1. Positive (semi) definite

a) A symmetric matrix has real eigenvalues.

The complex version of a inner produ~~c~~t

$$\langle s,t\rangle = s^T t^*, \quad t^* \text{ is the complex conjugate.}$$

Some properties

$$\langle Ks,t\rangle = \langle s, K^T t\rangle \quad \text{if } K \text{ is real}$$

$$\langle \lambda s,t\rangle = \lambda^* \langle s,t\rangle$$

$$\langle s,\lambda t\rangle = \lambda \langle s,t\rangle.$$

Observe for symmetric A that is real

$$\lambda^* \langle s,s\rangle = \langle \lambda s, s\rangle = \langle As,s\rangle$$
$$= \langle s, As\rangle = \langle s,\lambda s\rangle$$
$$= \lambda \langle s,s\rangle.$$

So $\lambda = \lambda^*$ since $\langle s,s\rangle \neq 0$.

b) If $\lambda_1, \dots, \lambda_n > 0$ then $\alpha^T K \alpha > 0 \quad \forall \alpha \in \mathbb{R}^n$.

$K = U \Lambda U^T$, $\quad \Lambda^{1/2}_{ii} = \lambda_i^{1/2}$.

$$\alpha^T U \Lambda^{1/2} \Lambda^{1/2} U^T \alpha = \alpha^T K \alpha,$$

$$r = \Lambda^{1/2} U^T \alpha$$

$$r^T = \alpha^T U (\Lambda^{1/2})^T = \alpha^T U \Lambda^{1/2}$$

$r^T r = \|r\|^2 \geq 0$.

i) if $\lambda_1, \dots, \lambda_n > 0 \Rightarrow r \neq 0 \Rightarrow \|r\|^2 > 0$

ii) if $\lambda_1, \dots, \lambda_n \geq 0 \Rightarrow r$ can be 0

so $\|r\|^2 \geq 0$.

The eigenvalues state the matrix is positive definite if $\lambda_1, \dots, \lambda_n > 0$ and if $\lambda_1, \dots, \lambda_n \geq 0$ then positive semi definite.

2. In words: If for any finite collection of size $n \in \mathbb{N}$ the matrix is positive definite, then the Kernel function is positive definite.

1) a) $K(u,v) = \langle u, v \rangle$

$= (\langle u,v \rangle + 1)^d$, $d > 0$, $d \in \mathbb{N}$

$K(u,v) = \exp(-\kappa \|u-v\|^2)$

are all positive (semi) definite

b) $K(u,v) = au^2 + 2buv + cv^2$ $u,v \in \mathbb{R}$

$a < 0$, $c < 0$.

Problem 2.

a. $$\arg\min_{f \in \mathcal{H}_K} \left[\frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-y_i f(x_i))) + \lambda \|f\|_K^2 \right]$$

$$f_\parallel \in \text{span}(\phi(x_1), \dots, \phi(x_n)) \text{ where}$$

$$\phi(x_i) = \begin{pmatrix} \sqrt{\lambda_1}\, \phi_1(x_i) \\ \vdots \\ \sqrt{\lambda_\ell}\, \phi_\ell(x_i) \end{pmatrix}$$

$$f_\perp \in \text{span}(\mathcal{H}_K) \ominus f_\parallel .$$

$$f = f_\perp + f_\parallel$$

From notes

$$\frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-y_i f(x_i))) = \frac{1}{n} \sum \log(1 + \exp(-y_i f_\parallel(x_i)))$$

because $f(x_i) = \langle f, \phi(x_i) \rangle$

$$= \langle f_\parallel, \phi(x_i) \rangle$$

$$\|f_{\parallel} + f_{\perp}\|_K^2 > \|f_{\parallel}\|_K^2 \quad \text{if} \quad f_{\perp} \neq 0.$$

So $f_{\perp} = 0$ for minimizer, or

$$f(x) = \sum_{i=1}^{n} \alpha_i K(x, x_i)$$

b) $$\arg\min_{\alpha \in \mathbb{R}^n} \left[\frac{1}{n} \sum \log\left(1 + \exp\left(-(y\, K\alpha)_i\right)\right) + \lambda\, \alpha^T K \alpha \right]$$

c) $K(u,v) = (1 + u^T v)^2, \quad u, v \in \mathbb{R}$

$$\phi(u) = \begin{pmatrix} 1 \\ \sqrt{2}\, u \\ u^2 \end{pmatrix}, \qquad \phi(v) = \begin{pmatrix} 1 \\ \sqrt{2}\, v \\ v^2 \end{pmatrix}$$

$$\phi^T(u)\, \phi(v) = 1 + 2uv + u^2 v^2$$
$$= (1 + uv)^2$$

Problem 3.

Go look at fourier.eng.hmc.edu/e161/lectures/gaussianprocess/node7.html

Problem 4.

a) Primal:

$$\min_{c} \ \frac{1}{2} c^T K c + C \sum \xi_i$$

$$\text{s.t.} \quad y_i \left(\sum_j c_j k(x_i, x_j)\right) \geq 1 - \xi_i, \quad i = 1, \ldots, n$$

$$\xi_i \geq 0$$

Dual:

$$L(c, \xi, \alpha, \eta) = \frac{1}{2} c^T K c + C \sum \xi_i - \sum_i \alpha_i \left[y_i \left(\sum_j c_j k(x_i, x_j) \right) - 1 + \xi_i \right]$$

$$- \sum_i \eta_i \xi_i$$

$$\frac{\partial L}{\partial \xi_i} = 0 \Rightarrow C - \alpha_i - \xi_i = 0 \Rightarrow$$

$$L^R(c, \alpha, \eta) = \frac{1}{2} c^T K c$$

$$- \sum \alpha_i \left[y_i \left(\sum_j c_j k(x_i, x_j) \right) - 1 \right]$$

also $0 \leq \alpha_i \leq C$

$$L^R(c,\alpha,\eta) = \frac{1}{2} c^T K c + \sum \alpha_i - \sum_i \alpha_i \Big[\sum_j c_j K(x_i, x_j)\Big]$$

$$\frac{\partial L^R}{\partial c} = 0 \Rightarrow \quad c_i = \alpha_i y_i$$

Dual

$$\min_{\alpha \in \mathbb{R}^n} \quad \sum \alpha_i - \alpha^T Q \alpha \,, \qquad Q_{ij} = K_{ij} \, y_i \, y_j$$

s.t. $\quad 0 \leq \alpha_i \leq C$.

b) KKT conditions

$$C - \alpha_i - \eta_i = 0$$

$$y_i \left(\sum_j y_j \alpha_j k(x_i, x_j) \right) - 1 + \xi_i \geq 0, \quad i=1,\dots,n$$

$$\alpha_i \left[y_i \left(\sum_j y_j \alpha_j K(x_i, x_j) \right) - 1 + \xi_i \right] = 0, \quad i=1,\dots,n$$

$$\eta_i \xi_i = 0 \quad , \quad i=1,\dots,n$$

$$\xi_i, \alpha_i, \eta_i \geq 0 \; ; \; i=1,\dots,n$$

$$\sum c_j K(x_i, x_j) - \sum y_i \alpha_j K(x_i, x_j) = 0, \quad i=1,\dots,n,$$