\[ \pi - \lambda \text{ Theorem} \]

- A class \( \mathcal{P} \) of subsets of \( \Omega (= X) \) is a \textbf{\( \pi \)-system} if it is closed under the formation of finite intersections: \([\mathcal{P}] \). \( A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P} \).
  - Examples:
    - \( \{\emptyset\} \)
    - \( \{\Omega\} \)
    - \( \{A\} \)
    - any algebra
    - any \( \sigma \)-field
    - \( \{(-\infty, x], x \in \mathbb{R}\} \)
    - \( \{(a_i, b_i] \times \cdots \times (a_n, b_n] : a_i, b_i \in \mathbb{R}\} \)
  - If \( \mathcal{P} \) is a \( \pi \)-system; so are \( \mathcal{P} \cup \{\Omega\}, \mathcal{P} \cup \{\emptyset\} \), \( \mathcal{P} \cup \{\emptyset, \Omega\} \).
  - For any class \( \mathcal{A} \) of subsets. Let \( \mathcal{P} \) be the class of all finite intersections of elements of \( \mathcal{A} \). Then, \( \mathcal{P} \) is a \( \pi \)-system.
    - Moreover, \( \sigma(\mathcal{P}) = \sigma(\mathcal{A}) \).

\textit{Proof.} We shall assume the trivial intersection is in \( \mathcal{P} \), i.e. \( A \subset \mathcal{P} \). Note that any finite intersection of elements in \( \mathcal{A} \) is in \( \sigma(\mathcal{A}) \). So, \( \mathcal{P} \subset \sigma(\mathcal{A}) \). From \( A \subset \mathcal{P} \subset \sigma(\mathcal{A}) \), we have \( \sigma(\mathcal{P}) = \sigma(\mathcal{A}) \).

- A class \( \mathcal{L} \) is a \textbf{\( \lambda \)-system} if it contains \( \Omega \) and is closed under the formation of complements and of finite and countable disjoint unions:
  - \( \Omega \in \mathcal{L} \)
  - \( A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L} \)
  - \( \text{Disjoint } A_1, A_2, \ldots \in \mathcal{L} \Rightarrow \bigcup_n A_n \in \mathcal{L} \).
    - Because of the disjointness condition in L3, the definition of \( \lambda \)-system is weaker (more inclusive) than that of \( \sigma \)-field.
    - L1 + L2 \( \Rightarrow \emptyset \in \mathcal{L} \). Combine with L2, then the countably infinite case of L3 implies the finite one (take \( A_n = \emptyset \) \( \forall n > N \)).
    - Apply L2 to L3, disjoint \( A_1, A_2, \ldots \in \mathcal{L} \Rightarrow \bigcap_n A_n^c \in \mathcal{L} \).
    - Under L1 and L3, we have L2 \( \equiv \mathcal{L} \) is closed under the formation of \textit{proper} differences: \( [L2'] A, B \in \mathcal{L} \) and \( A \subset B \Rightarrow B - A \in \mathcal{L} \).
      In fact, we have L2 + L3 \( \Rightarrow \) L2'. And, L2' + L1 \( \Rightarrow \) L2.
Proof. “⇒” Suppose \( A, B \in \mathcal{L}, A \subset B \). Then, \( B - A = \left( \bigcup_{\alpha \in \mathcal{L}} A^c \right) \in \mathcal{L} \).

In general, it is not true that it is closed under any difference.

- A \( \sigma \)-field is a \( \lambda \)-system.
  The reverse is not true

  - Example: Take \( \Omega = \{a_1, a_2, a_3, a_4\} \). Let \( \mathcal{L} = \{\emptyset, \Omega\} \cup \left( \frac{Q}{2} \right) \). Then, to be \( \sigma \)-algebra, it has to contain \( \{\{a_1, a_2\} \cup \{a_2, a_3\}\} \setminus \{a_1, a_3\} = \{a_i\} \).

- [Durrett’s Def] A class \( \mathcal{L} \) is a \( \lambda \)-system if \( L_1, L_2', L_3' \): \( A_n \in \mathcal{L} \) and \( A_n \not\supset A \) then \( A \in \mathcal{L} \).
  Proof. “⇒” \( L_2, L_3 \Rightarrow L_2' \). By \( L_2' \), because \( A_{i-1} \subset A_i \), we have \( B_i = A_i \setminus A_{i-1} \in \mathcal{L} \), \( i \geq 2 \). \( B_i = A_i \in \mathcal{L} \). Then, \( B_i \)’s are disjoint. So, by \( L_3' \),
  \[ A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathcal{L} \] and so we have \( L_3' \).
  This is a weaker condition than the \( \lambda \)-system defined above.

- A class that is both a \( \pi \)-system and a \( \lambda \)-system is a \( \sigma \)-field
  Proof. We only need to show that it is closed under countable union. Suppose \( A_1, A_2, \ldots \in \mathcal{L} \). Let \( B_i = A_i, B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i = A_n \cap \bigcap_{i=1}^{n-1} A_i^c \) for \( n > 1 \). Then, by
  \[ [P] \text{ and } [L2], B_n \in \mathcal{L}. \]
  Note that \( \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \). Because \( B_i \)'s are disjoint, by
  \[ [L3], \bigcup_{i=1}^{\infty} B_i \in \mathcal{L}. \]
  So, \( \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \in \mathcal{L} \).

- Dynkin’s \( \pi \)-\( \lambda \) theorem:
  If \( \mathcal{P} \) is a \( \pi \)-system and \( \mathcal{L} \) is a \( \lambda \)-system, then \( \mathcal{P} \subset \mathcal{L} \Rightarrow \sigma(\mathcal{P}) \subset \mathcal{L} \).
  Proof. [Bellingsley 42]
Let $\mathcal{L}_0$ be the $\lambda$-system generated by $\mathcal{P}$—that is, the intersection of all $\lambda$-systems containing $\mathcal{P}$. Then, $\mathcal{L}_0$ is also a $\lambda$-system which is contained in every $\lambda$-systems which contains $\mathcal{P}$. (same proof as the construction of generated $\sigma$-filed which based mainly on the facts that an element is in the intersection iff it is in every sets). In particular, $\mathcal{L}_0 \subset \mathcal{L}$ because $\mathcal{L}$ is one of the set in the intersection. (So, we have $\mathcal{P} \subset \mathcal{L}_0 \subset \mathcal{L}$.)

Claim 1: $\mathcal{L}_0$ is a $\pi$-system.

Proof of the claim 1:

For each $A$ (in $\mathcal{L}$ or in $2^\mathcal{X}$), let $\mathcal{L}_A = \{B : A \cap B \in \mathcal{L}_0\}$.

Claim 1.1: If $A \in \mathcal{L}_0$, then $\mathcal{L}_A$ is a $\lambda$-system. (*)

Proof of claim 1.1

To see this,

(1) $A \cap \Omega = A \in \mathcal{L}_0$ by assumption; so $\Omega \in \mathcal{L}_A$.

(2') If $B_1, B_2 \in \mathcal{L}_A$ and $B_1 \subset B_2$, then, by definition, $A \cap B_1, A \cap B_2 \in \mathcal{L}_0$. Now, because $\mathcal{L}_0$ is a $\lambda$-system, and $A \cap B_1 \subset A \cap B_2$, we have $A \cap B_2 - A \cap B_1 \in \mathcal{L}_0$ by [L2']. But $A \cap B_2 - A \cap B_1 = A \cap (B_2 - B_1)$. So, $B_2 - B_1 \in \mathcal{L}_A$.

(3) Finally, disjoint $B_n \in \mathcal{L}_A \Rightarrow$ disjoint $A \cap B_n \in \mathcal{L}_0 \Rightarrow 
A \cap \bigcup_n B_n = \bigcup_n (A \cap B_n) \in \mathcal{L}_0$ (by [L3]) $\Rightarrow \bigcup_n B_n \in \mathcal{L}_A$.

We will only need this for $A \in \mathcal{P} \subset \mathcal{L}_0$.

Claim 1.2: $A \in \mathcal{P} \Rightarrow \mathcal{L}_0 \subset \mathcal{L}_A$

Proof of claim 1.2

If $A \in \mathcal{P} \subset \mathcal{L}_0$, then for any $B \in \mathcal{P}$, we have $A \cap B \in \mathcal{P} \subset \mathcal{L}_0$ by definition of $\mathcal{P}$ being a $\pi$-system. So, $B \in \mathcal{L}_A$. Hence, $\mathcal{P} \subset \mathcal{L}_A$. So, $\mathcal{L}_A$ is a $\lambda$-system (by (*)) which contains $\mathcal{P}$. By construction of $\mathcal{L}_0$, $\mathcal{L}_0 \subset \mathcal{L}_A$. 

\[A\]
\[B_1\]
\[B_2\]
Note that by symmetry of the definition,
\[ A \in \mathcal{L}_C \iff C \in \mathcal{L}_A \iff A \cap C \in \mathcal{L}_0. \]

**Claim 1.3:** \( C \in \mathcal{L}_0 \Rightarrow \mathcal{L}_0 \subset \mathcal{L}_C \)

**Proof of claim 1.3**

Now, consider \( C \in \mathcal{L}_0 \). We know (from claim 1.2) that \( \forall A \in \mathcal{P} \), \( \mathcal{L}_0 \subset \mathcal{L}_A \). Combining \( \mathcal{L}_0 \subset \mathcal{L}_A \) with \( C \in \mathcal{L}_0 \) gives \( C \in \mathcal{L}_A \) which, as noted above, also implies \( A \in \mathcal{L}_C \). But this is true for any \( A \in \mathcal{P} \). So, \( \mathcal{P} \subset \mathcal{L}_C \). From (\(^*\)), we already know that \( \mathcal{L}_C \) is a \( \lambda \)-system. Hence, \( \mathcal{L}_C \) is a \( \lambda \)-system which contains \( \mathcal{P} \). By construction of \( \mathcal{L}_0 \), \( \mathcal{L}_0 \subset \mathcal{L}_C \).

This is true for any \( C \in \mathcal{L}_0 \). Hence,
\[ \forall C \in \mathcal{L}_0 \mathcal{L}_0 \subset \mathcal{L}_C \ (**) \]

Finally, let \( D_1, D_2 \in \mathcal{L}_0 \). Then, by (\( ** \)), \( D_1 \in \mathcal{L}_0 \subset \mathcal{L}_{D_1} \) (and vice versa). By definition of \( \mathcal{L}_{D_1} \), we then have \( D_1 \cap D_2 \in \mathcal{L}_0 \). So, \( \mathcal{L}_0 \) is a \( \pi \)-system.

Because \( \mathcal{L}_0 \) is both a \( \pi \)-system and \( \lambda \)-system, it is a \( \sigma \)-field. Because it is a \( \sigma \)-field which contains \( \mathcal{P} \), we have \( \sigma(\mathcal{P}) \subset \mathcal{L}_0 \) by the minimality of \( \sigma(\mathcal{P}) \).

Because we also have \( \mathcal{L}_0 \subset \mathcal{L} \), we conclude that \( \sigma(\mathcal{P}) \subset \mathcal{L} \).

- Useful for many uniqueness arguments.

**Monotone Class Theorem**

- **Definition:** A class \( \mathcal{C} \) of subsets of \( \Omega \) \((\mathcal{C} \subset 2^\Omega)\) is **closed** if
  - Under **finite intersections** if for when \( A_1, \ldots, A_n \in \mathcal{C} \), then \( \bigcap_{i=1}^n A_i \in \mathcal{C} \) as well (\( n \) arbitrary but finite).
  - Under **increasing limits** if whenever \( A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \) is a sequence of sets in \( \mathcal{C} \), then \( \bigcup_{n=1}^\infty A_n \in \mathcal{C} \) as well.
  - Under **differences** if whenever \( A, B \in \mathcal{C} \) with \( A \subset B \), then \( B \setminus A = B \cap A^c \in \mathcal{C} \).

- A \( \sigma \)-algebra is closed under finite intersections, increasing limits, and differences.

- Intersection of classes of sets closed under differences is again a class of that type. Intersection of classes of sets closed under finite intersection is again a class of that type.

- Intersection of classes of sets closed under increasing limits is again a class of that type.

**Proof.** Let \( \mathcal{C}_i, i \in I \) be classes which are closed under (1) differences, (2) finite differences, and (3) increasing limits.
(1) \( A, B \in \bigcap_{i \in I} C_i \Rightarrow A, B \in C_i \forall i \in I \Rightarrow B \setminus A \subseteq C_i \forall i \in I \Rightarrow B \setminus A \subseteq \bigcap_{i \in I} C_i \). (2)

\[
A_1, \ldots, A_n \in \bigcap_{i \in I} C_i \Rightarrow A_1, \ldots, A_n \in C_i \forall i \in I \Rightarrow \bigcap_{i = 1}^n A_n \in C_i \forall i \in I \Rightarrow \bigcap_{i = 1}^n A_n \in \bigcap_{i \in I} C_i .
\]

(3) \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \in \bigcap_{i \in I} C_i \Rightarrow A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \in C_i \forall i \in I \Rightarrow \bigcup_{n=1}^\infty A_n \in C_i \forall i \in I \Rightarrow \bigcup_{n=1}^\infty A_n \in \bigcap_{i \in I} C_i .
\]

- Let (*) be a property of the class \( C_i \), \( i \in I \). If \( \bigcap_{i \in I} C_i \) also has property (*), then if \( C \) satisfies (*) then there exists the smallest class \( B \) containing \( C \) which also satisfies (*).

\[
B = \bigcap_{\substack{C \subseteq A \text{ satisfies } (*)}} A .
\]

- A **monotone class** \( \mathcal{M} \) is a collection of sets (subsets of \( X \)) with the properties:

1. If \( A_i \in \mathcal{M} \), and \( A_1 \subseteq A_2 \subseteq \cdots \), then \( \bigcup_{i=1}^\infty A_i \in \mathcal{M} \).
   - closed under countable increasing unions
   - closed under the formation of monotone unions
   - \( A_n \uparrow A \Rightarrow A \in \mathcal{M} \)

2. If \( B_i \in \mathcal{M} \), and \( B_1 \supseteq B_2 \supseteq \cdots \), then \( \bigcap_{i=1}^\infty B_i \in \mathcal{M} \).
   - closed under countable decreasing intersections
   - closed under the formation of monotone intersections
   - \( A_n \downarrow A \Rightarrow A \in \mathcal{M} \)

- (1) + [ \( \mathcal{M} \) is closed under complement ] \( \Rightarrow \) (2)

*Proof.* If \( B_i \in \mathcal{M} \), and \( B_1 \supseteq B_2 \supseteq \cdots \), then by (3) \( A_i = B_i^c \in \mathcal{M} \) and increasing.

Hence, \( \bigcup_{i=1}^\infty A_i \in \mathcal{M} \). Apply (3) again, and we have \( \left( \bigcup_{i=1}^\infty A_i \right)^c = \bigcap_{i=1}^\infty B_i \in \mathcal{M} \).

- **\( \mathcal{F} \) is a \( \sigma \)-field** iff **\( \mathcal{F} \) is a monotone field.**

*Proof.* [See the \( \sigma \)-field section].

- A **\( \sigma \)**-algebra is a monotone class.

- Monotonicity is preserved under arbitrary intersection.
Proof. Let I be an index set possibly uncountable. Suppose $\forall \alpha \in I$, $\mathcal{M}_\alpha$ is monotone. Consider $\bigcap_{\alpha \in I} \mathcal{M}_\alpha \subseteq \mathcal{M} = \bigcap_{\alpha \in I} \mathcal{M}_\alpha$. Suppose $A_i \in \mathcal{M}_\alpha$. Then, $\forall \alpha \in I$ $A_i \in \mathcal{M}_\alpha$.

(1) Suppose $A_i \supseteq A$. Because $\mathcal{M}_\alpha$ is monotone, we know that $A \in \mathcal{M}_\alpha$. This is true $\forall \alpha \in I$. Hence, $A \in \mathcal{M}$.

(2) Same as (1), but change the assumption to $A_i \subseteq A$.

- Suppose $\mathcal{B}$ is a monotone class.
  - $\mathcal{D} = \{B \in 2^X : B^c \in \mathcal{B}\}$ and $\tilde{\mathcal{D}} = \{B \in \mathcal{B} : B^c \in \mathcal{B}\}$ are also monotone.

  Proof. Suppose $D_i \supseteq D$. By definition of $\mathcal{D}$, we have $D_i^c \in \mathcal{B}$.

  Note that

  $$
  D_i^c \supseteq D^c \left( \bigcap_{i=1}^n D_i^c = \left( \bigcup_{i=1}^n D_i \right)^c \right).$$

  Because $\mathcal{B}$ is monotone, $D^c \in \mathcal{B}$. Hence, $D \in \mathcal{D}$.

  For $\tilde{\mathcal{D}}$, we need to prove also that $D \in \mathcal{B}$. This is trivial because $\tilde{\mathcal{D}} \subseteq \mathcal{B}$.

  So, $D_i \in \tilde{\mathcal{D}} \Rightarrow D_i \in \mathcal{B}$. $\mathcal{B}$ is monotone, so, $D_i \supseteq D \Rightarrow D \in \mathcal{B}$.

  Similar argument for $D_i \supseteq D$. Switch $\supseteq$ and $\supseteq$ in the above argument.

- Let $A \subset X$. Then, $\mathcal{G}_A = \{C \in 2^X : A \cup C \in \mathcal{B}\}$ and $\tilde{\mathcal{G}}_A = \{C \in \mathcal{B} : A \cup C \in \mathcal{B}\}$ are monotone.

  Note that no requirement is imposed on $A$.

  Proof. Suppose $G_i \supseteq G$. Then, $A \cup G_i \in \mathcal{B}$.

  Note that $(A \cup G_i) \supseteq (A \cup G)$.

  Because $\mathcal{B}$ is monotone, $A \cup G \in \mathcal{B}$. So, $G \in \mathcal{G}_A$.

  For $\tilde{\mathcal{G}}_A$, same proof as above but need to show further that $G \in \mathcal{B}$. This is easy because $\tilde{\mathcal{G}}_A \subseteq \mathcal{B}$. So, $G_i \supseteq G \Rightarrow G_i \supseteq G$. Because $\mathcal{B}$ is monotone, $G \in \mathcal{B}$.

  Similar proof for $G_i \supseteq G$. Switch $\supseteq$ and $\supseteq$ in the above argument.

- Let $\mathcal{A}$ be a collection of subsets of $X$. Then, $\mathcal{G}_A = \{C \in 2^X : \forall A \in \mathcal{A}, A \cup C \in \mathcal{B}\}$ and $\tilde{\mathcal{G}}_A = \{C \in \mathcal{B} : \forall A \in \mathcal{A}, A \cup C \in \mathcal{B}\}$ are monotone.

  Note that no requirement is imposed on $\mathcal{A}$.
Proof. Same proof as above. Instead of working with one $A$; now, we have
$\forall A \in A \ A \cup G \in B$. Fix $A$, the above argument gives $A \cup G \in B$. This is
true $\forall A \in A$; so, $G \in G_A$.

Alternative Proof

$G_A = \bigcap_{A \in A} G_A$? Monotonicity is preserved under intersection.

- Let $A$ be a class of subsets of $\Omega$. Define

$$\mathcal{M}(A) = \bigcap \{ \mathcal{M} : A \subset \mathcal{M}, \text{\mathcal{M} is monotone} \}$$

i.e. the intersection of all monotone classes containing $A$.

Then, $\mathcal{M}(A)$ is the smallest monotone class which contains $A$, in the sense that
$\mathcal{M}(A)$ is the subset of all monotone class that contains $A$.

Proof. Monotonicity is preserved under arbitrary intersection. Also, any $x$ in $A$ is in
every classes that are intersected. Hence, it is also in the intersection.

- $A \subset \mathcal{M}(A) \subset \sigma(A)$

Proof. By definition $\mathcal{M}(A) = \bigcap \{ \mathcal{M} : A \subset \mathcal{M}, \text{\mathcal{M} is monotone} \}$. Note that $\sigma(A)$ is
a monotone class which contains $A$. Therefore, It is one of those $\mathcal{M}$ being
intersected. Thus, $\mathcal{M}(A) \subset \sigma(A)$.

- Let $A$ be an algebra of subsets of $\Omega$. Then,

- $\mathcal{M}(A)$ is a $\sigma$-algebra.

Proof. Let $B = \mathcal{M}(A)$. Then, $B$ is the smallest monotone class which contains
$A$.

To show that $B$ is a $\sigma$-algebra, it suffices to show that $B$ is a field.

1) $\emptyset \in A \subset B$.

2) Let $\mathcal{D} = \{ B \in B : B^c \in B \}$. First, note that $\mathcal{D} \subset B$. Also, $B$ monotone
$\Rightarrow \mathcal{D}$ monotone. Furthermore, because $A$ is an algebra, $\forall A \in A$ we
know that $A^c \in A \subset B$. Hence, $A \subset \mathcal{D}$. Therefore, $\mathcal{D}$ is a monotone
class that contain $A$. $B$ is the smallest monotone class containing $A$; so, $B \subset \mathcal{D}$. We then conclude that $\mathcal{D} = B$.

3) Let $\mathcal{G}_A = \{ C \in 2^A : \forall A \in A, A \cup C \in B \}$. Then, $\mathcal{G}_A$ is monotone. Note
that for $\tilde{A} \in A$, we have $\forall A \in A \ A \cup \tilde{A} \in A \subset B$ because $A$ is a
field. Hence, $A \subset \tilde{G}_A$. By minimality of $B$, we have $B \subset \mathcal{G}_A$. Hence,
$\forall B \in B \ \forall A \in A, A \cup B \in B \ \text{(**)}$

Let $\mathcal{G}_B = \{ C \in 2^A : \forall A \in B, A \cup C \in B \}$. Then, $\mathcal{G}_B$ is monotone. Note
that (***) also tells $\forall A \in A \ \forall B \in B, A \cup B \in B$ (; we just switch the
two $\forall$’s). Therefore, $A \subset \tilde{G}_B$. By minimality of $B$, we have $B \subset \mathcal{G}_B$. 
Hence, by definition, $\forall B \in \mathcal{B}$ because $B \in \mathcal{G}_B$, we have $\forall A \in \mathcal{B}$ $A \cup B \in \mathcal{B}$. So, $\mathcal{B}$ is closed under finite union. So, $\mathcal{B}$ is a $\sigma$-algebra which contains $\mathcal{A}$.

**Alternative Proof**

2) We can define $\hat{\mathcal{D}} = \{B \in 2^X : B^c \in \mathcal{B}\}$, then we still have $\hat{\mathcal{D}}$ is a monotone class containing $\mathcal{A}$. The conclusion changes to $\mathcal{B} \subset \hat{\mathcal{D}}$. So, $\mathcal{B}$ is closed under complement.

3) Fix $A_0 \in \mathcal{A}$. Consider $C_{A_0} = \{B \in \mathcal{B} : B \cup A_0 \in \mathcal{B}\}$. Then,

- $C_{A_0} \subset \mathcal{B}$ and $C_{A_0}$ is a monotone class.
- For any $A \in \mathcal{A}$, because $\mathcal{A}$ is an algebra, $A \cup A_0 \in \mathcal{A}$. Now, because $\mathcal{A} \subset \mathcal{B}$, we have both $A$ and $A \cup A_0$ are in $\mathcal{B}$. This is true for any $A \in \mathcal{A}$; so, $A \subset C_{A_0}$.

So, $C_{A_0}$ is a monotone class containing $\mathcal{A}$.

We know that $\mathcal{B}$ is the smallest monotone class containing $\mathcal{A}$; hence, $\mathcal{B} \subset C_{A_0}$. We conclude that $C_{A_0} = \mathcal{B}$.

So, for any $A \in \mathcal{A}$, $C_A = \mathcal{B}$.

Next, fix $B_0 \in \mathcal{B}$, and consider $C_{B_0} = \{B \in \mathcal{B} : B \cup B_0 \in \mathcal{B}\}$. Then,

- $C_{B_0} \subset \mathcal{B}$ and $C_{B_0}$ is a monotone class.
- For any $A \in \mathcal{A}$, we have just shown that $C_A = \mathcal{B}$. Hence, $B_0 \in \mathcal{B} \Rightarrow B_0 \in C_A \Rightarrow B_0 \cup A \in \mathcal{B}$. Also, $A \in \mathcal{B}$ because $\mathcal{A} \subset \mathcal{B}$. So, $A \in C_{B_0}$. This is true for any $A \in \mathcal{A}$; hence, $\mathcal{A} \subset C_{B_0}$.

From $\mathcal{A} \subset C_{B_0} \subset \mathcal{B}$ and that $\mathcal{B}$ is the smallest monotone class containing $\mathcal{A}$, we conclude that $C_{B_0} = \mathcal{B}$.

Now, for any $B_1, B_2 \in \mathcal{B}$, we have $B_{i} \in \mathcal{B}$ which contains $B_2$, and hence $B_1 \cup B_2 \in \mathcal{B}$. So, $\mathcal{B}$ is closed under finite union.

- **Monotone Class Lemma** [Bartle p 116]: $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$ (if $\mathcal{A}$ is an algebra).

It is the smallest $\sigma$-algebra containing $\mathcal{A}$ and the smallest monotone class containing $\mathcal{A}$.

**Proof.** Note that any $\sigma$-algebra is monotone. Suppose $\mathcal{B}$ is another $\sigma$-algebra which contains $\mathcal{A}$; then, $\mathcal{B}$ is monotone and contain $\mathcal{A}$. Because $\mathcal{M}(\mathcal{A})$ is the smallest monotone class containing $\mathcal{A}$, $\mathcal{M}(\mathcal{A}) \subset \mathcal{B}$. Therefore, $\mathcal{M}(\mathcal{A})$ is also the smallest $\sigma$-algebra containing $\mathcal{A}$.
• **Halmos’s monotone class theorem:**
  If $\mathcal{M}$ is a monotone class containing the field $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \mathcal{M}$.

  In fact, $\sigma(\mathcal{A}) \subset \mathcal{M}(\mathcal{A}) \subset \mathcal{M}$.

  **Proof.** $\mathcal{M}(\mathcal{A})$ is the smallest monotone class containing $\mathcal{A}$. So, $\mathcal{M}(\mathcal{A}) \subset \mathcal{M}$.

  $\mathcal{M}(\mathcal{A})$ is a $\sigma$-algebra containing $\mathcal{A}$. $\sigma(\mathcal{A})$ is the smallest $\sigma$-algebra containing $\mathcal{A}$. Hence, $\sigma(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$.

• **Monotone Class Theorem**

  Let $\mathcal{C}$ be a class of subsets of $\Omega \ (\mathcal{C} \subset 2^\Omega)$, closed under finite intersections and containing $\Omega$. Let $\mathcal{B}$ be the smallest class containing $\mathcal{C}$ which is closed under increasing limits and by difference. Then, $\mathcal{B} = \sigma(\mathcal{C})$.

  **Proof.** The intersection of classes of sets closed under increasing limits and differences is again a class of that type. So, by taking the intersection of all such classes, there always exists a smallest class containing $\mathcal{C}$ which is closed under increasing limits and by differences. Let this smallest class be $\mathcal{B}$.

  Define, for each $B$, $\mathcal{B}_B = \{ A : A \in \mathcal{B}, A \cap B \in \mathcal{B} \}$. Then, $\mathcal{B}_B$ is closed under increasing limits and by difference.

  Let $A_1, A_2 \in \mathcal{B}_B$. Then $A_1 \in \mathcal{B}$, $A_1 \cap B \in \mathcal{B}$, $A_2 \in \mathcal{B}$, $A_2 \cap B \in \mathcal{B}$.

  To prove $A_1 \backslash A_2 \in \mathcal{B}_B$, we need (1) $A_1 \cap A_2^c \in \mathcal{B}$ which is obvious because $A_1 \in \mathcal{B}$, $A_2 \in \mathcal{B}$, and $\mathcal{B}$ is closed under differences, (2) $A_1 \cap A_2^c \cap B \in \mathcal{B}$, which is obvious because $A_1 \cap A_2^c \cap B = (A_1 \cap B) \backslash A_2$ because $A_1 \cap B \in \mathcal{B}$, $A_2 \in \mathcal{B}$, and $\mathcal{B}$ is closed under differences.

  Let $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \in \mathcal{B}_B$. Then, $A_1 \in \mathcal{B}$, $A_i \cap B \in \mathcal{B}$ $\forall i \in \mathbb{N}$. To prove $\bigcup_{n=1}^\infty A_n \in \mathcal{B}_B$, we need (1) $\bigcup_{n=1}^\infty A_n \in \mathcal{B}$ which is obvious because $\mathcal{B}$ is closed under increasing limits, (2) $\left( \bigcup_{n=1}^\infty A_n \right) \cap B \in \mathcal{B}$ which is also obvious because $\left( \bigcup_{n=1}^\infty A_n \right) \cap B = \bigcup_{n=1}^\infty (A_n \cap B)$ and all $A_i \cap B \in \mathcal{B}$.

  Fix $C_0 \in \mathcal{C}$. Then, for each $C \in \mathcal{C}$, one also has $C \cap C_0 \in \mathcal{C} \subset \mathcal{B}$ because $\mathcal{C}$ is closed under finite intersection. So, $C \in \mathcal{B}_{C_0}$. Since this is true for all $C \in \mathcal{C}$, we have $C \subset \mathcal{B}_{C_0}$. By definition, we also have $\mathcal{B}_{C_0} \subset \mathcal{B}$. Hence, $C \subset \mathcal{B}_{C_0} \subset \mathcal{B}$. Now, both $\mathcal{B}$ and $\mathcal{B}_{C_0}$ contain $\mathcal{C}$ and are closed under increasing limits and by differences. Because $\mathcal{B}$ should be the smallest class which has these properties, so $\mathcal{B}_{C_0} \subset \mathcal{B} \Rightarrow \mathcal{B}_{C_0} = \mathcal{B}$.
Now, consider a fix $B \in \mathcal{B}$. Note that we have just shown that for all $C \in \mathcal{C}$, $B_C = B$, and hence $B \in B_C$, which implies, by definition of $B \in B_C$, $B \cap C \in \mathcal{B}$. Next, note that $C \in \mathcal{C} \subset \mathcal{B}$, so $C \in \mathcal{B}$. Because $C \in \mathcal{B}$, and $B \cap C \in \mathcal{B}$, then $C \in B_B$. So, we have $C \subset B_B \subset \mathcal{B}$, and hence (same argument as above where $B$ being the smallest) $B_B = \mathcal{B}$. This is true for any $B \in \mathcal{B}$.

Let $B_1, B_2 \in \mathcal{B}$, then $B_1 \setminus B_2 \in \mathcal{B}$ which contains $B_2$, and hence $B_1 \cap B_2 \in \mathcal{B}$. So, $\mathcal{B}$ is closed under finite intersection. Because $\Omega \subset \mathcal{C} \subset \mathcal{B}$, $\Omega \in \mathcal{B}$ and for any $B \in \mathcal{B}$, $B^c = \Omega \setminus B \in \mathcal{B}$ because $\mathcal{B}$ is closed under differences. Because $\mathcal{B}$ is closed under finite intersection and complementation, it is also closed under finite union by De Morgan’s law. Now, for $B_1, B_2, \ldots \in \mathcal{B}$, define $A_n = \bigcup_{i=1}^n B_i$.

Then, whenever $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ is a sequence of sets in $\mathcal{B}$. Because $\mathcal{B}$ is closed under increasing limits, we have $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n \in \mathcal{B}$. These properties show that $\mathcal{B}$ is a $\sigma$-algebra.

Note that any $\sigma$-algebra is closed under increasing limits and differences. So, if there exists a $\sigma$-algebra smaller than $\mathcal{B}$ and containing $\mathcal{C}$, then $\mathcal{B}$ would not be the smallest one closed under increasing limits and differences and containing $\mathcal{C}$. This contradicts the construction of $\mathcal{B}$. Hence, $\mathcal{B} = \sigma(\mathcal{C})$.

- **Monotone Class Theorem for function** [Jacod & Protter p. 37]:

  Let $\mathcal{M}$ be a class of functions mapping a given space $\Omega$ into $\mathbb{R}$.

  Let $\sigma(\mathcal{M})$ denote that smallest $\sigma$-algebra on $\Omega$ that makes all of the functions in $\mathcal{M}$ measurable: $\sigma(\mathcal{M}) = \{ f^{-1}(\Lambda) \}; \Lambda \in \mathcal{B}(\mathbb{R})$.

  Suppose $\mathcal{M}$ is closed under multiplication: $f, g \in \mathcal{M} \Rightarrow fg \in \mathcal{M}$.

  Let $\mathcal{H} = \sigma(\mathcal{M})$. Let $\mathcal{H}$ be a vector space of functions with $\mathcal{M} \subset \mathcal{H}$.

  Suppose $\mathcal{H}$ contains the constant functions and is such that whenever $(f_n)_{n \geq 1}$ is a sequence in $\mathcal{H}$ such that $0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots$

  then if $f = \lim_{n \to \infty} f_n$ is bounded, then $f \in \mathcal{H}$.

  Then, $\mathcal{H}$ contains all bounded, $\mathcal{A}$-measurable functions.