Intro to Monte Carlo, Part II

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Module 5
Rejection Sampling

Rejection sampling is a method for drawing random samples from a distribution whose p.d.f. can be evaluated up to a constant of proportionality.

Difficulties? You must design a good proposal distribution (which can be difficult, especially in high-dimensional settings).
Uniform Sampler

Goal: Generate samples from Uniform(A), where A is complicated.

Example: $X \sim \text{Uniform(Mandelbrot)}$.

How? Consider $I_X(A)$.

Figure 1: A complicated function $A$, called the Mandelbrot!
Proposition

- Suppose $A \subset B$.
- Let $Y_1, Y_2, \ldots \sim \text{Uniform}(B)$ iid and
- $X = Y_k$ where $k = \min\{k : Y_k \in A\}$,

Then it follows that

$$X \sim \text{Uniform}(A).$$

Proof: Exercise. Hint: Try the discrete case first and use a geometric series.
Figure 2: (Left) How to draw uniform samples from region $A$? (Right) Draw uniform samples from $B$ and keep only those that are in $A$. 
General Rejection Sampling Algorithm

Goal: Sample from a complicated pdf $f(x)$.

Suppose that

$$f(x) = \frac{\tilde{f}(x)}{\alpha}, \alpha > 0$$

Algorithm:

1. Choose a proposal distribution $q$ such that $c > 0$ with

   $$cq(x) \geq \tilde{f}(x).$$

2. Sample $X \sim q$, sample $Y \sim \text{Unif}(0, cq(X))$ (given $X$)

3. If $Y \leq \tilde{f}(X)$, $Z = X$,

   Otherwise we reject and return to step (2).

Output: $Z \sim f$

Proof: Exercise.
Figure 3: Visualizing just f (hard to sample from).
Figure 4: Visualizing just $f$ and $\tilde{f}$. 
Figure 5: Visualizing $f$ and $\tilde{f}$. Now we look at enveloping $q$ over $f$. 
Figure 6: Visualizing $f$ and $\tilde{f}$. Now we look at enveloping $cq$ over $\tilde{f}$.
Figure 7: Recalling the sampling method and accept/reject step.
Figure 8: Entire picture and an example point $X$ and $Y$. 

$X \sim q$

$Y \sim \text{Unif}(0, cq)$

$Y \leq \tilde{f}$

accept
Suppose we want to generate random variables from the Beta(5.5,5.5) distribution.

There are no direct methods for generating from Beta(a,b) if a,b are not integers.

One possibility is to use a Uniform(0,1) as the trial distribution. A better idea is to use an approximating normal distribution.

Do this as an exercise on your own.

In lab: you’ll go through both importance sampling and rejection sampling.