Sta 711: Homework 1

Fields and σ-fields

1. Enumerate the class $\aleph$ of all σ-fields $\mathcal{F}$ on the three-point set $\Omega = \{a, b, c\}$ that contain the singleton $\{a\}$, *i.e.*, that satisfy $\mathcal{C} \subset \mathcal{F}$ for $\mathcal{C} := \big\{\{a\}\big\}$. What is $\sigma(\mathcal{C})$?

2. Prove that for any two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on any set Ω, the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a field.

3. Find a set Ω and two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on Ω for which $\mathcal{F}_1 \cup \mathcal{F}_2$ is *not* a field.

4. Suppose a collection $\{\mathcal{F}_n : \ n \in \mathbb{N}\}$ of σ-fields on a set Ω satisfies the relation $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for every $j \in \mathbb{N}$. Does it follow that $\cup \mathcal{F}_j$ is a field? (the answer is "yes"— show why)

5. Under the same conditions, must $\cup \mathcal{F}_j$ be a σ-field? (this one is "no"— find a counter-example. The idea is to find a sequence $A_n \in \mathcal{F}_n$ with $\cup_n A_n \notin \mathcal{F}_j$ for every j, hence $\cup_n A_n \notin \cup_j \mathcal{F}_j$).

Dyadic Rational Probability Spaces

For problems 6–9, let $\Omega = \mathbb{Q}_2 := \big\{j/2^n :\ j \in \{1, 2, \cdots, 2^n\},\ n \in \mathbb{N}\big\}$ be the dyadic rational numbers in the half-open unit interval, and let

$$\mathcal{C} = \{(0, b] \cap \mathbb{Q}_2 :\ b \in \mathbb{Q}_2,\ 0 < b \le 1\} \tag{1}$$

denote the collection of half-open intervals **of dyadic rationals** $(0, b] = \{q \in \mathbb{Q}_2 :\ 0 < q \le b\}$ with left endpoint zero. **Every Ω on this page contains *only* dyadic rational numbers.**

Recall that a real-valued set function P on a σ-algebra $\mathcal{G}$ of subsets of a space Ω is a "probability measure" (PM) if and only if it satisfies the three rules:

- $(\forall A \in \mathcal{G})\ \mathsf{P}(A) \ge 0$;
- $(\forall \{A_i\} \subset \mathcal{G},\ A_i \cap A_j = \emptyset)\ \mathsf{P}(\cup A_i) = \sum \mathsf{P}(A_i)$;
- $\mathsf{P}(\Omega) = 1$.

6. Let $n \in \mathbb{N}$ be a FIXED positive integer (like three) and set

   $$\mathcal{B}_n := \{(0, j/2^n],\ j \in \{0, 1, ..., 2^n\}\},$$

   the collection of half-open intervals in Ω of dyadic rationals from zero up to an integral multiple of 2^{-n}. Describe the elements of the σ-field

   $$\mathcal{F}_n := \sigma(\mathcal{B}_n)$$

   generated by $\mathcal{B}_n$, for fixed $n \in \mathbb{N}$. How many elements does $\mathcal{B}_n$ have? How many distinct elements does $\mathcal{F}_n$ have? What are they? Suggestion: Try $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ first, by hand. Is there a *partition* that generates $\mathcal{F}_n$?

7. What is the field $\mathcal{F}_0 := \mathcal{F}(\mathcal{C})$ of subsets of $\mathbb{Q}_2$ generated by the class $\mathcal{C}$ of Eqn (1)? (hint: Do problems (4) and (6) first). Try to describe it in just a few words, without using any symbols besides $\mathbb{Q}_2$. Don't just echo the definition!

8. Describe simply and clearly in no more than five words or symbols (seriously, *three* should be enough) the σ-field $\mathcal{F} := \sigma(\mathcal{C})$ of subsets of $\mathbb{Q}_2$ generated by $\mathcal{C}$. Don't just echo the definition!

9. Define a set function λ_0 on $\mathcal{C}$ by

   $$\lambda_0\big(\ (0, b]\ \big) = b$$

   Show that there does **not** exist a probability measure λ on $(\mathbb{Q}_2, \mathcal{F})$ that extends λ_0, *i.e.*, one for which $\lambda\big(\ (0, b]\ \big) = b$ for all $b \in \mathbb{Q}_2$ (Hint: Exactly what does the function $F(x) := \lambda\big(\ (0, x]\ \big),\ 0 \le x \le 1$ look like near $x \in \mathbb{Q}_2$, for *any* PM λ on $\mathbb{Q}_2$?).

Last edited: September 3, 2018