Finding the normalizing constant

$X \sim Bin(n, \theta)$

$\theta \sim Beta(a, b)$

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Lecture 2

①

what is the normalizing constant in the posterior $\pi(\theta|x)$?

$$\pi(\theta|x) = \frac{\overbrace{\pi(x|\theta)}^{\text{likelihood}}\ \overbrace{\pi(\theta)}^{\to \text{ prior}}}{\underbrace{m(x)}_{\text{marginal}}}$$

$$= \frac{\binom{n}{x}\theta^{x}(1-\theta)^{n-x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}}{m(x)}$$

Remark: To find to normalizing constants, we will group all "values" that are a function of the data or constant, ie not a function of θ. → random.

$$= \frac{\binom{n}{x}}{m(x)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left\{\theta^{x}(1-\theta)^{n-x}\theta^{a-1}(1-\theta)^{b-1}\right\}$$

→ kernel of Beta($x+a$, $n-x+b$) =: Beta(a', b')

$$= \frac{\frac{\binom{n}{x}}{m(x)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}{\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}}\left\{\underbrace{\theta^{x+a-1}(1-\theta)^{n-x+b-1}\times\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}}_{\text{This is a Beta}(x+a,\ n-x+b)\text{ density}}\right\}$$

Hence, the normalizing constant is
(Simplification should be possible).

②

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what is

$$m(x) = \int \underbrace{\pi(x|\theta)\pi(\theta)}_{\text{kernel of Beta}(a',b')} d\theta$$

$$= \frac{1}{\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}}$$

This implies that the normalizing constant $C(x) =$

$$\frac{\binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}{\cancel{\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}}} \times \cancel{\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}}$$

$$= \frac{\binom{n}{x}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$