Stationary Infinitely-Divisible Markov Processes with Non-negative Integer Values

Robert L Wolpert

Department of Statistical Science
Duke University

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Prelude

Progress isn’t made by early risers. It’s made by lazy men trying to find easier ways to do something.

- Robert Heinlein, *Time Enough for Love*
Acknowledgment

This is collaborative work with Larry Brown, one part of a larger effort exploring and characterizing classes of stationary infinitely-divisible time series and processes.

It’s partially supported by grants from the NSF and NASA and our collaboration has been facilitated by the generosity of the Wharton Department of Statistics and of SAMSI. Thanks!
**Q**: How can we model processes or time series of *counts* $X_t$ (e.g., dial-ins to a call-center), w/ serial autocorrelation?

**A**: Obvious idea: $X_t \sim \text{Po}(\Lambda_t)$, with random $\Lambda_t \geq 0$.

**X**: That sounds hard— even if $\Lambda_t \geq 0$ is Markov, $X_t$ would only be *hidden* Markov. Any easier ideas?

**A**: Okay— how about Markov $X_t$?

**Q**: Great. What other properties should we impose?
Modeling Correlated Count Data

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Motivation

Conditions

Solutions

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ID Count Models

For either **discrete** \((t \in \mathcal{T} = \mathbb{Z})\) or **continuous** \((t \in \mathcal{T} = \mathbb{R})\) time, model count data \(X_t\) such that for each \(\vec{t} = (t_1, t_2, ..., t_p) \in \mathcal{T}^p, p \in \mathbb{N}\), the joint distribution

\[ p_{\vec{t}}(\vec{x}) = \mathbb{P}[X_{t_i} = x_i] \]

is:

1. Supported on the non-negative integers \(\mathbb{Z}^p_+\);
3. Stationary;
4. Infinitely-divisible (not just the marginals);
5. Time-reversible.

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Why Markov?

- To specify the distribution of a general $Z_+\text{-valued}$ process $X_t$ would require specifying all finite-dimensional marginal distributions

$$p_{\vec{t}}(\vec{x}) = P[X_{t_i} = x_i], \quad \vec{t} \in \mathcal{T}^p, \quad \vec{x} \in \mathcal{X}^p, \quad p \in \mathbb{N}$$

That sounds hard.

- For Markov processes we need only to specify the marginal

$$p_t(x) = P[X_t = x]$$

and transition distributions

$$q_{st}(y \mid x) = P[X_t = y \mid X_s = x].$$

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Why Markov?

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Why Stationary?

• Under stationarity, the Markov specification of marginal

\[ p(x) = P[X_t = x] \]

and transition distributions

\[ q_s(y \mid x) = P[X_{t+s} = y \mid X_t = x] \]

are simpler because they don’t depend on \( t \).

• We’ll worry later about temporal patterns; we can use stationary processes as building-blocks.

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Why Infinitely Divisible?

• A random vector $\vec{Y} \in \mathbb{R}^p$ is ID if for every $n \in \mathbb{N}$ we can write
  \[
  \vec{Y} = \vec{Y}_1^{(n)} + \cdots + \vec{Y}_n^{(n)} \text{ w/iid } \vec{Y}_j^{(n)}.
  \]

• Calls arrive from otherwise similar individuals of different
  - Hair color;
  - Zip codes;
  - Gender;
  - Profession;
  - Experience;
  - SES;
  - Age;
  - SSN (mod 10);

• Lévy-Khinchine characterization (for counts):
  \[
  \log \mathbb{E} \left[ e^{i\omega \cdot \vec{Y}} \right] = \sum_j \left( e^{i\omega \cdot u_j} - 1 \right) \nu_j,
  \quad \{u_j\} \subset \mathbb{R}^p, \quad \{\nu_j\} \subset \mathbb{R}_+
  \]
  \[
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• Simplicity: All we need specify is $\nu(du)$ on $\mathbb{Z}_+^2$. 
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Why Time-Reversible?

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Recap

SO— what are all \textbf{discrete time} \((t \in T = \mathbb{Z})\) and \textbf{continuous time} \((t \in T = \mathbb{R})\) time series with distributions that are:

1. Supported on the non-negative integers \(\mathbb{Z}_+\);  
2. Markov;  
3. Stationary;  
4. Infinitely-divisible (not just the marginals); and  
5. Time-reversible?
The usual solutions I

- Trivial:

\[ X_t \equiv X_0 \sim p(x), \text{ an arbitrary ID distribution} \]
The usual solutions II

- Trivial:

\[ X_t \equiv X_0 \sim p(x), \text{ an arbitrary ID distribution, or:} \]
\[ X_t^{\text{iid}} \sim p(x), \text{ an arbitrary ID distribution} \]
The usual solutions III

- Well-known: First, recall “Bivariate Poisson” distribution.

Set:

\[ X_1 = \zeta_1 + \zeta_{12}, \quad X_2 = \zeta_{12} + \zeta_2 \]

\[ \zeta_1, \zeta_2 \overset{iid}{\sim} \text{Po}(\lambda(1 - \rho)) \quad \bot \quad \zeta_{12} \sim \text{Po}(\lambda \rho); \]

\[ X_1, X_2 \sim \text{Po}(\lambda), \quad \text{Cov}(X_1, X_2) = \lambda \rho. \]

- Equivalent recursive construction:

\[ X_1 \sim \text{Po}(\lambda) \quad X_2 \mid X_1 \sim \text{Bi}(X_1, \rho) + \text{Po}(\lambda(1 - \rho)) \]
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The usual solutions III: Thinning Construction

- The recursive (or **Thinning**) bivariate rep’n

\[ X_2 \mid X_1 \sim \text{Bi}(X_1, \rho) + \text{Po}(\lambda(1 - \rho)) \]

leads to recursive Markov prescription (E McKenzie ’85, etc.):

\[ X_{t_0} \sim \text{Po}(\lambda) \]
\[ X_{t+1} \mid F_t \sim \text{Bi}(X_t, \rho) + \text{Po}(\lambda(1 - \rho)), \quad t \geq t_0 \]

- But— is it ID? Yes. Because...
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The usual solutions III: Another Construction

Fix $s, u \in \mathcal{T} := \mathbb{Z}$ with $s \leq u$; construct $X_t$ as finite sum:

$$X_t = \sum_{i=u-t}^{u-s} \sum_{j=i-u+t}^{i} \zeta_{ij}, \quad \zeta_{ij} \overset{\text{ind}}{\sim} \text{Po}(\lambda_{ij})$$

with intensities $\lambda_{ij} = \lambda \rho^j (1 - \rho)^2$, $0 \leq j < i$

Figure: Illustration of Poisson process $X_t$
Is that it?

Are these three the only time series whose joint distribution

\[ p_{\vec{t}}(\vec{x}) = P[X_{t_i} = x_i] \]

is:

1. Supported on the non-negative integers \( \mathbb{Z}_+ \);
2. Markov;
3. Stationary;
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Theorem

Let $X_t$ be a nonnegative integer-valued process indexed by time $t \in \mathbb{Z}$ that is Markov, Stationary, (multivariate) Infinitely Divisible, and Time-reversible. Then the joint distribution of \{X_t\} is one of the four possibilities:

1. $X_t \equiv X$ for some $X$ with arbitrary ID dist’n on $\mathbb{Z}_+$; or
2. $X_t \overset{iid}{\sim} \mu_0$ for some arbitrary ID dist’n $\mu_0$ on $\mathbb{Z}_+$; or
3. $X_t \sim \text{Po}(\lambda)$ with Bivariate Poisson 2-marginals.; or
4. $X_t \sim \text{NB}(\alpha, p)$ with Negative Trinomial 2-marginals.
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The Negative Trinomial distribution

• For $\alpha > 0$ and $0 \leq p, q, r \leq 1$ with $p + q + r = 1$, let $X_0, X_1$ have joint pmf:

$$P[X_0 = j, X_1 = k] = \binom{-\alpha}{j, k} r^\alpha (-p)^j (-q)^k$$

$$= \frac{\Gamma(\alpha + j + k)}{\Gamma(\alpha) j! k!} r^\alpha p^j q^k,$$

with negative binomial univariate marginals and conditionals

$$X_0 \sim \text{NB}\left(\alpha, \frac{r}{r+p}\right), \quad X_1 \mid X_0 \sim \text{NB}(\alpha + X_0, r + p)$$

• Note that $(X_0, X_1)$ is ID...

and Stationary if $p = q \leq \frac{1}{2}$, w/correlation $\rho = p/(1 - p)$.
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Proof of Theorem

• By Stationarity & Markov property, probability generating function

\[ \phi(s, t, u) = \mathbb{E} s^{X_0} t^{X_1} u^{X_2} \]

determines distribution of entire process;

• By Lévy-Khinchine, for some \( \nu_{ijk} \geq 0 \),

\[ \log \phi(s, t, u) = \sum_{\mathbb{Z}^3_+} (s^i t^j u^k - 1) \nu_{ijk} \]

• Poisson representation: for indep. \( N_{ijk} \sim \text{Po}(\nu_{ijk}) \),

\[ X_0 = \sum iN_{i++} \quad X_1 = \sum jN_{+j+} \quad X_2 = \sum kN_{++k} \]
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- By Time Reversibility,
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- By Stationarity,
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Continuous Time

- The continuous-time case turns out to be easier than the discrete one!
- The same four solutions arise: constant, iid, Poisson, and Negative Binomial.
  - The Po case is a Linear Death branching process, with immigration;
  - The NB case is a Linear Birth/Death branching process, with immigration.
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What’s Next?

- Stationary Markov ID Processes abound, with every ID marginal distribution— No, Ga, St, Po, NB, etc.
- Could be useful— e.g., stationary Gamma processes $\Lambda_t \sim \text{Ga}(\alpha, \beta)$ for point process rates $X_t \sim \text{Po}(\Lambda_t)$.
- The families are rich— e.g., we know of at least four distinct stationary Gamma processes $\Lambda_t$ with identical marginals and covariance structures!
- Characterizing all of them could take us until Larry’s 80th birthday... or longer... so stay tuned!
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Thanks!

More details (references, this talk in .pdf, related work) are available at

http://www.stat.duke.edu/~rlw/.

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Happy Birthday, Larry!