Practice exam solutions

Pr. 1 A hand in bridge consists of thirteen cards dealt out from a well mixed deck. A deck as 52 cards with four suits (hearts, clubs, diamonds, and spades).

(a) What is the probability that the bridge hand contains exactly 5 hearts?

$$p = \frac{\binom{13}{5}\binom{39}{8}}{\binom{52}{13}}.$$

(b) What is the probability that the bridge hand contains exactly 5 hearts?

$$p = \frac{\binom{13}{5}\binom{13}{5}\binom{26}{3}}{\binom{52}{13}}.$$

(d) What is the probability that the hand contains exactly 5 cards from at least one suit?

$$\begin{aligned}
\Pr(\text{not 5 cards from clubs suit}) &= 1 - \Pr(\text{5 cards from red suit}) \\
&= 1 - \frac{\binom{13}{5}\binom{39}{8}}{\binom{52}{13}} \\
&= \Pr(\text{not 5 cards from hearts suit}). \\
&= \Pr(\text{not 5 cards from diamonds suit}). \\
&= \Pr(\text{not 5 cards from spades suit}).
\end{aligned}$$

Now

$$\Pr(\text{exactly 5 cards from at least one suit}) = 1 - \Pr(\text{not 5 cards from any suit}).$$

and

$$\Pr(\text{not 5 cards from any suit}) = [\Pr(\text{not 5 cards from clubs suit})]^4,$$

so

$$\Pr(\text{exactly 5 cards from at least one suit}) = 1 - \left(1 - \frac{\binom{13}{5}\binom{39}{8}}{\binom{52}{13}}\right)^4.$$

(e) Given the first three cards are 3 hearts what is the probability the next two are spades ?

$$p = \frac{13}{49} \times \frac{12}{48}.$$

Pr. 2 Two distributions

$$\begin{aligned} Y &\sim \text{Geometric}(p) \\ Y &\sim \text{Poisson}(\lambda). \end{aligned}$$

1. Consider the following process

(1) $X \sim \text{Be}(p = .3)$.

(2) If $X = 1$ then $Y \sim \text{Geometric}(p)$
If $X = 0$ then $Y \sim \text{Poisson}(\lambda)$

What is $\mathbf{E}[Y]$?

$$\begin{aligned} \mathbf{E}[Y] &= \sum_y y\mathbf{P}[Y = y|X = 1]\mathbf{P}[X = 1] + \sum_y y\mathbf{P}[Y = y|X = 0]\mathbf{P}[X = 0] \\ &= .3 \times \sum_y y\mathbf{P}[Y = y|X = 1] + .7 \times \sum_y y\mathbf{P}[Y = y|X = 0] \\ &= .3 \times \frac{1-p}{p} + .7 \times \lambda. \end{aligned}$$

2. My n observations were either generated by either of the following experiments

$$\begin{aligned} \text{case } a : Y_1, ..., Y_n &\overset{iid}{\sim} \text{Geometric}(p) \\ \text{case } b : Y_1, ..., Y_n &\overset{iid}{\sim} \text{Poisson}(\lambda). \end{aligned}$$

Compute $\mathbf{P}(a \mid Y_1, ..., Y_n)$ and $\mathbf{P}(b \mid Y_1, ..., Y_n)$.
First

$$\begin{aligned} \ell_a &= \mathbf{P}(y_1, ..., y_n \mid a) = \prod_{i=1}^n [(1-p)^{y_i} p] \\ \ell_b &= \mathbf{P}(y_1, ..., y_n \mid b) = \prod_{i=1}^n \left[\frac{e^{-\lambda}\lambda^{y-i}}{y_i!}\right]. \end{aligned}$$

also $\mathbf{P}(a) = .3$ and $\mathbf{P}(b) = .7$. By Bayes rule

$$\begin{aligned} \mathbf{P}(a \mid Y_1, ..., Y_n) &= \frac{.3 \times \ell_a}{.3 \times \ell_a + .7 \times \ell_b} \\ \mathbf{P}(b \mid Y_1, ..., Y_n) &= \frac{.7 \times \ell_b}{.3 \times \ell_a + .7 \times \ell_b}. \end{aligned}$$

3. I run the following experiment n times to obtain $Z_1, ..., Z_n$

(1) $X_1, ..., X_5 \overset{iid}{\sim} \text{Exp}(\lambda)$

(2) $Z = \max\{X_1, ..., X_5\}$.

What is $\mathbf{E}(Z)$?

First

$$\begin{aligned} \mathbf{P}(Z \leq t) &= \mathbf{P}(\{X_1 \leq t\}, ..., \{X_5 \leq t\}) \\ &= [\mathbf{P}(\{X \leq t\})]^5. \end{aligned}$$

Also

$$\mathbf{P}(\{X \leq t\}) = 1 - e^{-\lambda t}.$$

Note that

$$\begin{aligned} \mathbf{E}(Z) &= \int_{t=0}^{\infty} \mathbf{P}(\{Z \geq t\}) dt \\ \mathbf{P}(\{Z \geq t\}) &= 1 - \mathbf{P}(Z \leq t) \\ &= 1 - (1 - e^{-\lambda t})^5 \\ \mathbf{E}(Z) &= \int_{t=0}^{\infty} [1 - (1 - e^{-\lambda t})^5] dt. \end{aligned}$$

Pr. 3 $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$

1. $Z = \frac{\sum X_i}{n}$, what is $\mathbf{E}(Z)$ and $\mathbf{V}(Z)$.
$\mathbf{E}(Z) = \mu, \quad \mathbf{V}(Z) = \frac{\sigma^2}{n}$.

2. How do I transform the random variable Z into a standard normal variable W ?

$$W = \frac{Z - \mu}{\sqrt{\frac{\sigma^2}{n}}}.$$

3. Now suppose that $X_1, ..., X_n \overset{iid}{\sim} \mathrm{Bin}(m, p)$ For what value of m, p would Z approximate the distribution in part (1) ?
$pm = \mu$ and $mp(1 - p) = \sigma^2$

4. I run the following experiment n times to obtain $X_, ..., X_n$

(1) $\mu \sim \mathrm{U}[1, 3]$

(2) $X_i \mid \mu \sim \mathrm{N}(\mu, 1)$

What is $\mathbf{E}(X)$?

$$\mathbf{E}(X) = \mathbf{E}[\mathbf{E}(X \mid \mu)].$$

So

$$\mathbf{E}(X \mid \mu) = \mu,$$

and

$$\frac{1}{2}\int_1^3 \mu d\mu = \frac{1}{4}\mu^2|_1^3 = 2.$$

Pr. 4 Don't do.

Pr. 5 A new drug for leukemia works 25% of the time in patients 55 and older, and 50% of the time in patients younger than 55. A test group consists of 17 patients 55 and older and 12 patients younger than 55.

1. A patient is chosen uniformly at random from the test group, the drug is administered, and it is a success. What is the probability the patient was in the older group?
Compute $\mathbf{P}(\text{older} \mid \text{success})$. Prior probabilities

$$\mathbf{P}(\text{older}) = \mathbf{\frac{17}{29}}, \quad \mathbf{P}(\text{younger}) = \mathbf{\frac{12}{29}}.$$

Likelihoods:

$$Y = 1 \text{ or success} \mid \text{older} \sim \text{Bernoulli}(.25), \quad Y = 1 \text{ or success} \mid \text{younger} \sim \text{Bernoulli}(.50).$$

So by Bayes rule

$$\mathbf{P}(\text{older} \mid \text{success}) = \frac{\mathbf{P}(\text{older}, \mathbf{Y} = \mathbf{1})}{\mathbf{P}(\mathbf{Y} = \mathbf{1})} = \frac{\mathbf{.25 \times \frac{17}{29}}}{\mathbf{.25 \times \frac{17}{29} + .5 \times \frac{12}{29}}}.$$

2. A subgroup of 4 patients are chosen and the drug is administered to each. What is the probability that the drug works in all four patients?

Compute $\mathbf{P}(Y_1, ..., Y_4 = 1)$.

$$\begin{aligned}\mathbf{P}(Y_1, ..., Y_4 = 1) \;=\;& \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 4 \text{ older})\mathbf{P}(4 \text{ older}) + \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 3 \text{ older})\mathbf{P}(3 \text{ older}) + \\ & \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 2 \text{ older})\mathbf{P}(2 \text{ older}) + \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 1 \text{ older})\mathbf{P}(1 \text{ older}) + \\ & \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 0 \text{ older})\mathbf{P}(0 \text{ older}).\end{aligned}$$

$$\begin{aligned}\mathbf{P}(Y_1, ..., Y_4 = 1 \mid 4 \text{ older}) &= .25^4 \\ \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 3 \text{ older}) &= .25^3 \times .5 \\ \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 2 \text{ older}) &= .25^2 \times .5^2 \\ \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 1 \text{ older}) &= .25^1 \times .5^3 \\ \mathbf{P}(Y_1, ..., Y_4 = 1 \mid 0 \text{ older}) &= .5^4\end{aligned}$$

$$\mathbf{P}(0 \text{ older}) = \frac{12 \times 11 \times 10 \times 9}{29 \times 28 \times 27 \times 26}.$$

$$\mathbf{P}(4 \text{ older}) = \frac{19 \times 18 \times 17 \times 16}{29 \times 28 \times 27 \times 26}.$$

$$\mathbf{P}(1 \text{ older}) = \frac{\binom{12}{3}\binom{19}{1}}{\binom{29}{4}}.$$

$$\mathbf{P}(2 \text{ older}) = \frac{\binom{12}{2}\binom{19}{2}}{\binom{29}{4}}.$$

$$\mathbf{P}(3 \text{ older}) = \frac{\binom{12}{1}\binom{19}{3}}{\binom{29}{4}}.$$

Pr. 6 You have a choice of placing your investments in one of six funds. The return in a year in dollars for the six firms are

$$X_1 \sim \text{Exp}(\lambda), \quad X_2, ..., X_6 \overset{iid}{\sim} \text{N}(\mu, \sigma^2)$$

1. What is the probability that at least one the funds returned $\geq \$1$? Set event e to be the event at least one of the funds returned $\geq \$1$.

$$\begin{aligned} \mathbf{P}(e) &= 1 - \left[(1 - \mathbf{P}(X_1 < t = 1)) \times (1 - \mathbf{P}(X_2 < t = 1)) \times \cdots \times (1 - \mathbf{P}(X_6 < t = 1))\right]. \\ &= 1 - \left[(1 - e^{-\lambda}) \times \Phi\left(\frac{1-\mu}{\sigma_2}\right)^5\right]. \end{aligned}$$

In the above we used the complement of none of the firms returning $t = 1$.

2. What is $\mathbf{E}(\max\{X_1, ..., X_6\})$?

Define $Z = \max\{X_1, ..., X_6\}$. Note

$$\begin{aligned} \mathbf{E}(\max\{X_1, ..., X_6\}) &= \mathbf{E}(Z) \\ &= \int_0^\infty \mathbf{P}(Z > t)dt. \end{aligned}$$

Note that

$$\mathbf{P}(Z > t) = \mathbf{P}(e).$$

So

$$\mathbf{E}(\max\{X_1, ..., X_6\}) = \int_{t=0}^\infty \left[1 - \left[(1 - e^{-\lambda t}) \times \Phi\left(\frac{t-\mu}{\sigma_2}\right)^5\right]\right] dt.$$