Practice Examination $\#3$

Sta 230: Probability

October 4, 2012

This is a closed-book exam so do not refer to your notes, the text, or any other books (please put them on the floor). You may use a single sheet of notes or formulas and a calculator, but materials may not be shared. A formula sheet and four blank worksheets are attached to the exam.

You must show your work to get partial credit. Even correct answers will not receive full credit without justification.

Please give all numerical answers to at least two correct digits or as exact fractions reduced to lowest terms. Write your solutions as clearly as possible and make sure it's easy to find your answers (circle them if necessary), since you will not receive credit for work that I cannot understand or find. Good Luck!

If you find a question confusing please **ask me** to clarify it.

This is a practice exam. You will have to do 4 problems on the real exam. Here I gave you six for practice.

Problem 1 : A hand in bridge consists of thirteen cards dealt out from a well shuffled deck. A deck as 52 cards with four suits (hearts, clubs, diamonds, and spades).

a. What is the probability that the bridge hand contains exactly 5 hearts?

b. What is the probability that the bridge hand contains exactly 5 hearts and 5 spades?

c. What is the probability that the bridge hand contains exactly 5 hearts and 5 spades?

d. What is the probability that the hand contains exactly 5 cards from at least one suit?

e. Given the first three cards are 3 hearts what is the probability the next two are spades ?

Problem 2: There are two possible distributions that you are drawing data from

$$\begin{aligned} Y &\sim \text{Geometric}(p) \\ Y &\sim \text{Poisson}(\lambda) \end{aligned}$$

1. I run the following experiment n times identically and independently

   (1) $X \sim \text{Be}(p = .3)$

   (2) If X is 1 then draw $Y \sim \text{Geometric}(p)$
   If X is 0 then draw $Y \sim \text{Poisson}(\lambda)$.

   What is $\mathbf{E}[Y]$?

2. My n observations were either generated by either of the following experiments

$$\begin{aligned} \text{case a} : Y_1, ..., Y_n &\overset{iid}{\sim} \text{Geometric}(p) \\ \text{case b} : Y_1, ..., Y_n &\overset{iid}{\sim} \text{Poisson}(\lambda) \end{aligned}$$

   Compute the formula for $\Pr(\text{case a} \mid Y_1, ..., Y_n)$ and $\Pr(\text{case b} \mid Y_1, ..., Y_n)$.

3. I run the following experiment n times to obtain $Z_1, ..., Z_n$

   (1) $X_1, ..., X_5 \overset{iid}{\sim} \text{Poisson}(\lambda)$

   (2) $Z_i = \max\{X_1, ..., X_5\}$.

   What is $\mathbf{E}(Z)$?

Problem 3 : $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$

1. Define $Z = \frac{\sum_i X_i}{n}$, what is $\mathbf{E}[Z]$ and $\mathbf{V}[Z]$?
2. How do I compute transform the random variable Z into a standard normal variable W ?
3. Now suppose that $X_1, ..., X_n \overset{iid}{\sim} \mathrm{Bin}(m, p)$ again define $Z = \frac{\sum_i X_i}{n}$. For what value of m, p would Z approximate the distribution in part (1) ?
4. I run the following experiment n times to obtain $X_1, ..., X_n$

   (1) $\mu \sim \mathrm{U}[1, 3]$

   (2) $X_i \mid \mu \sim \mathrm{N}(\mu, 1)$.

   What is $\mathbf{E}(X)$?

Problem 4 : Stark's Pond contains 10 trout and 5 bluegill fish. Kyle catches a random number of fish (call the number X, where $X \sim \mathrm{U}(\{1, \ldots, 4\})$. Once caught, that fish is removed from the pond and cannot be caught again. Each new fish comes uniformly from the remaining fish.

1. What is the chance that Kyle catches all trout?
2. Suppose all the fish that Kyle caught were trout. Given this information, what is the probability that he caught exactly 5 fish?
3. What is the expected number of trout caught ?

Problem 5 : . A new drug for leukemia works 25% of the time in patients 55 and older, and 50% of the time in patients younger than 55. A test group consists of 17 patients 55 and older and 12 patients younger than 55.

1. A patient is chosen uniformly at random from the test group, the drug is administered, and it is a success. What is the probability the patient was in the older group?
2. A subgroup of 4 patients are chosen and the drug is administered to each. What is the probability that the drug works in all four patients?

Problem 6 : . You have a choice of placing your investments in one of six funds. The return in a year in dollars for the six firms are

$$X_1 \sim \text{Exp}(\lambda), \quad X_2, ..., X_6 \overset{iid}{\sim} \text{N}(\mu, \sigma^2).$$

1. What is the probability that at least one the funds returned $\geq \$1$?
1. What is $\mathbf{E}[\max\{X_1, ..., X_6\}]$?

(Nearly) Blank Work-Sheet #1 (of 4)

(Nearly) Blank Work-Sheet #2 (of 4)

(Nearly) Blank Work-Sheet #3 (of 4)

(Nearly) Blank Work-Sheet #4 (of 4)

Distributions

Binomial	$\Pr[X=x]$	$=$	$\binom{n}{x} p^x q^{(n-x)},\ x=0,1,\ldots,n$	$\mu=np,\ \sigma^2=npq$
Poisson	$\Pr[X=k]$	$=$	$\lambda^k\, e^{-\lambda}/k!,\ k=0,1,\ldots$	$\mu=\lambda,\ \sigma^2=\lambda$
Poisson process	$\Pr[X=k \mid t]$	$=$	$(\alpha t)^k\, e^{-\alpha t}/k!,\ k=0,1,\ldots$	
Geometric	$\Pr[X=x]$	$=$	$pq^x,\ y=0,1,2,\ldots$	$\mu=q/p,\ \sigma^2=q/p^2$
Hypergeometric	$\Pr[X=x]$	$=$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},\ x=0,1,2,\ldots$	$\mu=n\frac{M}{N}$
				$\sigma^2=\frac{N-n}{N-1}n\frac{M}{N}(1-\frac{M}{N})$
Exponential	$\Pr[X\le x]$	$=$	$1-e^{-\lambda x},\ x\in(0,\infty)$	$\mu=1/\lambda,\ \sigma^2=1/\lambda^2$
	$f(x)$	$=$	$\lambda e^{-\lambda x},\ x>0;\ 0,\ x\le 0$	
Uniform	$\Pr[X\le x]$	$=$	$(x-a)/(b-a),\ x\in(a,b)$	$\mu=\frac{a+b}{2},\ \sigma^2=\frac{(b-a)^2}{12}$
	$f(x)$	$=$	$\frac{1}{b-a},\ a<x<b;\ 0,\ x\notin(a,b)$	
Normal	$\Pr[X\le x]$	$=$	$\Phi(\frac{x-\mu}{\sigma}),\ x\in\mathbb{R}$	$\mu=\mu,\ \sigma^2=\sigma^2$
	$f(x)$	$=$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2},\ x\in\mathbb{R}$	

$$P_k^n = \frac{n!}{(n-k)!} = \overbrace{(n)(n-1)\cdots(n-k+1)}^{k \text{ terms}}$$

$$\text{“}n\text{ choose }k\text{”: } \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)(n-1)\cdots(n-k+1)}{(k)(k-1)\cdots(1)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
$$\sum_{k=0}^{\infty} ar^k = a/(1-r) \qquad \sum_{k=0}^{n} ar^k = (a - ar^{n+1})/(1-r)$$
$$\Pr[X \le x] = F(x) = \int_{-\infty}^{x} f(t)\,dt \qquad f(x) = F'(x)$$
$$\mathsf{E}[g(X)] = \int g(x) f(x)\,dx \qquad \mathsf{E}[g(X)] = \sum_x g(x) P(x)$$
$$\mu = \mathsf{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \sigma^2 = \mathsf{E}[(X-\mu)^2] = \mathsf{E}[X^2] - \mu^2$$

A **and** $B : A \cap B$ **not** $A : A^c$ A **or** $B : A \cup B$

$$\mathsf{P}[A|B] = \mathsf{P}[A \cap B]/\mathsf{P}[B] \qquad \mathsf{P}[B|A] = \mathsf{P}[A \cap B]/\mathsf{P}[A]$$
$$\mathsf{P}[A \cap B] = \mathsf{P}[A|B]\,\mathsf{P}[B] \qquad \mathsf{P}[A \cup B] = \mathsf{P}[A] + \mathsf{P}[B] - \mathsf{P}[A \cap B]$$

Independent $\Leftrightarrow \mathsf{P}[A \cap B] = \mathsf{P}[A]\,\mathsf{P}[B]$; Exclusive $\Leftrightarrow \mathsf{P}[A \cap B] = 0$

Bayes: $\mathsf{P}[A_i|B] = \frac{\mathsf{P}[B|A_i]\mathsf{P}[A_i]}{\mathsf{P}[B|A_1]\mathsf{P}[A_1] + \ldots + \mathsf{P}[B|A_k]\mathsf{P}[A_k]}$ $A_i \cap A_j = \emptyset, \quad \mathsf{P}[A_1] + \ldots + \mathsf{P}[A_k] = 1$

Sterling: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

		Continuous	Discrete
Mean	$\mu = \mathsf{E}[X] =$	$\int_{-\infty}^{\infty} x\, f(x) dx$	$= \sum x p(x)$
Variance	$\sigma^2 = \mathsf{Var}[X] =$	$\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$	$= \sum (x-\mu)^2 p(x)$
		$= \mathsf{E}[X^2] - \mathsf{E}[X]^2$	$= \mathsf{E}[X^2] - \mathsf{E}[X]^2$
Expectation	$\mathsf{E}[g(X,Y)] =$	$\iint g(x,y)\, f(x,y) dx\, dy$	$= \sum_{x,y} g(x,y)\, p(x,y)$
Covariance	$\mathsf{Cov}[X,Y] =$	$\iint (x-\mu_x)(y-\mu_y)\, f(x,y) dx\, dy$	$= \sum_{x,y} (x-\mu_x)(y-\mu_y)\, p(x,y)$
		$= \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$	$= \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$
Correlation coefficient	$\rho[X,Y] =$	$\mathsf{Cov}[X,Y]/\sigma_X \sigma_Y$	
Conditional	$f(x\|y) =$	$f(x,y)/f_2(y)$	$= p(x,y)/p_2(y)$
Marginal	$f_1(x) =$	$\int_{-\infty}^{\infty} f(x,y) dy$	$= \sum_y p(x,y)$
	$f_2(y) =$	$\int_{-\infty}^{\infty} f(x,y) dx$	$= \sum_x p(x,y)$
Independent		$\Leftrightarrow f(x,y) = f_1(x)\, f_2(y)$	$\Leftrightarrow p(x,y) = p_1(x)\, p_2(y)$
		$\Leftrightarrow f(x\|y) = f_1(x)$	$\Leftrightarrow p(x\|y) = p_1(x)$
		$\Leftrightarrow f(y\|x) = f_2(y)$	$\Leftrightarrow p(y\|x) = p_2(y)$

Conf. Intervals	$\mu = \bar{x} \pm z_{\alpha/2} S/\sqrt{n}$	Large Sample
	$= \bar{x} \pm t_{\alpha/2} S/\sqrt{n}$	Small Sample (normal)
	$p = \hat{p} \pm z_{\alpha/2} \sqrt{\hat{p}\,\hat{q}\,/n}$	Population Proportion
Diff. of Means	$\mu_1 - \mu_2 = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$	Large Samples
	$= (\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$	Small Samples
	$= \overline{x_1 - x_2} \pm z_{\alpha/2} s_\Delta/\sqrt{n}$	Matched Pairs, large n
	$= \overline{x_1 - x_2} \pm t_{\alpha/2} s_\Delta/\sqrt{n}$	Matched Pairs, small n, normal

Sample Mean	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$	
Sample Variance	$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$	Deg fdm: $\nu = n - 1$
Regression slope	$b_1 = \hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$	
Regression offset	$b_0 = \hat{\beta}_0 = \frac{\sum y_i - \hat{\beta}_1 \sum x_i}{n} = \bar{y} - \hat{\beta}_1 \bar{x}$	
Sum Square Error	$SSE = \sum (y_i - \hat{y}_i)^2$	
Estimate of variance	$\hat{\sigma^2} = \frac{SSE}{n-2}$	
TSS	$SST = \sum (y_i - \bar{y})^2$	
CD	$r^2 = 1 - \frac{SSE}{SST}$	
F-dist	$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$ F-dist with $m-1$,$n-1$ d.o.f.	