Convergence time to equilibrium for large finite Markov chains

A. D. Manita*

November 1996

Revised in February 1997

Abstract

For a sequence of finite Markov chains $\mathcal{L}(N)$ we introduce the notion of "convergence time to equilibrium" $T(N)$. For sequences that are obtained by truncating some countable Markov chain $\mathcal{L}$ we find the convergence time to equilibrium in terms of Lyapunov function of Markov chain $\mathcal{L}$. We apply this result to queueing systems with a limited number of customers: a priority system with several customer types and the Jackson network.

Key words: *convergence time to equilibrium, Lyapunov functions, nonreversible Markov chains, Monte Carlo Markov chains, priority systems, Jackson network*

1 Introduction

At the present time to solve various problems of probability theory and statistical physics one uses widely computer simulations of Markov chains [1]. Main problem of these algorithms is to estimate the number of steps it takes the simulated Markov chain to

*Faculty of Mathematics and Mechanics, Moscow State University, Moscow, 119899, Russia. E-mail: manita@mech.math.msu.su , URL: http://mech.math.msu.su/~manita

approach its stationary distribution. Currently this activity is usually called the *Monte Carlo Markov Chains*. The Metropolis and Hastings algorithms are examples of the most popular models in this field.

It is well known that a finite irreducible aperiodic Markov chain converges to equilibrium exponentially fast, i.e., uniformly in initial distributions the distance between distribution of the chain at time t and its stationary distribution can be bounded as $K \cdot |\lambda_2|^t$ where λ_2 is the second largest (in modulus) eigenvalue of the transition matrix. From a viewpoint of simulation of *large* Markov chains this bound has the following disadvantage: both quantities K and λ_2 depend in a complicated way on a "size" of the chain. So to estimate a convergence rate of a Markov chain with large number of states one should estimate two quantities K and λ_2 as functions of the number of states. Such approach dominates in the overwhelming majority of currently available papers. Progress in this field is related mostly with reversible Markov chains having many nice additional properties which make their study easier (see survey [2]) and random walks on finite groups [3].

In this paper we consider sequences $\{\mathcal{L}(N)\}$ of finite Markov chains with state spaces $X(N)$ such that $|X(N)| \to \infty$, $N \to \infty$. In Section 2 for such sequences we introduce a notion of *convergence time to equilibrium* $T(N)$. This notion is a mathematical formalization of "the number of steps of the algorithm needed to approach the stationary distribution" as a function of the "size" of the Markov chain.

In Section 3 we consider sequences of finite Markov chains $\mathcal{L}(N)$ that are truncations of some countable Markov chain $\mathcal{L}$. It is assumed that the chain $\mathcal{L}$ possesses a Lyapunov function which provides the geometric ergodicity of the chain. We find the convergence time to equilibrium for truncated chains. It is significant that we *do not assume* reversibility of the Markov chain. Using the Lyapunov function technique, we show in Section 3.4 that stationary distributions of the truncated chain $\mathcal{L}(N)$ and the countable chain $\mathcal{L}$ are close if N is large. We remark that proof of this fact is very easy in the reversible case [4] and becomes nontrivial in the nonreversible situation.

Our approach can be applied to many interesting stochastic models. We find the convergence time to equilibrium for two important queueing models which were never considered before in this respect: priority system (Section 4.2) and Jackson network (Section 4.3) with limited queues. Our interest in these models is motivated as follows. Models with infinite buffers for queueing customers are mathematical idealizations. In actual practice queueing systems have bounded buffers.

Results of this paper were announced on Chebyshev conference [5] (Moscow University, May 1996).

The author is grateful to Prof. V.A. Malyshev who awakened my interest to this field for constant attention to this work. The author is grateful also to Prof. F.M. Spieksma (Leiden University, the Netherlands) for useful discussions.

2 Definition and notation

Consider a sequence $\mathcal{L}(N)$, $N = 1, 2, \ldots$, of finite homogeneous Markov chains. Let $X(N)$ be a state space of $\mathcal{L}(N)$. Our general assumption is $|X(N)| \to \infty$ $(N \to \infty)$.

We assume that any chain $\mathcal{L}(N)$ is irreducible and aperiodic. Let $\pi(N) = (\pi_\alpha(N), \alpha \in X(N))$ be its stationary distribution and P_N be a transition matrix of $\mathcal{L}(N)$.

Denote by $\mu = (\mu_\alpha, \alpha \in X(N))$ an initial distribution of the chain $\mathcal{L}(N)$. Then the distribution of the chain at time t is equal to μP_N^t, where P_N^t is the t-th power of matrix P_N. Let ν and ρ be distributions on $X(N)$. *The variation distance* between them is defined as

$$\| \nu - \rho \| = \sup_{B \subset X(N)} |\nu(B) - \rho(B)| \equiv \frac{1}{2} \sum_{x \in X(N)} |\nu_x - \rho_x|.$$

Definition 2.1 *A function $T(N)$ is called the convergence time to equilibrium (CTE) if the following condition holds: for any function $\psi(N) \uparrow \infty$*

$$\sup_{\mu} \| \mu P_N^{T(N)\psi(N)} - \pi(N) \| \to 0, \quad N \to \infty. \tag{1}$$

We say that a function $T(N)$ is the minimal convergence time to equilibrium if for any $T'(N)$ that satisfies (1) we have $T(N) = O(T'(N))$ as $N \to \infty$.

Remark 2.1. It follows from definition that minimal CTE is determined up to multiplication on bounded function.

Remark 2.2. In [6, 7] authors suggested an interesting probabilistic approach to estimate convergence rate of Markov chains. They introduced the following notion. Let $\{\xi_t, t \in \mathbf{Z}_+\}$ be a Markov chain and π be its stationary distribution. A randomized stopping time τ is called a *time to stationarity* if random variable ξ_τ has distribution π and does not depend on τ. It was proved that the variance distance between distribution of ξ_t and stationary distribution π is bounded by $\mathsf{P}\{\tau > t\}$. It was proved also that time to stationarity exists for any ergodic Markov chain. Therefore, if we know the distribution of τ, we can find convergence time to equilibrium $T(N)$ from Definition 2.1. But practically this way is very dificult because construction of τ proposed in [6] is purely theoretical result which needs complete information on distribution of ξ_t for all t.

In the subsequent sections we show that Lyapunov functions provide an efficient method for finding $T(N)$.

3 Convergence time to equilibrium

3.1 Countable Markov chains and Lyapunov functions

In this section we consider Markov chains $\mathcal{L}(N)$ that are truncations of some homogeneous countable Markov chain $\mathcal{L}$ with state space X. Assume that chain $\mathcal{L}$ is irreducible and aperiodic. Denote by ξ_t a state of the chain $\mathcal{L}$ at time t. We use notation:

$$\begin{aligned} p_{\alpha\beta} &= \mathsf{P}\{\xi_{t+1} = \beta | \xi_t = \alpha\}, \\ p^n_{\alpha\beta} &= \mathsf{P}\{\xi_{t+n} = \beta | \xi_t = \alpha\}. \end{aligned}$$

We assume that the following Conditions 3.1–3.4 hold.

Condition 3.1. There exist a function $f(\alpha) \geq 0$, $\alpha \in X$, a *finite* set of states B and $\epsilon > 0$ such that

$$\mathsf{E}(f(\xi_{t+1})|\xi_t = \alpha) - f(\alpha) < -\epsilon, \qquad \alpha \in X \backslash B.$$

Condition 3.2. There exists $d > 0$ such that

$$p_{\alpha\beta} = 0, \quad \text{if} \quad |f(\alpha) - f(\beta)| > d.$$

Condition 3.3. $\forall \gamma > 0$

$$\#\{\alpha : \ f(\alpha) \leq m\} \cdot e^{-\gamma m} \longrightarrow 0, \qquad (m \to \infty).$$

(Condition 3.3 corresponds to condition ii) from definition 7.11 in [8].)

Condition 3.4. There exist $n^\circ > 0$ and $\delta^\circ > 0$ such that

$$p_{\alpha\beta} > 0 \quad \Longrightarrow \quad p_{\beta\alpha}^{n^\circ} > \delta^\circ.$$

The function $f(\alpha)$ is usually called a *Lyapunov function.* It is well known that if MC $\mathcal{L}$ satisfies Condition 3.1 then it is *ergodic* (see [9]).

If Conditions 3.1 and 3.2 hold then for MC $\mathcal{L}$ a so-called exponential bound holds (see Lemma 1.2 in [9] or Theorem 2.1.8 in [8]).

If Conditions 3.1–3.4 holds then MC $\mathcal{L}$ converges *exponentially fast* to its stationary distribution (see [8, 9]). Namely, there exist $\delta, C > 0$ and $C_1 > 1$ such that

$$\sum_{\beta \in X} |p_{\alpha\beta}^n - \pi_\beta| \ \leq \ C \cdot C_1^{f(\alpha)} \cdot e^{-\delta n},$$

where $(\pi_\alpha, \alpha \in X)$ denotes the stationary distribution of $\mathcal{L}$.

Remark 3.1. If a Markov chain satisfies the following condition:

$$\sum_{\beta \in X} |p_{\alpha\beta}^n - \pi_\beta| \ \leq \ C_\alpha \cdot q^n, \qquad C_\alpha > 0, \ 0 < q < 1,$$

then one says that MC is exponentially ergodic. Geometrical ergodicity was studied by many authors [9–15]. Among the recent results we point out paper [16] which is devoted to the following problem: how rate of convergence of countable MC depends on perturbations of transition probabilities in finite number of states.

3.2 Truncations of countable Markov chains

Fix some monotonic positive function

$$g(N) \uparrow \infty, \quad N \to \infty.$$

Definition 3.1 *Let $\mathcal{L}$ be a countable Markov chain, and $f(\cdot)$ be its Lyapunov function. The g–truncation of chain $\mathcal{L}$ is the sequence of Markov chains $\mathcal{L}(N)$ with state spaces $X(N)$ and transition probabilities $p_{\alpha\beta}(N)$, $\alpha, \beta \in X(N)$, of the following form:*

$$X(N) = \{\alpha \in X : \ f(\alpha) \le g(N)\},$$

$$p_{\alpha\beta}(N) = \begin{cases} p_{\alpha\beta}, & \text{if } \beta \ne \alpha, \\ p_{\alpha\alpha} + \sum\limits_{\delta \notin X(N)} p_{\alpha\delta}, & \text{if } \alpha = \beta, \end{cases}$$

where $p_{\alpha\beta}$ are transition probabilities of $\mathcal{L}$.

We assume below that the following condition holds.

Condition 3.5. For any N MC $\mathcal{L}(N)$ is irreducible and aperiodic. There exists N_0 such that for all $\mathcal{L}(N)$, $N \ge N_0$, Condition 3.4 holds with the same n° and δ°.

Denote by $(\pi_\alpha(N), \alpha \in X(N))$ the stationary distribution of the truncated chain $\mathcal{L}(N)$.

Now we state main result of this section.

Theorem 3.1 *Suppose a sequence $\{\mathcal{L}(N)\}$ is the g–truncation of some countable chain $\mathcal{L}$. Suppose Conditions* 3.1–3.5 *hold. Then the minimal convergence time to equilibrium $T(N)$ is equal to $g(N)$.*

3.3 Proof of Theorem 3.1

First note that convergence time to equilibrium is not less than $g(N)$. Indeed, consider two different states α_1 and α_2 such that

$$f(\alpha_1) = a_1 g(N), \quad f(\alpha_2) = a_2 g(N), \quad 0 < a_2 < a_1 < 1.$$

It follows easily from Condition 3.2 that time needed to reach the state α_2 from the state α_1 is not less than $(a_1 - a_2)g(N)/d$.

Let us prove now that any increasing function $\psi(N) \uparrow \infty$

$$\sup_{\alpha\in X(N)} \sum_{\beta\in X(N)} |p_{\alpha\beta}^{g(N)\psi(N)}(N) - \pi_\beta(N)| \longrightarrow 0, \qquad N \to \infty.$$

The following lemma holds.

Lemma 3.1 *If the countable MC $\mathcal{L}$ satisfies Conditions* 3.1–3.3*, then any finite chain $\mathcal{L}(N)$ that is a g–truncation of $\mathcal{L}$ satisfies Conditions* 3.1–3.3 *with the same $f(\cdot), B, \epsilon, d$.*

Proof of Lemma 3.1*.* Let us prove that Condition 3.1 holds. Taking into account Definition 3.1, we get

$$\begin{aligned}\sum_{\beta\in X(N)} p_{\alpha\beta}(N)f(\beta) - \sum_{\beta} p_{\alpha\beta}f(\beta) &= \sum_{\beta\notin X(N)} p_{\alpha\beta}f(\alpha) - \sum_{\beta\notin X(N)} p_{\alpha\beta}f(\beta) = \\ &= \sum_{\beta\notin X(N)} p_{\alpha\beta}(f(\alpha) - f(\beta)) \ \le\ 0.\end{aligned}$$

Conditions 3.2 and 3.3 can be checked straightforwardly. This concludes the proof of the lemma. □

Let us use now a theorem about exponential convergence to the stationary distribution in terms of Lyapunov function [8, 9]. Taking into account Lemma 3.1, we have the following conclusion: there exist $\delta, C > 0$ and $C_1 > 1$ not depending on N such that

$$\sum_{\beta\in X(N)} |p_{\alpha\beta}^t(N) - \pi_\beta(N)| \ \le\ C \cdot C_1^{f(\alpha)} \cdot e^{-\delta t}.$$

Hence,

$$\begin{aligned}\sup_{\alpha\in X(N)} \sum_{\beta\in X(N)} |p_{\alpha\beta}^{g(N)\psi(N)}(N) - \pi_\beta(N)| &\le C \cdot \left(\sup_{\alpha\in X(N)} C_1^{f(\alpha)}\right) \cdot e^{-\delta g(N)\psi(N)} \le \\ &\le C \cdot C_1^{g(N)} e^{-\delta g(N)\psi(N)} = \\ &= C \cdot \left(\frac{C_1}{\exp(\delta\psi(N))}\right)^{g(N)}.\end{aligned}$$

This completes the proof of the theorem. □

3.4 Stationary distributions of countable and truncated Markov chains

We prove here the following proposition.

Proposition 3.1 *Under the conditions of Theorem* 3.1*, there exist* $C_2, \delta_2 > 0$ *such that*

$$\sum_{\beta \in X(N)} |\pi_\beta - \pi_\beta(N)| \le C_2 \exp(-\delta_2 \cdot g(N)).$$

Let us introduce some notation:

$${}_\gamma p^n_{\alpha\beta} = \mathsf{P}\left\{\xi_n = \beta, \xi_{n-1} \neq \gamma, \dots, \xi_1 \neq \gamma \,|\, \xi_0 = \alpha\right\},$$

$${}_\gamma p^*_{\alpha\beta} = \sum_{n=1}^{\infty} {}_\gamma p^n_{\alpha\beta}.$$

Similar probabilities for the chain $\mathcal{L}(N)$ are denoted by ${}_\gamma p^n_{\alpha\beta}(N)$ and ${}_\gamma p^*_{\alpha\beta}(N)$. We use also standard notation

$$f^n_{\alpha\alpha} \stackrel{\text{def}}{=} {}_\alpha p^n_{\alpha\alpha}, \qquad f^n_{\alpha\alpha}(N) \stackrel{\text{def}}{=} {}_\alpha p^n_{\alpha\alpha}(N).$$

Recall some results of the theory of Markov chains.

Result 1. Suppose MC $\mathcal{L}$ satisfies Conditions 3.1 and 3.2. Then for any prefixed state α_0 there exist functions $\tilde f(\alpha) \ge c > -\infty$, $\tilde k(\alpha) \in \mathbf{N}$, $\sup_{\alpha \in X} \tilde k(\alpha) = \tilde b < \infty$, and constants $\tilde d > 0$ and $\epsilon_1 > 0$ such that the following conditions are satisfied:

Condition 3.1°: Inequality

$$\mathsf{E}(\tilde f(\xi_{t+\tilde k(\alpha)})|\xi_t = \alpha) - \tilde f(\alpha) < -\epsilon_1$$

holds for all $\alpha \in X \backslash \{\alpha_0\}$;

Condition 3.2°: $|\tilde f(\alpha) - \tilde f(\beta)| > \tilde d \quad \Longrightarrow \quad p_{\alpha\beta} = 0.$

Remark 3.2. Proof of this statement is given in [8,9] (see Lemma 3.3 in [9], for instance). It can be shown that one can choose the functions $\tilde{f}$, $\tilde{k}$ and the constants $\tilde{b}$, $\tilde{d}$, ϵ_1 in such a manner that Conditions 3.1° and 3.2° hold for any truncated chain $\mathcal{L}(N)$. Moreover, this choice is such that $\tilde{f}(\alpha) = f(\alpha)$, $\alpha \neq \alpha_0$, and $\tilde{f}(\alpha_0) < f(\alpha_0)$.

Result 2. If Conditions 3.1° and 3.2° hold, then for MC $\mathcal{L}$ exponential estimate holds. This means that probability that MC does not visit the state α_0 for time n is exponentially small in n.

It follows that

$$ {}_0p^n_{0\beta} \le D \cdot e^{-\delta_1 n}, \quad {}_0p^n_{0\beta}(N) \le D \cdot e^{-\delta_1 n}, \tag{2} $$

$$ f^n_{0\beta} \le D \cdot e^{-\delta_1 n}, \quad f^n_{0\beta}(N) \le D \cdot e^{-\delta_1 n} \tag{3} $$

for some $D > 0$, $\delta_1 > 0$. (To have shorter notation we write 0 instead of α_0.)

Proof of Proposition 3.1. It is well known [17] that for any ergodic Markov chain

$$ \frac{\pi_\beta}{\pi_\alpha} = {}_\alpha p^*_{\alpha\beta}. $$

We get

$$ \begin{aligned} \sum_{\beta\in X(N)} |\pi_\beta - \pi_\beta(N)| &= \sum_{\beta\in X(N)} |\pi_0 \cdot {}_0p^*_{0\beta} - \pi_0(N) \cdot {}_0p^*_{0\beta}(N)| = \\ &= \sum_{\beta\in X(N)} |(\pi_0 - \pi_0(N)){}_0p^*_{0\beta} + \pi_0(N)({}_0p^*_{0\beta} - {}_0p^*_{0\beta}(N))| \le \\ &\le |\pi_0 - \pi_0(N)| \sum_{\beta\in X(N)} {}_0p^*_{0\beta} + \sum_{\beta\in X(N)} |{}_0p^*_{0\beta} - {}_0p^*_{0\beta}(N)|. \end{aligned} $$

Let us remark that

$$ \sum_{\beta\in X(N)} {}_0p^*_{0\beta} = \sum_{\beta\in X(N)} \frac{\pi_\beta}{\pi_0} < \frac{1}{\pi_0}. $$

By Condition 3.2° for $n < g(N)/\tilde{d}$ we have ${}_0p^n_{0\beta} = {}_0p^n_{0\beta}(N)$. Combining this with (2), we get

$$ \sum_{\beta\in X(N)} |{}_0p^*_{0\beta} - {}_0p^*_{0\beta}(N)| = $$

$$= \sum_{\beta \in X(N)} \left| \sum_{n=1}^{\infty} {}_0p^n_{0\beta} - \sum_{n=1}^{\infty} {}_0p^n_{0\beta}(N) \right| =$$

$$= \sum_{\beta \in X(N)} \left| \sum_{n<g(N)/\tilde{d}} ({}_0p^n_{0\beta} - {}_0p^n_{0\beta}(N)) + \sum_{n \geq g(N)/\tilde{d}} ({}_0p^n_{0\beta} - {}_0p^n_{0\beta}(N)) \right| \leq$$

$$= \sum_{\beta \in X(N)} \sum_{n \geq g(N)/\tilde{d}} 2 \cdot D \cdot e^{-\delta_1 n} \leq D_1 \cdot |X(N)| \cdot e^{-\delta_1 g(N)/\tilde{d}}.$$

By Condition 3.3 the r.h.s. of the last inequality can be bounded from above by $D_2 \exp(-\delta_2 g(N))$, where $0 < \delta_2 < \delta_1/\tilde{d}$. Now Proposition 3.1 follows from the below Lemma 3.2. □

Lemma 3.2 *For any $\delta_2 < \delta_1/\tilde{d}$ there exists $D_3 > 0$ such that*

$$|\pi_0 - \pi_0(N)| \leq D_3 \cdot \exp(-\delta_2 g(N)).$$

Proof of Lemma 3.2. Since $0 < \pi_0 < 1$ and $0 < \pi_0(N) < 1$, we have

$$|\pi_0 - \pi_0(N)| \leq \left| \frac{1}{\pi_0} - \frac{1}{\pi_0(N)} \right|.$$

recall that for the chain $\mathcal{L}$ the quantity $1/\pi_0$ is equal to the expectation of the recurrence time to state 0 if the initial state is 0. Similar statement holds also for the chains $\mathcal{L}(N)$.

Consider

$$\frac{1}{\pi_0} - \frac{1}{\pi_0(N)} = \sum_{n=1}^{\infty} n f^n_{00} - \sum_{n=1}^{\infty} n f^n_{00}(N).$$

By Condition 3.2°, we have

$$f^n_{00} = f^n_{00}(N) \qquad \text{for} \quad n < \frac{g(N)}{\tilde{d}}.$$

Taking into account exponential estimate (3), we get

$$\left| \frac{1}{\pi_0} - \frac{1}{\pi_0(N)} \right| \leq \sum_{n \geq g(N)/\tilde{d}} 2 \cdot n \cdot D \exp(-\delta_1 n) \leq D_3 \exp(-\delta_2 g(N)).$$

Lemma 3.2 is proved. □

3.5 Some generalization: class of admissible truncations

The aim of this subsection is to generalize Theorem 3.1 on more wide class of sequences $\mathcal{L}(N)$.

As before, we fix some monotonic positive function $g(N) \uparrow \infty$, $N \to \infty$. The following notion of *admissible* truncation generalizes the notion of g–truncation from Definition 3.1.

Definition 3.2 *Let $\mathcal{L}$ be a countable MC, $f(\cdot)$ be its Lyapunov function. Assume that Conditions* 3.1–3.4 *hold. We say that the sequence of MCs $\mathcal{L}(N)$ with state spaces $X(N)$ and transition probabilities $p_{\alpha\beta}(N)$, $\alpha, \beta \in X(N)$ is the admissible truncation of the chain $\mathcal{L}$ if the following conditions hold:*

1. *There exist $b_2 > b_1 > 0$ such that*

$$\{\alpha \in X : \ f(\alpha) \le b_1 g(N)\} \subset X(N) \subset \{\alpha \in X : \ f(\alpha) \le b_2 g(N)\};$$

2. *$p_{\alpha\beta}(N) = p_{\alpha\beta}$, for $\alpha, \beta \in X^\circ(N)$, where*

$$X^\circ(N) = \{\alpha \in X : \ p_{\alpha\beta} > 0 \ \Rightarrow \ \beta \in X(N)\};$$

3. $\sum\limits_{\beta \in X(N)} p_{\alpha\beta}(N) f(\beta) - f(\alpha) < -\epsilon, \quad \alpha \in \partial X(N) \equiv X(N) \backslash X^\circ(N)$. *(In other words, for any chain $\mathcal{L}(N)$ Condition* 3.1 *holds with the same B and ϵ not depending on N.)*

Theorem 3.2 *Suppose $\mathcal{L}(N)$ is an admissible truncation of countable MC $\mathcal{L}$ and Conditions* 3.1–3.5 *hold. Then minimal convergence time to equilibrium $T(N)$ is equal to $g(N)$.*

The proof is similar to the proof of Theorem 3.1.

If a sequence $\mathcal{L}(N)$ is an admissible truncation then the statement of Proposition 3.1 is also true.

4 Examples

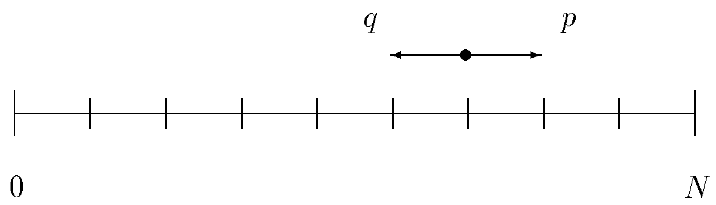
4.1 Asymetric random walk on $[0, N]$

Let $\mathcal{L}(N)$ be a random walk on $X(N) = \{0, 1, 2, \ldots, N\}$ with jump probabilities

$$p_{n,n+1} = p, \quad p_{n,n-1} = q \quad \text{for } 0 < n < N$$

$$\text{and} \quad p_{0,1} = p_{N,N} = p, \quad p_{0,0} = p_{N,N-1} = q.$$

Assume that $q > p$.



$p < q$

Proposition 4.1 *The minimal convergence time to equilibrium is equal to N.*

Proof of Proposition 4.1. Consider a countable Markov chain $\mathcal{L}$ with state space $X = \mathbf{Z}_+$ and jump probabilities $p_{0,0} = q$ and $p_{n,n+1} = p$, $p_{n,n-1} = q$ for $n > 0$. Let $f(x) = x$.

It is easy to prove that the pair $(\mathcal{L}, f(x))$ satisfies Conditions 3.1–3.4 from Section 3 whenever $\epsilon < q - p$, $A = \{0\}$, $d = 1$.

Consider a function $g(N) = N$. It follows easily that the sequence of Markov chains $\mathcal{L}(N)$ introduced above is the g–truncation of $\mathcal{L}$ in the sense of Definition 3.1.

Therefore, the proposition follows from Theorem 3.1. □

4.2 Priority systems

4.2.1 Priority system with several types of customers

Consider a priority queueing system with one server node and K types of customers: $1, 2, \ldots, K$. We assume the arrival streams and the service times to be independent. The exogenous arrival stream of type i is a Poisson stream with parameter λ_i and the service time for a customer of type i is exponentially distributed with parameter μ_i ($i = 1, \ldots, K$). The server chooses which customer from the queue to serve according to the discipline *absolute priority with preemptive resume*. If a customer finishes service, he leaves the system.

Service discipline is a rule that prescribes which customer from the queue should be chosen by the server. Priority disciplines are a special class of service disciplines which deal with situation when customers are of different types and these types are of different importance. We enumerate customer types in order of decreasing of their importance. To be precise, type i has priority over customers of type j if $i < j$. In other words, customers of type 1 have priority over customers of types $2, 3, \ldots$, customers of of type 2 have priority over customers of types $3, 4, \ldots$, and so on. A priority discipline prescribes also what to do in situation when the server is occupied by some customer but at the same time a new customer with more high priority arrives. In the case of *absolute* priority customer with higher priority interrupts the service of the customer with lower priority and immediately occupies the server node.

A service discipline is called the absolute priority *with preemptive resume* if the following rule holds: after system becomes free of customers with higher priority the interrupted customer comes back to the server node and has to be served for a time which is equal to the service time remained after previous interruption of this customer.

Denote by n_i the number of customers of type i in the system. Our queueing system can be considered as Markov process on state space $\mathbf{Z}_+^K = \{(n_1, \ldots, n_K) : \ n_i \in \mathbf{Z}_+\}$. In fact, this Markov process is a maximally homogeneous continuous time random walk.

Let $e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)$. Then jump rates $\lambda_{\alpha\beta}$ have the following form:

$$\lambda_{\alpha\beta} = \begin{cases} \lambda_i, & \text{if } \beta - \alpha = e_i, \\ \mu_j, & \text{if } \beta - \alpha = -e_j \text{ and } \alpha = (0, \ldots, 0, n_j, \ldots, n_K), n_j > 0, \\ 0, & \text{for all other } \beta \neq \alpha. \end{cases}$$

We assume that the following condition holds:

$$\sum_{k=1}^{K} \frac{\lambda_k}{\mu_k} < 1. \tag{4}$$

Condition (4) is necessary and sufficient for ergodicity of the queueing system under consideration (see [18]).

4.2.2 Countable Markov chain and Lyapunov function

Consider a discrete time countable MC $\mathcal{L}$ on state space $\mathbf{Z}_+^K$ with transition probabilities

$$\begin{aligned} p_{\alpha\beta} &= w\lambda_{\alpha\beta} \quad (\alpha \neq \beta), \\ p_{\alpha\alpha} &= 1 - \sum_{\beta\neq\alpha} p_{\alpha\beta}, \end{aligned}$$

where $w = \min\limits_{\alpha} \left(\sum\limits_{\beta} \lambda_{\alpha\beta} \right)^{-1}$.

Consider the following vectors

$$\begin{aligned} v_1 &= (\mu_1 - \lambda_1, -\lambda_2, \ldots, -\lambda_K), \\ v_2 &= \left(0, \mu_2\left(1 - \frac{\lambda_1}{\mu_1}\right) - \lambda_2, -\lambda_3, \ldots, -\lambda_K\right), \\ &\ldots \\ v_K &= \left(0, \ldots, 0, \mu_K\left(1 - \sum_{j=1}^{K-1} \frac{\lambda_j}{\mu_j}\right) - \lambda_K\right). \end{aligned}$$

Let V be a matrix with columns $v_1, \ldots, v_K$. This matrix is triangular and invertible if condition (4) holds. It is easy to check that all elements of the matrix V^{-1} are *nonnegative.*

We use notation $e = (1, \ldots, 1) \in \mathbf{R}_+^K$ and

$$(x, y) \stackrel{\text{def}}{=} \sum_{i=1}^{K} x_i y_i \quad \text{for } x, y \in \mathbf{R}_+^K.$$

Proposition 4.2 *Suppose condition* (4) *holds; then the MC* $\mathcal{L}$ *and the linear function*

$$f(\alpha) = (n, \alpha), \qquad \textit{where} \quad n \stackrel{\text{def}}{=} (V^{-1})^{\top} e, \tag{5}$$

satisfy Conditions 3.1–3.4.

Remark 4.1. This construction of Lyapunov function is related with properties of a second vector field for the chain $\mathcal{L}$ (terminology of [8]). For priority system under consideration the second vector field was found explicitly in [18]. We remark only that the vectors $v_1, \dots, v_K$ defined above coincide with the second vector field on ergodic faces.

The following Lemmas are needed for the proof of Proposition 4.2. They can be proved by direct calculations.

Lemma 4.1 $f(v_i) = 1, \quad f(e_i) > 0, \quad \forall i = 1, \dots, K.$

Lemma 4.2 *Let*

$$M_i = w(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - \mu_i, \lambda_{i+1}, \dots, \lambda_K). \tag{6}$$

Then

$$M_1 = -w \cdot v_1, \qquad M_{m+1} - M_m = -\frac{w}{1 - \sum\limits_{j=1}^{m-1} \dfrac{\lambda_j}{\mu_j}} (v_{m+1} - v_m) \quad (m \geq 2).$$

Proof of Proposition 4.2. Let us prove that Condition 3.1 holds.

Let $M(\alpha)$ be the mean jump from the point α:

$$M(\alpha) \stackrel{\text{def}}{=} \sum_\beta (\beta - \alpha) p_{\alpha\beta} \equiv w \sum_\beta (\beta - \alpha) \lambda_{\alpha\beta}.$$

For any $\Lambda \subset \{1, \dots, K\}$ define the face B_Λ as follows:

$$B_\Lambda = \{(n_1, \dots, n_K) \in \mathbf{Z}_+^K : \ n_i > 0, i \in \Lambda, \ n_j = 0, j \notin \Lambda\}. \tag{7}$$

It follows from definition of the chain $\mathcal{L}$ that vector field $M(\alpha)$ is constant on each face of the form $B_{\{i,i+1,\dots,K\}}$. Moreover, $M(\alpha) = M_i$, if $\alpha \in B_{\{i,i+1,\dots,K\}}$, where M_i is given by (6).

By linearity of the function $f(\cdot)$ we get

$$\mathsf{E}(f(\xi_{t+1}) | \xi_t = \alpha) - f(\alpha) = f(M(\alpha)).$$

Hence, to prove that Condition 3.1 holds, we need to show that $f(M_i) < 0$ for all $i = 1, \ldots, K$.

Indeed, it follows from Lemmas 4.1 and 4.2 that $f(M_1) = f(-w \cdot v_1) = -w$. By Lemma 4.2 we obtain

$$M_i = M_1 + \sum_{m=1}^{i-1} (M_{m+1} - M_m) = -w \cdot v_1 - \sum_{m=1}^{i-1} \frac{w}{1 - \sum\limits_{j=1}^{m-1} \frac{\lambda_j}{\mu_j}} (v_{m+1} - v_m)$$

for $i > 1$. By linearity of $f(\cdot)$, we get

$$f(M_i) = f(-w \cdot v_1) = -w.$$

Conditions 3.2–3.4 can be checked straightforwardly.

This completes the proof of Proposition 4.2. □

4.2.3 Priority system with limited queues

Consider the same priority system as before with the following additional limitation: the number of customer of each type does not exceed N. If a new customer of type i arrive at the time when there are already N customers of type i in the system, then this customer is lost.

This system is described by a continuous time Markov process on state space $X(N) = \underbrace{[0, N] \times \cdots \times [0, N]}_{K} = [0, N]^K$ with jump probabilities

$$\lambda_{\alpha\beta}(N) = \begin{cases} \lambda_i, & \text{if } \beta - \alpha = e_i, \alpha = (n_1, \ldots, n_K), n_i < N, \\ \mu_j, & \text{if } \beta - \alpha = -e_j \text{ and } \alpha = (0, \ldots, 0, n_j, \ldots, n_K), 0 < n_j \le N, \\ 0, & \text{for all other } \beta \neq \alpha. \end{cases}$$

As before, consider a related finite MC $\mathcal{L}(N)$ with state space $X(N) = [0, N]^K$ and transition probabilities

$$p_{\alpha\beta}(N) \;\; = \;\; w\lambda_{\alpha\beta}(N) \quad (\alpha \neq \beta),$$

$$p_{\alpha\alpha}(N) \;=\; 1 - \sum_{\beta \neq \alpha} p_{\alpha\beta}(N),$$

where $w = \min_\alpha \left(\sum_\beta \lambda_{\alpha\beta}(N) \right)^{-1} = \min_\alpha \left(\sum_\beta \lambda_{\alpha\beta} \right)^{-1}$.

Our aim is to find CTE for the sequence of chains $\{\mathcal{L}(N), N = 1, 2, \ldots\}$.

Proposition 4.3 *Suppose condition* (4) *holds. Them the minimal convergence time to equilibrium* $T(N)$ *for the sequence* $\mathcal{L}(N)$ *is equal to* N.

Proof of Proposition 4.3. Let $f(\alpha)$ be the function defined in (5). Let us show that $f(\cdot)$ is a Lyapunov function for each $\mathcal{L}(N)$ and, moreover, Condition 3.1 holds uniformly in N.

Since the function $f(\alpha)$ is linear, it is sufficient to prove that

$$f(M^{(N)}(\alpha)) < f(M_k),$$

where

$$M^{(N)}(\alpha) = \sum_\beta (\beta - \alpha) p_{\alpha\beta}(N), \quad \alpha = (n_1, \ldots, n_K), \quad \max_j n_j = N,$$

and $k = \min\{j : \; n_j > 0\}$.

Indeed,

$$M^{(N)}(\alpha) = M_k - w \sum_{\substack{j: \\ n_j = N}} \lambda_j e_j.$$

This implies that

$$f(M^{(N)}(\alpha)) - f(M_k) = -w \sum_{\substack{j: \\ n_j = N}} \lambda_j f(e_j) < 0,$$

since $f(e_j) > 0 \quad \forall j$ by Lemma 4.1.

Let us remark that there exist $0 < b_1 < b_2$ such that

$$\{\alpha \in \mathbf{Z}_+^K : \; f(\alpha) \leq b_1 N\} \subset X(N) \subset \{\alpha \in \mathbf{Z}_+^K : \; f(\alpha) \leq b_2 N\}.$$

So we are under conditions of Theorem 3.2. Now Proposition 4.3 easily follows. □

4.3 Jackson network

4.3.1 Definition

Network consist of n nodes: $1, \ldots, n$. All customers are of the same type. The arrival stream in node i from outside the network is Poisson λ_i distributed. A node at any instant is serving no more than one customer. Customers waiting for service at a node form a queue. Service times of different customers are independent. Service time at node i is exponentially (μ_i) distributed. A customer that finishes service at node i routes to node j with probability r_{ij}, or he leaves the network with probability r_{i0},

$$r_{i0} + \sum_{j=1}^{n} r_{ij} = 1 \quad \forall i.$$

The stochastic evolution of the network is a continuous time Markov process with states $\alpha = (l_1, l_2, \ldots, l_n)$, where l_i is the number of customers at node i. To be precise, Jackson network is a random walk on $\mathbf{Z}_+^n$ with jump rates

$$\lambda_{\alpha\beta} = \begin{cases} \mu_{0i}, & \text{if } \beta - \alpha = e_i, \\ \mu_{i0}, & \text{if } \beta - \alpha = -e_i, \\ \mu_{ij}, & \text{if } \beta - \alpha = -e_i + e_j, \quad 1 \le i, j \le n,\ i \ne j, \end{cases}$$

where

$$\mu_{0i} \stackrel{\text{def}}{=} \lambda_i, \quad \mu_{i0} \stackrel{\text{def}}{=} \mu_i r_{i0}, \quad \mu_{ij} \stackrel{\text{def}}{=} \mu_i r_{ij},$$

$$e_i \stackrel{\text{def}}{=} (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0), \qquad e_0 \stackrel{\text{def}}{=} (0, \ldots, 0).$$

Let $\mathcal{M}$ be an auxiliary MC on state space $\{0, 1, \ldots, n\}$ with transition probabilities r_{il}, $i = \overline{1, n}$, $l = \overline{0, n}$, and absorption at 0.

Condition 4.1. For any initial state the MC $\mathcal{M}$ hits the absorbing state 0 with probability 1.

Classical result due to Jackson states that Jackson network is ergodic iff $\nu_i < \mu_i \quad \forall i$, where $(\nu_i,\ i=\overline{1,n})$ is a solution to the following system of equation:

$$\nu_j = \lambda_j + \sum_{i=1}^{n} \nu_i r_{ij}, \qquad j=\overline{1,n}.$$

Existence and uniqueness of this solution follows from Condition 4.1.

4.3.2 Countable Markov chain

Consider a discrete time Markov chain $\mathcal{L}$ on state space $\mathbf{Z}_+^n$ with the following transition probabilities:

$$p_{\alpha\beta} = w \cdot \lambda_{\alpha\beta},$$

where $w = \min\limits_{\alpha} \left(\sum_\beta \lambda_{\alpha\beta}\right)^{-1}$.

Let us introduce vectors

$$f_i = w \sum_{j=0}^{n} \mu_{ij}(-e_i+e_j), \qquad i=0,1,\ldots,n. \tag{8}$$

Denote by Π a parallelepiped in $\mathbf{R}_+^n$:

$$\Pi = \{f_0 + c_1 f_1 + \cdots + c_n f_n \in \mathbf{R}_+^n :\ 0 \le c_i \le 1\}.$$

It was proved in [8] that Jackson network is ergodic iff the parallelepiped Π is not degenerate and $0 \in \mathbf{R}^n$ is its internal point.

We shall assume in the sequel that Jackson network is *ergodic*.

4.3.3 Vector field

For any $k \in \{1,\ldots,n\}$ consider a vector $n_k \in \mathbf{R}^n$ that satisfies the following conditions:

$$(n_k, f_0) = 1, \qquad (n_k, f_i) = 0 \quad \forall i \ne k, 0. \tag{9}$$

Lemma 4.3 *Suppose the chain $\mathcal{L}$ is ergodic. Then $(n_k, f_k) < -1 \quad \forall k = \overline{1,n}$.*

Proof of Lemma 4.3. Since the chain $\mathcal{L}$ is ergodic, the point 0 belongs to the interior of the parallelepiped Π. Hence, there exist $0 < \epsilon_i < 1$, $i = 1, \ldots, n$, such that

$$0 = f_0 + \epsilon_1 f_1 + \cdots + \epsilon_n f_n.$$

Multiplying both sides by n_k, we obtain

$$0 = (n_k, f_0) + \epsilon_k (n_k, f_k) = 1 + \epsilon_k (n_k, f_k).$$

This implies that $(n_k, f_k) = -1/\epsilon_k < -1$. □

For any $x = (x_1, \ldots, x_n) \in \mathbf{R}_+^n$ consider $\Lambda(x) = \{k : \ x_k > 0\}$.

Consider a mean jump from the point α:

$$M(\alpha) = \sum_\beta (\beta - \alpha) p_{\alpha\beta}.$$

Since the chain $\mathcal{L}$ is maximally homogeneous, this vector field is constant on each face $B_\Lambda \in \mathbf{Z}_+^n$ (see (7)), namely,

$$M(\alpha) = f_0 + \sum_{i \in \Lambda(\alpha)} f_i \stackrel{\text{def}}{=} M_{\Lambda(\alpha)}$$

(see also [8]).

Lemma 4.4 *There exists $\delta_1 > 0$ such that $\forall \alpha \in \mathbf{Z}_+^n$*

$$(n_k, M_{\Lambda(\alpha)}) < -\delta_1 \qquad \forall k \in \Lambda(\alpha).$$

Proof of Lemma 4.4. Indeed,

$$(n_k, M_{\Lambda(\alpha)}) \ = \ (n_k, f_0 + \sum_{i \in \Lambda(\alpha)} f_i) \ = \ 1 + (n_k, f_k).$$

Taking into account Lemma 4.3 and putting $\delta_1 = \frac{1}{2} \min_{k=\overline{1,n}} (-(n_k, f_k) - 1)$, we get statement of the Lemma. □

4.3.4 Construction of Lyapunov function

In this subsection we give an *explicit* construction of Lyapunov function for the chain $\mathcal{L}$. We follow the strategy presented in [8].

Consider functions

$$h(x) = \max_{k=\overline{1,n}} (n_k, x) \tag{10}$$

and

$$h_\kappa(x) = \max_{k=\overline{1,n}} [(n_k, x) + \kappa |n_k|],$$

where $0 < \kappa < 1$ is some fixed number.

It is easy to see that the distance

$$\text{dist }(\{h(x) = 1\}, \{h_\kappa(x) = 1\}) = \kappa,$$

since $h_\kappa(x) = \max_{k=\overline{1,n}} \left[\left(n_k, x + \kappa \frac{n_k}{|n_k|} \right) \right]$.

Denote $B(x;r) = \{y \in \mathbf{R}^n : |y - x| \le r\}$. Consider a convex set

$$F^* = \left(\bigcup_{\substack{y \in \mathrm{R}^n: \\ h_\kappa(y) \le 1}} B(y;\kappa) \right) \bigcap \mathbf{R}^n_+$$

and define a function $f : \mathbf{R}^n_+ \to \mathbf{R}_+$ by the formula

$$f(x) = \min\{\lambda \ge 0 : \lambda F^* \ni x\}. \tag{11}$$

Consider a set $\partial F^* = \{w \in \mathbf{R}^n_+ : f(w) = 1\}$. Let $n(w)$ be the outward normal to ∂F^* at point w. Suppose the vector $n(w)$ is normed as follows

$$(n(w), w) = 1.$$

From construction of the set F^* it follows that

$$n(w) = \sum_{i=1}^{n} a_i(w) n_i. \tag{12}$$

It is easy to see that functions $a_i(w) : \partial F^* \to \mathbf{R}^1$ satisfy to the following properties:

1. $a_i(w) \geq 0$;
2. $a_i(w) = 0$, $i \notin \Lambda(w)$;
3. functions $a_i(w)$ are smooth and do not equal to zero simultaneously.

Property 2 follows from Lemma 4.5 which will be given at the end of this subsection.

It follows from properties 1–3 that there exists $b > 0$ such that

$$\sum_{i \in \Lambda(x)} a_i(w) > b > 0 \qquad \forall w \in \partial F^*. \tag{13}$$

Proposition 4.4 *Suppose the chain* $\mathcal{L}$ *is ergodic. Then the function* $f(x)$ *defined by* (11) *and the chain* $\mathcal{L}$ *satisfy Conditions* 3.1–3.4.

Proof of Proposition 4.4. Proof of Condition 3.1 uses the following property of the function $f(x)$: *there exist* <u>*linear*</u> *functions* $g_x(\cdot)$ *such that* $\forall r > 0$

$$\sup_{x:\, f(x)=a} \ \sup_{y \in B(x;r)} |f(y) - g_x(y)| \longrightarrow 0 \qquad (a \to \infty).$$

In [8]this property was called *the principle of local linearity*. By construction, the function $f(x)$ defined by (11) satisfies this property. Moreover,

$$g_x(y) = (n\,(x/f(x))\,, y)\,.$$

From local linearity it is easy to obtain

$$\sup_{\alpha:\, f(\alpha)=a} \left| \sum_{\beta} p_{\alpha\beta} f(\beta) - f(\alpha + M(\alpha)) \right| \longrightarrow 0 \qquad (a \to \infty). \tag{14}$$

Let us show now that for sufficiently large x the vector field $M(x)$ is directed inside level surfaces of the function $f(x)$:

$$f(x + M(x)) - f(x) < -\delta_2 < 0. \tag{15}$$

By local linearity, it is sufficient to show that

$$g_x(x+M(x)) - g_x(x) < -\delta_3 < 0. \tag{16}$$

Indeed,

$$\begin{aligned} g_x(x+M(x)) - g_x(x) &= g_x(M(x)) = (n(x/f(x)), M(x)) \\ &= \left(\sum_{i\in\Lambda(x)} a_i(x/f(x)) n_i, M(x) \right) \\ &= \sum_{i\in\Lambda(x)} a_i(x/f(x)) \cdot (n_i, M(x)) \\ &= -\delta_1 \sum_{i\in\Lambda(x)} a_i(x/f(x)) \le -\delta_1 b \overset{\text{def}}{=} -\delta_3 < 0. \end{aligned} \tag{17}$$

We have used Lemma 4.4 and inequality (13).

It follows from (14) and (15) that Condition 3.1 holds.

Conditions 3.2–3.4 can be checked straightforwardly.

Proposition 4.4 is proved. □

Lemma 4.5 *Suppose* $h(\cdot)$ *is defined by* (10). *Then the function* $h(\cdot)$ *can be represented in the following form:*

$$h(x) = \max_{k\in\Lambda(x)} (n_k, x).$$

Moreover, there exists $K > 0$ *such that*

$$h(x) - \max_{i\notin\Lambda(x)} (n_i, x) > K \cdot h(x).$$

Proof of Lemma 4.5. Let x be such that $|\Lambda(x)| < n$. Let us note that Lemma follows from the following representation:

$$x = h(x) \left(f_0 + \sum_{i\notin\Lambda(x)} \delta_i(x) f_i + \sum_{j\in\Lambda(x)} \gamma_j(x) f_j \right), \tag{18}$$

where $\delta_i(x) \ge K > 0 \ \forall i \notin \Lambda(x)$, $\gamma_j(x) \ge 0 \ \forall j \in \Lambda(x)$ and $\prod_{j\in\Lambda(x)} \gamma_j(x) = 0$. Indeed, assume that representation (18) holds. Let $l \notin \Lambda(x)$. Taking into account (9) and Lemma 4.3, we have

$$\begin{aligned} (n_l, x) &= h(x)(1 + \delta_l(x)(n_l, f_l)) \\ &\le h(x)(1 + K(n_l, f_l)) \\ &= f(x) - K \cdot f(x). \end{aligned}$$

Therefore, to finish the proof it is sufficient to show that representation (18) holds. Assume that $|\Lambda(x)| < n$ and consider a set

$$\{y \in \mathbf{R}^n_+ : \ h(y) = h(x)\}.$$

This set is a subset of the boundary of a parallelepiped $h(x)\Pi$ and contains the point x. Hence, we have the following representation:

$$x = h(x)\left(f_0 + \sum_{i \notin \Lambda(x)} \delta_i(x) f_i + \sum_{j \in \Lambda(x)} \gamma_j(x) f_j\right),$$

where $\delta_i(x) \ge 0$, $\gamma_j(x) \ge 0$ and $\prod_{i \notin \Lambda(x)} \delta_i(x) \prod_{j \in \Lambda(x)} \gamma_j(x) = 0$. Let us show that there exists $K > 0$ such that $\delta_i(x) \ge K > 0$ $\forall i \notin \Lambda(x)$.

Indeed, it follows from definition of the vectors f_i that

$$f_i = -a_i e_i + \sum_{j \neq i} b_{ij} e_j, \qquad i = 1, \ldots, n,$$

for some $a_i > 0$, $b_{ij} \ge 0$.

Suppose $l \notin \Lambda(x)$. Let us remark that $x_l = 0$ but the l-th coordinate of the vector f_0 is equal to λ_l. It is clear that $\delta_l(x) \ge \lambda_l / a_l > 0$. Putting $K = \min_l \dfrac{\lambda_l}{a_l}$, we conclude that representation (18) holds.

Lemma 4.5 is proved. □

4.3.5 Monotonicity of the Lyapunov function

We need in the sequel the following result.

Lemma 4.6 *Suppose a routing matrix $R = (r_{ij})_{i,j=1}^n$ is irreducible; then*

$$(n_k, e_i) > 0 \quad \forall i, k \in \{1, \ldots, n\}.$$

Proof of Lemma 4.6. Fix some k. By construction (see (9)), $(n_k, f_j) = 0$, $j \neq k$. Moreover, by Lemma 4.3 $(n_k, f_k) \stackrel{\text{def}}{=} -d_k < -1$. It follows from (8) that the following vector

$$y = (y_1, \ldots, y_n), \quad y_i = w \cdot (n_k, e_i),$$

is a solution to the system of linear equations

$$\begin{cases} -\mu_k y_k + \sum\limits_{j \neq k} \mu_k r_{kj} y_j = -d_k, \\ -\mu_i y_i + \sum\limits_{j \neq i} \mu_i r_{ij} y_j = 0, \quad i \neq k. \end{cases}$$

In matrix form this system can be represented as

$$M(R - I)\, y = -d_k e_k,$$

where $M = \mathrm{diag}\,[\mu_1, \mu_2, ..., \mu_n]$ is a diagonal matrix, I is the identity matrix.

Hence,

$$\begin{aligned} y &= (I - R)^{-1} M^{-1} d_k e_k = \\ &= (I + R + R^2 + \cdots + R^m + \cdots) \cdot (\mu_k^{-1} d_k) e_k. \end{aligned} \tag{19}$$

Convergence of this series follows from Condition 4.1. Since the matrix R is irreducible, there exists m_0 such that $r_{ij}^{m_0} > 0$ $\forall i, j = \overline{1, n}$. From (19) it follows easily now that $y_j > 0$ $\forall j = \overline{1, n}$.

This completes proof of the Lemma. □

Proposition 4.5 *The function $f(x)$ defined in* (11) *satisfies the following monotonicity property:*

$$f(x) < f(y), \quad \text{if } x_i < y_i \ \forall i = \overline{1, n},$$

$(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbf{R}^n)$.

Proof of Proposition 4.5 is similar to the arguments in (15)–(17) and uses Lemma 4.6.

4.3.6 Jackson networks with limited queues

Consider the following modification of Jackson network: the number α_i of customer at any node i can not exceed N. If a customer comes to a node j with $\alpha_i = N$, then this customer is lost.

A state space is

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in X(N) = \underbrace{[0, N] \times \cdots \times [0, N]}_{n} = [0, N]^n.$$

The system is described by a continuous time Markov process on $X(N)$ with jump rates

$$\lambda_{\alpha\beta}(N) = \begin{cases} \mu_{0i}, & \text{if } \beta - \alpha = e_i, \quad \alpha_i < N, \\ \mu_{i0} + \sum\limits_{\substack{j: \\ \alpha_j = N}} \mu_{ij}, & \text{if } \beta - \alpha = -e_i, \\ \mu_{ij}, & \text{if } \beta - \alpha = -e_i + e_j, \quad 1 \le i, j \le n, \, i \ne j, \, \alpha_j < N. \end{cases}$$

Consider a related finite MC $\mathcal{L}(N)$ on $X(N) = [0, N]^n$ with transition probabilities

$$\begin{aligned} p_{\alpha\beta}(N) &= w\lambda_{\alpha\beta}(N) \quad (\alpha \ne \beta), \\ p_{\alpha\alpha}(N) &= 1 - \sum_{\beta \ne \alpha} p_{\alpha\beta}(N), \end{aligned}$$

where $w = \min\limits_{\alpha} \left(\sum\limits_{\beta} \lambda_{\alpha\beta}(N) \right)^{-1} = \min\limits_{\alpha} \left(\sum\limits_{\beta} \lambda_{\alpha\beta} \right)^{-1}$.

Further exposition is similar to the subsection 4.2.3. Our aim is to find convergence time to equilibrium for the sequence of MCs $\{\mathcal{L}(N), N = 1, 2, \ldots\}$.

Proposition 4.6 *Suppose the denumerable MC $\mathcal{L}$ is ergodic. The convergence time to equilibrium $T(N)$ for the sequence $\mathcal{L}(N)$ is equal to N.*

Proof of Proposition 4.6. Let us show that the function $f(x)$ defined by (11) is a Lyapunov function for any finite chain $\mathcal{L}(N)$ and, moreover, Condition 3.1 holds uniformly in N.

It is sufficiently to check Condition 3.1 for points of the "boundary":

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \partial X(N) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \max_i \alpha_i = N.$$

Consider a mean jump from the boundary:

$$M^{(N)}(\alpha) = \sum_\beta (\beta - \alpha) p_{\alpha\beta}(N), \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad \max_j \alpha_j = N.$$

By *local linearity* of function $f(x)$, it is sufficient to show that

$$g_\alpha(\alpha + M^{(N)}(\alpha)) - g_\alpha(\alpha) \; < \; g_\alpha(\alpha + M(\alpha)) - g_\alpha(\alpha). \tag{20}$$

By linearity of functions $g_\alpha(\cdot)$ and by representation (12), inequality (20) will follow from

$$(n_k, M^{(N)}(\alpha)) < (n_k, M(\alpha)), \quad \forall k \in \Lambda(\alpha). \tag{21}$$

Note that

$$M^{(N)}(\alpha) - M(\alpha) = - \sum_{\substack{i: \\ \alpha_i = N}} z_i e_i$$

for some $z_i > 0$. The application of Lemma 4.6 yields now (21).

To finish the proof of Proposition it is sufficient to note that there exist $0 < b_1 < b_2$ such that

$$\{\alpha \in \mathbf{Z}_+^n : \; f(\alpha) \le b_1 N\} \subset X(N) \subset \{\alpha \in \mathbf{Z}_+^n : \; f(\alpha) \le b_2 N\},$$

and to apply Theorem 3.2. □

References

[1] SOKAL A. D. *Monte Carlo Methods in Statistical Mechanics: Foundations and New Algorithms.* Troisieme Cycle de la Physique, En Suisse Romande, Semestre d'été 1989.

[2] INGRASSIA S. Geometric approaches to the estimation of the spectral gap of reversible Markov chains. Combinatorics, Probability and Computing, 1993, No. 2, p. 301–321.

[3] DIACONIS P. *Group representations in Probability and Statistics.* IMS Lectures Series, Vol. 11, Istitute of Mathematical Statistics, Hayward, California, 1988.

[4] LIGGETT T. M. *Interacting particle systems.* Springer, New York (1985).

[5] MANITA A. D. *On equilibrium convergence time for large finite Markov chains.* Bull. Moscow University, Math. and Mech., N. 6, 56–59 (1996).

[6] ALDOUS D., DIACONIS P. *Strong uniform times and finite random walks.* Adv. in Appl. Math., Vol. 8, 69–97 (1987).

[7] DIACONIS P., FILL J. A. *Strong stationary times via a new form of duality.* Ann. Probab., Vol. 18, No. 4, 1483–1522 (1990).

[8] FAYOLLE G., MALYSHEV V. A., MENSHIKOV M. V. *Topics in constructive theory of countable Markov chains.* Cambridge Univ. Press, 1994.

[9] MALYSHEV V. A., MENSHIKOV M. V. *Ergodicity, continuity and analyticity of countable Markov chains.* Trans. Moscow Math. Society, Vol. 39, 3–48, 1979.

[10] VERE-JONES D. *Geometric ergodicity in denumerable Markov chains.* Quart. J. Math. Oxford, Vol. 13, 7–28 (1962).

[11] POPOV N. N. *Conditions for geometric ergodicity of countable Markov chains.* Soviet Math. Dokl., Vol. 18, 676–679 (1977).

[12] SPIEKSMA F. M. *Geometrically ergodic Markov chains and the optimal control of queues.* Ph.D. Thesis, Leiden University, 1990.

[13] HORDIJK A., SPIEKSMA F. M. *On ergodicity and reccurence properties of a Markov chain with an application.* Adv. Appl. Prob., V. 29, (1992).

[14] SPIEKSMA F. M., TWEEDIE R. L. *Strengthening ergodicity to geometric ergodicity for Markov chains.* Stoch. Models, Vol. 10 (1994).

[15] MEYN S. P., TWEEDIE R. L. *Computable bounds for convergence rates of Markov chains.* Annals of Applied Probability, Vol. 4, (1994).

[16] MALYSHEV V. A., SPIEKSMA F. M. *Intrinsic convergence rate of countable Markov chains.* Markov Processes and Related Fields, Vol. 1, No. 2 (1995).

[17] CHUNG K. L. *Markov chains with stationary transition probabilities.* Springer-Verlag, Berlin (1967).

[18] FILIN A. V., MALYSHEV V. A., MANITA A. D. *Probabilistic models for computer architectures.* Fundamental and Applied Mathematics, 1996.