Intro to Bayesian Methods: Part II

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Bayesian Methods and Modern Statistics: STA 360/601

Lecture 2
Last Time

- Why should we learn about Bayesian concepts?
- Natural if thinking about unknown parameters as random.
- They naturally give a full distribution when we perform an update.
- Today: An example about students and sleep.
Bayesian Motivation

Parameters

\[ P(X|\theta) = \text{Probability}[\text{data}|\text{pattern}] \]

Inference idea

\[ \text{data} = \text{underlying pattern} + \text{independent noise} \]

[Picture: Peter Orbanz, Columbia University]
Review of the Beta-Binomial

\[ X \mid \theta \sim \text{Binomial}(n, \theta) \]  \hspace{1cm} (1)
\[ \theta \sim \text{Beta}(a, b) \]  \hspace{1cm} (2)

Find \( \pi(\theta \mid X) \)
Review of the Beta-Binomial

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Find \( \pi(\theta | X) \)

\[ \pi(\theta | x) \propto p(x|\theta)p(\theta) \]
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Find \( \pi(\theta \mid X) \)

\[
\pi(\theta|x) \propto p(x|\theta)p(\theta) \\
\propto \binom{n}{x} \theta^x (1 - \theta)^{n-x} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1 - \theta)^{b-1}
\]
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\]
\[
\propto \theta^x (1 - \theta)^{n-x} \theta^{a-1}(1 - \theta)^{b-1}
\]

\[ \Rightarrow \theta | x \sim \text{Beta}(x + a, n - x + b) \]
Review of the Beta-Binomial

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\propto \theta^{x+a-1}(1-\theta)^{n-x+b-1} \Rightarrow
\]
Review of the Beta-Binomial

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\propto \theta^x (1 - \theta)^{n-x} \theta^{a-1}(1 - \theta)^{b-1} \\
\propto \theta^{x+a-1}(1 - \theta)^{n-x+b-1} \implies \]

\[
\theta|x \sim \text{Beta}(x + a, n - x + b). 
\]
How Much Do You Sleep

We are interested in a population of American college students and the proportion of the population that sleep at least eight hours a night, which we denote by $\theta$. 
The Gamecock, at the USC printed an internet article “College Students Don’t Get Enough Sleep” (2004).
- Most students spend six hours sleeping each night.

2003: University of Notre Dame’s paper, Fresh Writing.
- The article reported took random sample of 100 students:
  - “approximately 70% reported to receiving only five to six hours of sleep on the weekdays,
  - 28% receiving seven to eight,
  - and only 2% receiving the healthy nine hours for teenagers.”
Have a random sample of 27 students is taken from UF.
11 students record that they sleep at least eight hours each night.
Based on this information, we are interested in estimating $\theta$. 
From USC and UND, believe it’s probably true that most college students get less than eight hours of sleep.

Want our prior to assign most of the probability to values of \( \theta < 0.5 \).

From the information given, we decide that our best guess for \( \theta \) is 0.3, although we think it is very possible that \( \theta \) could be any value in \([0, 0.5]\).
Given this information, we believe that the median of $\theta$ is 0.3 and the 90th percentile is 0.5.

Knowing this allows us to estimate the unknown values of $a$ and $b$.

How do we actually calculate $a$ and $b$?
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Knowing this allows us to estimate the unknown values of $a$ and $b$.

How do we actually calculate $a$ and $b$?

We would need to solve the following equations:

\[
\int_0^{0.3} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1 - \theta)^{b-1} \, d\theta = 0.5
\]

\[
\int_0^{0.5} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1 - \theta)^{b-1} \, d\theta = 0.9
\]

In non-calculus language, this means the 0.5 quantile (50th percentile) = 0.3. The 0.9 quantile (90th percentile) = 0.5.

The equations are written as percentiles above!
We can easily solve this numerically in R using a numerical solver BBsolve.

The documentation for this package is not great, so beware.

Since you won’t have used this command before, you’ll need to install the package BB and load the library.
Here is the code in R to find $a$ and $b$.

```r
## install the BBsolve package
install.packages("BB", repos="http://cran.r-project.org")
library(BB)

## using percentiles
myfn <- function(shape){
test <- pbeta(q = c(0.3, 0.5), shape1 = shape[1],
             shape2 = shape[2]) - c(0.5, 0.9)
return(test)
}
BBsolve(c(1,1), myfn)

## using quantiles
fn = function(x){qbeta(c(0.5,0.9),x[1],x[2])-c(0.3,0.5)}
BBsolve(c(1,1),fn)
```
Using our calculations from the Beta-Binomial our model is

\begin{align*}
X \mid \theta & \sim \text{Binomial}(27, \theta) \\
\theta & \sim \text{Beta}(3.3, 7.2) \\
\theta \mid x & \sim \text{Beta}(x + 3.3, 27 - x + 7.2) \\
\theta \mid 11 & \sim \text{Beta}(14.3, 23.2)
\end{align*}
th = seq(0,1,length=500)
a = 3.3
b = 7.2
n = 27
x = 11
prior = dbeta(th,a,b)
like = dbeta(th,x+1,n-x+1)
post = dbeta(th,x+a,n-x+b)
pdf("sleep.pdf",width=7,height=5)
plot(th,post,type="l",ylab="Density",lty=2,lwd=3,
xlab = expression(theta))
lines(th,like,lty=1,lwd=3)
lines(th,prior,lty=3,lwd=3)
legend(0.7,4,c("Prior","Likelihood","Posterior"),
lty=c(3,1,2),lwd=c(3,3,3))
dev.off()
Figure 1: Likelihood $p(X|\theta)$, Prior $p(\theta)$, and Posterior Distribution $p(\theta|X)$
What questions might we want to ask?

- What is the posterior mean and variance? (Homework 1).
- Confidence intervals
- If new data comes in, how do we predict whether these students are getting less than 8 hours of sleep?

Above questions we’ll cover in Module 3!
Coming up next

Make sure you’re feeling very familiar with R. See the Intro to Bayes Lab (with solutions).

Module 2: Decision theory: how do we think about loss, risk (general decision making)?

Wednesday: Lab with decision theory.