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Joint distributions

**Definition**

Let $X, Y$ be two discrete random variables. The **joint pdf** $p(x, y)$ is defined by

$$p(x, y) = \mathbb{P}(X = x \text{ and } y = Y),$$

and for a set $A$

$$\mathbb{P}[(x, y) \in A] = \sum_{x, y \in A} p(x, y).$$
Example

An evil Leprechaun is chopping off fingers and toes at night.

People wake up with $X = 1, 2$ fingers and $Y = 2, 3, 4$ toes.
Example

Jointly distributed random variables

<table>
<thead>
<tr>
<th></th>
<th>$y = 2$</th>
<th>$y = 3$</th>
<th>$y = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 1$</td>
<td>.2</td>
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</tr>
<tr>
<td>$x = 2$</td>
<td>.05</td>
<td>.15</td>
<td>.3</td>
</tr>
</tbody>
</table>

$\Pr(y > 2) = p(x = 1, y = 3) + p(x = 1, y = 4) + p(x = 2, y = 3) + p(x = 2, y = 4)$

$\Pr(y > 2) = .1 + .2 + .15 + .3 = .75$

$\Pr(x = 2, y < 4) = p(x = 2, y = 3) + p(x = 2, y = 2)$

$\Pr(x = 2, y < 4) = .15 + .05 = .2$
Definition

The marginal distributions of $p(x, y)$ denoted by $p_X(x)$ and $p_Y(y)$ are given by

\[ p_X(x) = \sum_y p(x, y) \]
\[ p_Y(y) = \sum_x p(x, y). \]
Example

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
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<th>$y = 4$</th>
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<td>.5</td>
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Jointly distributed random variables

Discrete random variables

Continuous random variables

Covariance

A statistic

Sampling Distributions
### Example

Jointly distributed random variables

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<td>.15</td>
<td>.3</td>
</tr>
<tr>
<td>$p_Y(y)$</td>
<td>.25</td>
<td>.25</td>
<td>.5</td>
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Continuous random variables

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Sampling Distributions

Artin Armagan and Sayan Mukherjee

Joint distributions and the central limit theorem
Jointly distributed random variables

**Continuous random variables**

**Covariance**

A statistic

**Sampling Distributions**

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**Joint distributions**

**Definition**

Let $X$, $Y$ be two continuous random variables. The **joint pdf** $p(x, y)$ as the function that for any set $A$

$$\mathbb{P}[(x, y) \in A] = \int\int_A p(x, y) \, dx \, dy.$$ 

If $A = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$

$$\mathbb{P}[(x, y) \in A] = \int_a^b \int_c^d p(x, y) \, dx \, dy.$$
Example

For \((x, y) \in [0, 1] \times [0, 1]\)

\[ p(x, y) = \frac{6}{5}(x + y^2). \]
Jointly distributed random variables

Matlab code

```matlab
x=0:.01:1;
y=0:.01:1;
m = length(x);
z=zeros(m,m);
for i=1:m
    for j=1:m
        z(i,j) = 6*(x(i)+y(j)*y(j))/5;
    end
end
surf(x,y,z);
h=gca;
set(h,'FontSize',[20]);
xlabel('x');
ylabel('y');
zlabel('p(x,y)');
```

Artin Armagan and Sayan Mukherjee

Joint distributions and the central limit theorem
Marginal distributions

Definition

The marginal distributions of \( p(x, y) \) denoted by \( p_X(x) \) and \( p_Y(y) \) are given by

\[
\begin{align*}
p_X(x) &= \int_{-\infty}^{\infty} p(x, y) \, dy \\
p_Y(y) &= \int_{-\infty}^{\infty} p(x, y) \, dx.
\end{align*}
\]
Example

For \((x, y) \in [0, 1] \times [0, 1]\)

\[ p(x, y) = \frac{6}{5}(x + y^2), \]
\[ p_x(x) = \frac{6}{5}x + \frac{2}{5}. \]
Example

For \((x, y) \in [0, 1] \times [0, 1]\)

\[
p(x, y) = \frac{6}{5}(x + y^2),
\]

\[
p_Y(y) = \frac{6}{5}y^2 + \frac{3}{5}
\]
Example

There is a beer-pong challenge between genders.

\[ X: \text{ proportion of tipsy females when the game is over } \]

\[ Y: \text{ proportion of barfing males when the game is over } \]
Example

The set of possible values for \((X, Y)\) is the rectangle
\[ D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \]
and the joint pdf is
\[
p(x, y) = \begin{cases} 
\frac{6}{5}(x + y^2) & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

What is the probability that sum of the proportions of tipsy females and barfing males is less than or equal to .7?

\[ A = \{x + y \leq .7\}. \]
Example

The set $A = \{x + y \leq .7\}$ is the area we integrate the pdf over

$$
\mathbb{P}[(x, y) \in A] = \int_A \int p(x, y) \, dx \, dy,
$$

$$
= \int_{0}^{.7} \left[ \int_{0}^{.7-x} p(x, y) \, dy \right] \, dx,
$$

$$
= \int_{0}^{.7} \left[ \int_{0}^{.7-x} \frac{6}{5}(x + y^2) \, dy \right] \, dx,
$$

$$
= \int_{0}^{.7} \left[ \int_{y=0}^{.7-x} \frac{6}{5}x + \frac{2}{5}y^3 \bigg|_{0}^{.7-x} \right] \, dx
$$

$$
= \int_{0}^{.7} \left[ \frac{6}{5}x + \frac{2}{5}(.7 - x)^3 \right] \, dx,
$$

change of variables $u = .7 - x$ and $du = -dx$

$$
= - \int_{.7}^{0} \left[ \frac{6}{5}(.7 - u) + \frac{2}{5}u^3 \right] \, du,
$$

$$
= \int_{0}^{.7} \left[ \frac{6}{5}(.7 - u) + \frac{2}{5}u^3 \right] \, du.
$$
Independence

**Definition**

The random variables $X$ and $Y$ are independent if for all values of $x, y$

$$p(x, y) = p_X(x)p_Y(y).$$
Jointly distributed random variables

Conditional density

Definition

Let $X$ and $Y$ be two continuous rvs with joint distribution $p(x, y)$ and marginal $p_X(x)$. Then for any value $x$ with $p_X(x)$ the conditional pdf of $Y$ given $X$ is

$$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)}, \quad -\infty < x < \infty.$$
Examples

Are $X$ and $Y$ independent?

Compute

\[ p_X(x) = \int_0^1 \frac{6}{5} (x + y^2) \, dy, \]

\[ p_Y(y) = \int_0^1 \frac{6}{5} (x + y^2) \, dx, \]

is

\[ p_X(x) \times p_Y(y) = \frac{6}{5} (x + y^2). \]
Examples

If 20% of the females are tipsy, then what is the probability that more than 60% of the males will have to refund?

Compute

\[ p_{Y|X}(y|x = .2) = \frac{p(x = .2, y)}{p_x(x = .2)}. \]

Then integrate to obtain \( P(Y > .6|X = .2) \).
Expection of joint distribution

**Definition**

Let $X$ and $Y$ be two continuous rvs with joint distribution $p(x, y)$ the expected value of a function $h(x, y)$ is

$$
E[h(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) p(x, y) \, dx \, dy.
$$
Expection of joint distribution

**Definition**

Let $X$ and $Y$ be two discrete rvs with joint distribution $p(x, y)$ the expected value of a function $h(x, y)$ is

$$
E[h(x, y)] = \sum_x \sum_y h(x, y) p(x, y).
$$
Covariance

Definition

Let $X$ and $Y$ be two continuous rvs with joint distribution $p(x, y)$

\[
\text{Cov}(X, Y) = \mathbb{E}[(x - \mu_X)(y - \mu_Y)],
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) p(x, y) \, dx \, dy.
\]
Covariance

**Definition**

*Let $X$ and $Y$ be two discrete rvs with joint distribution $p(x, y)$*

\[
\text{Cov}(X, Y) = \mathbb{E}[(x - \mu_X)(y - \mu_Y)],
\]

\[
= \sum_x \sum_y (x - \mu_X)(y - \mu_Y) p(x, y).
\]
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2-Dimensional Gaussian

Definition

Given a random vector with two components $z = [x, y]^T$ from a normal distribution with mean $\mu = [\mu_x, \mu_y]^T$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \text{cov}_{X,Y} \\ \text{cov}_{Y,X} & \sigma_y^2 \end{bmatrix}.$$ 

The pdf of this random vector is

$$p(z) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp \left( -\frac{1}{2} (z - \mu)^T \Sigma^{-1} (z - \mu) \right).$$
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Independence example

Let \( \Sigma \) be a covariance matrix where

\[
\Sigma = \begin{bmatrix}
\sigma_X^2 & 0 \\
0 & \sigma_Y^2
\end{bmatrix}.
\]

Then

\[
|\Sigma|^{1/2} = \sqrt{\sigma_X^2 \sigma_Y^2} = \sigma_X \sigma_Y.
\]

The joint density function of \( Z \) is given by

\[
p(z) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left( -\frac{1}{2} (z - \mu)^T \Sigma^{-1} (z - \mu) \right)
\]

where

\[
p(z) = \frac{1}{\sqrt{2\pi} \sigma_X} \frac{1}{\sqrt{2\pi} \sigma_Y} \exp\left( -(x - \mu_X)^2 / 2\sigma_X^2 + (y - \mu_Y)^2 / 2\sigma_Y^2 \right)
\]

and

\[
p(z) = \frac{1}{\sqrt{2\pi} \sigma_X} \exp\left( -(x - \mu_X)^2 / 2\sigma_X^2 \right) \cdot \frac{1}{\sqrt{2\pi} \sigma_Y} \exp\left( -(y - \mu_Y)^2 / 2\sigma_Y^2 \right).
\]
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Independence example

\[ p(x, y) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp \left( -\frac{(x - \mu_X)^2}{2\sigma_X^2} \right) \cdot \frac{1}{\sqrt{2\pi}\sigma_Y} \exp \left( -\frac{(y - \mu_Y)^2}{2\sigma_Y^2} \right). \]

If \( X, Y \) are independent

\[ p(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp \left( -\frac{(x - \mu_X)^2}{2\sigma_X^2} \right), \]

\[ p(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp \left( -\frac{(y - \mu_Y)^2}{2\sigma_Y^2} \right). \]

Thus we have independence

\[ p(z) = p(x, y) = p(x)p(y). \]
Examples

\[ \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
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Examples

\[ \Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \]
Examples

\[ \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \]
Examples

\[ \Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix} \]
**Examples**

\[ \Sigma = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 1 \end{bmatrix} \]
Jointly distributed random variables

Matlab code I

x=-5:.1:5;
y=-5:.1:5;
mu = [0;0];
C = [1,-.9;-.9,1];
z=normal2d(x,y,mu,C);
surf(x,y,z);
h=gca;
set(h,'FontSize',[20]);
xlabel('x');
ylabel('y');
zlabel('p(x,y)');
function z = normal2d(x,y,mu,C)

m = length(x);
n = length(y);
z = zeros(n,m);
c = 1/(2*pi*sqrt(det(C)));
S = inv(C);
for i=1:n
    for j= 1:m
        xvec=[x(j);y(i)];
        z(i,j) = c * exp(-0.5 * (xvec-mu)' * S * (xvec-mu));
    end
end
Correlation

Definition

The correlation coefficient of $X$ and $Y$ is

$$
\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.
$$
Properties

Proposition

1. If $X$ and $Y$ are independent, $\rho = 0$. But $\rho = 0$ does not imply independence.

2. $\rho = \pm 1$ iff $Y = aX + b$ with $a \neq 0$. 
A statistic is a quantity that can be computed from (sample) data. For example the sample mean of \( x_1, x_2, x_3, x_4 \)

\[
\bar{x} = \frac{1}{4}(x_1 + x_2 + x_3 + x_4).
\]

The statistic is a random variable, for example \( \bar{X} \) and hence has a pdf

\[ p(\bar{X}). \]
The distribution of the sample mean

Proposition

Let $X_1, ..., X_n$ be drawn identically and independently (iid) from a distribution with mean $\mu$ and standard deviation $\sigma$.

The sample mean $\bar{X}$ and sum $S_n$ are

$$\bar{X} = \frac{S_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n}.$$

1. $E[\bar{X}] = \mu$ and $E[S_n] = n\mu$
2. $V[\bar{X}] = \sigma^2/n$ and $V[S_n] = n\sigma^2$. 
Proof of proposition

\[
\mathbb{E}[S_n] = \mathbb{E}\left[ \sum_{i=1}^{n} X_i \right] \\
= \sum_{i=1}^{n} \mathbb{E}[X_i] \quad \text{linearity} \\
= \sum_{i=1}^{n} \mu \quad \text{definition} \\
= n\mu \quad \text{definition}. \\
\]

\[
\mathbb{E}[\bar{X}] = \frac{\mathbb{E}[S_n]}{n} = \mu.
\]
Proof of proposition

\[ \text{Var}(S_n) = \text{Var} \left( \sum_{i=1}^{n} X_i \right) \]

\[ = \sum_{i=1}^{n} \text{Var}(X_i) \quad \text{independence} \]

\[ = \sum_{i=1}^{n} \sigma^2 \quad \text{definition} \]

\[ = n\sigma^2 \quad \text{definition}. \]
\[ \mathbb{V}[\bar{X}] = \mathbb{V} \left( \frac{\sum_{i=1}^{n} X_i}{n} \right) \]

\[ \mathbb{V}[\bar{X}] = \frac{1}{n^2} \mathbb{V} \left( \sum_{i=1}^{n} X_i \right) \text{ linearity} \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{V}[X_i] \text{ independence} \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 \text{ definition} \]

\[ = \frac{\sigma^2}{n} \text{ definition.} \]
Normal Population Case

Proposition

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean $\mu$ and standard deviation $\sigma$. Then for any $n$ $\bar{X}$ is normally distributed with mean $\mu$ and standard deviation $\sigma/\sqrt{n}$. 
Proposition

Let $X_1, ..., X_n$ be a random sample from a distribution with mean $\mu$ and standard deviation $\sigma$. Then for $n$ large enough ($n > 30$), $\bar{X}$ is normally distributed with mean $\mu$ and standard deviation $\sigma / \sqrt{n}$. 
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CLT

http://www.stat.sc.edu/~west/javahtml/CLT.html
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Pictures

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Matlab code

```matlab
a=-1;
b=1;
m=500000;
hold on;

xbar1 = usamp(1,m,a,b);
xbar2 = usamp(3,m,a,b);
xbar3 = usamp(30,m,a,b);
xbar3 = usamp(80,m,a,b);

[cts1,ind1] = normh(xbar1);
[cts2,ind2] = normh(xbar2);
[cts3,ind3] = normh(xbar3);
[cts4,ind4] = normh(xbar4);

plot(cts1,ind1,'b')
plot(cts2,ind2,'m')
plot(cts3,ind3,'r')
plot(cts4,ind4,'g')

h=gca;
set(h,'FontSize',[20]);
legend('n=1','n=3','n=30','n=80');
s=sprintf('m=%d',m);
title(s);
hold off
```

Artin Armagan and Sayan Mukherjee
Joint distributions and the central limit theorem
function [cts,inds] = normh(x)

m = length(x);
[inds,cts] = hist(x,50);
del = (cts(2)-cts(1));
inds = inds/(m*del);

function xbar = usamp(n,m,a,b)

xbar = zeros(m,1);
for i=1:m
xs = rand(n,1)*(b-a)+a;
xbar(i,1) = mean(xs);
end
Let $X_1, \ldots, X_n$ be iid draws from a Bernoulli distribution with mean $p$.
Notice that a new random variable $X = X_1 + \ldots + X_n$ is binomially distributed with probability of success $p$ and number of trials $n$. Then, for $n$ large enough, this binomial distribution can approximated by a normal distribution through the central limit theorem.