\text{span}(M[1,] \ldots M[m_1,])$

② Let $D = U^T M V$ (works if U, V are ordered in terms of evals of MM^T and $M^T M$)

Extension to tensors:

Idea: Obtain "SVD" for $M \in \mathbb{R}^{m_1 m_2 m_3}$ as follows:

① Let $U_{(1)}$ = orthog basis for cols of $M_{(1)}$
= $\text{SVD}(M_{(1)})\$u$
R_1 = $\dim(\text{span}(\text{cols of } M_{(1)}))$ = matrix rank of $M_{(1)}$

⋮

$U_{(3)}$ = orthog basis for cols of $M_{(3)}$
= $\text{SVD}(M_{(3)})\$u$
R_3 = matrix rank of $M_{(3)}$

Def: The multilinear rank of M is (R_1, R_2, R_3)

They are not equal, in general!!

Higher order SVD

Let $S = M \times \{U_1^T, U_2^T, U_3^T\}$

$$s = (U_3^T \otimes U_2^T \otimes U_1^T)\, m$$

Then $S \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ is the "core array" of M

Also,

$$(U_3 \otimes U_2 \otimes U_1)\, s = m$$

$$S \times \{U_1, U_2, U_3\} = M$$

This is called the "higher order SVD" or HOSVD.

The core array S satisfies some properties:

- "all-orthogonality"
- ordering
- mode-specific singular value recovery.

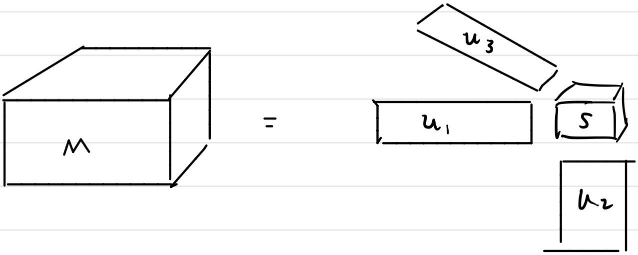
Truncated TSVD:

Model: $Y = M + E$, rank $(M) = (R_1 \ldots R_K)$

OLS: $\hat{M} = \arg\min_{M:\, \text{rank} = R_1 \ldots R_K} \|Y - M\|^2$

M has n-rank (R_1, R_2, R_3) if

$$M = S \times \{U_1, U_2, U_3\} \qquad U_k \in \mathbb{R}^{n_k \times R_k}, \quad S \in \mathbb{R}^{R_1 \times R_2 \times R_3}$$



Estimates:

1. "truncated" HOSVD – not OLS/MLE
2. "HOOI" iteratively update U_1, U_2, U_3, S.
   → converges to MLE/OLS.