0. Introduction

The structure and solidarity properties of general Markov chains satisfying
the measure-theoretic condition of \( \varphi \)-irreducibility, for some \( \varphi \), are now
well known (see, for example, Orey [8], Revuz [11]), and show a striking
similarity to these properties for irreducible Markov chains on a countable
state space. If the state space is topological then the condition of \( \varphi \)-
irreducibility is somewhat unnatural; however, it has so far proved
difficult to achieve results of the same strength when \( \varphi \)-irreducibility is
replaced by other, more topologically motivated, conditions.

Several authors have given results bearing some resemblance to the
'\( \varphi \)-solidarity' properties for Markov chains whose transition probabilities
are weakly continuous (see, for example, Foguel [2], Jamison [5], Rosen-
blatt [13, 14]). In this paper we propose a different form of continuity
condition, namely the existence of a continuous component in the
transition mechanism of the chain.

Our condition is automatically satisfied by strongly Feller chains, i.e.
in the special case where the transition probabilities are strongly
continuous; but in general it is much weaker than the strong Feller
condition.

In § 4 we shall show that if this 'continuous component' is non-trivial
in suitable ways then certain local topological irreducibility, recurrence
or positivity conditions imply the corresponding \( \varphi \)-solidarity properties.
These results are analogous to those known for 'spread-out' random walks,
and we show in §5 that in fact our continuous component exists for
random walks if and only if the increment distribution is spread-out;
therefore our condition can be seen as an exact generalization of this random
walk concept for more general Markov chains.

In § 6 we turn to chains which do not necessarily have any topological
irreducibility structure. The main theorem of this section, and of the
whole paper, states that, when the chain has a suitably non-trivial
continuous component and the topology is \( T_1 \) with a countable basis,
the state space admits a decomposition into a countable number of
stochastically closed Harris sets (i.e. the restriction of the chain to each

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of these sets is \( \varphi \)-recurrent for some \( \varphi \) and into a ‘transient’ part of the space which is not properly essential. Under a more stringent non-triviality condition the topological nature of the sets appearing in the decomposition can be identified. Several corollaries to the main theorem are derived, exhibiting amongst other things additional conditions which imply various types of recurrence for the whole chain.

In §7 we specialize to the situation where the space is compact. Here we are able to show that there is a finite positive number of Harris sets in the space and that the transient part of the space is actually inessential; whilst if the continuous component condition is strengthened slightly, the Harris sets are all positive recurrent and the transient part of the space becomes uniformly transient.

1. Preliminaries

Throughout this paper \((X, \mathcal{F})\) will be a measurable space and \((X_n)_{n \in \mathbb{N}}\) will be a discrete time Markov process on the state space \((X, \mathcal{F})\) with stationary and stochastic transition probabilities \(P(x, A)\), where \(x \in X\), \(A \in \mathcal{F}\). We denote by \(P_{\varepsilon} x\) the probability measure for the chain with initial distribution \(\varepsilon_x\), the Dirac measure at \(x \in X\). The \(n\)-step transition probabilities are denoted by \(P^n(x, A)\), where \(x \in X\), \(A \in \mathcal{F}\), \(n \in \mathbb{N}^+\), and their sum is denoted by

\[
G(x, A) = \sum_{n=1}^{\infty} P^n(x, A).
\]

We write \(\tau_A = \inf\{n \in \mathbb{N}^+ \mid X_n \in A\}\) for the first hitting time on a set \(A \in \mathcal{F}\) and \(\eta_A = \text{card}\{n \in \mathbb{N}^+ \mid X_n \in A\}\) for the number of visits to \(A\). The taboo probabilities are denoted by

\[
\mathbb{P}^n(x, B) = \mathbb{P}_x\{X_n \in B, \tau_A > n\},
\]

and we write

\[
L(x, A) = \mathbb{P}_x\{\tau_A < \infty\}, \quad Q(x, A) = \mathbb{P}_x\{\eta_A = \infty\}.
\]

In the sequel the following basic properties of \(G\), \(L\), and \(Q\) are frequently used:

1. (1.1) if \(L(y, A) \leq 1 - \varepsilon\) for every \(y \in A\) then \(G(x, A) \leq \varepsilon^{-1}\) for every \(x \in X\);

2. (1.2) if \(L(x, A) > 0\) and \(L(y, B) > 0\) for every \(y \in A\) then also \(L(x, B) > 0\);

3. (1.3) if \(L(x, A) > 0\) and \(Q(y, B) < 1\) for every \(y \in A\) then also \(Q(x, B) < 1\);

4. (1.4) if \(\inf_{x \in A} L(x, B) > 0\) then \(Q(x, A) \leq Q(x, B)\) for every \(x \in X\).

For the proof of (1.4) see Orey [8, Proposition 1.5.1, p. 22].
In terms of the above concepts we now summarize the 'measure-theoretic' irreducibility and recurrence concepts that we shall use; for a more thorough view see, for example, Orey [8], Revuz [11].

Definition 1.1. Let \( \varphi \) be a non-trivial, \( \sigma \)-finite measure on \((X, \mathcal{F})\). The chain is called \( \varphi \)-irreducible (respectively \( \varphi \)-recurrent) if
\[
L(x, A) > 0 \quad \text{(respectively } Q(x, A) = 1)\]
for every \( x \in X \) and every \( \varphi \)-positive \( A \in \mathcal{F} \). The chain is called a Harris chain if it is \( \varphi \)-recurrent for some \( \varphi \).

Definition 1.2. The chain is called recurrent (weakly or \( 1 \)-recurrent) if
\[
G(x, A) = \infty
\]
for every \( x \in X \) and every \( \varphi \)-positive \( A \in \mathcal{F} \).

If the chain is recurrent it has a \( \sigma \)-finite invariant measure \( \pi \gg \varphi \) which is unique up to a multiplicative constant (see Theorem 4 of Tweedie [16]).

Definition 1.3. A recurrent chain is called positive recurrent if \( \pi(X) < \infty \) where \( \pi \) is invariant; otherwise it is said to be null recurrent.

A recurrent chain is positive recurrent if and only if
\[
\pi(A) > 0 \quad \text{implies} \quad \limsup_{n \to \infty} P^n(x, A) > 0 \quad \text{for every } x \in X,
\]
and null recurrent if and only if
\[
\pi(A) < \infty \quad \text{implies} \quad \lim_{n \to \infty} P^n(x, A) = 0 \quad \text{for } \pi\text{-a.e. } x \in X
\]
(see Tweedie [17]).

Definition 1.4. A set \( A \in \mathcal{F} \) is called stochastically closed if \( A \neq \emptyset \) and \( L(x, A^c) = 0 \) for every \( x \in A \).

Definition 1.5. A stochastically closed set is called indecomposable if it does not contain two disjoint stochastically closed subsets.

Definition 1.6. A set \( A \in \mathcal{F} \) is called \( \varphi \)-irreducible (respectively \( \varphi \)-recurrent) if \( \varphi(A) > 0 \) and \( L(x, B) > 0 \) for every \( x \in A \) and every \( \varphi \)-positive subset \( B \) of \( A \) (respectively if \( \varphi(A) > 0 \), \( A \) is stochastically closed, and \( Q(x, B) = 1 \) for every \( x \in A \) and every \( \varphi \)-positive subset \( B \) of \( A \)). A set \( A \) is called a Harris set if it is \( \varphi \)-recurrent for some \( \varphi \).

Definition 1.7. A set \( A \in \mathcal{F} \) is called inessential if \( Q(x, A) = 0 \) for every \( x \in X \); otherwise \( A \) is called essential. An essential set is called properly (or absolutely) essential if it cannot be represented as a countable union of inessential sets.
2. Topological definitions and continuous components

Now we turn to the topological situation. We will assume throughout that \( X \) is a topological space and \( \mathcal{F} \) contains all the open sets of \( X \); however, we do not need to assume that \( \mathcal{F} \) is the Borel \( \sigma \)-field of \( X \). Following Rosenblatt [14] we define

**Definition 2.1.** A point \( x \in X \) is called recurrent, a point of sure return, or conservative if, respectively,
\[
Q(x, N) = 1, \quad L(x, N) = 1, \quad \text{or} \quad G(x, N) = \infty
\]
for every neighbourhood \( N \) of \( x \).

**Definition 2.2.** A point \( x \in X \) is called positive if
\[
\limsup_{n \to \infty} P^n(x, N) > 0
\]
for every neighbourhood \( N \) of \( x \).

Let \( T: X \times \mathcal{F} \to \mathbb{R}^+ \) be a finite kernel, that is, \( T(x, \cdot) \) is a finite measure on \( (X, \mathcal{F}) \) for every \( x \in X \), and \( T(\cdot, A) \) is \( \mathcal{F} \)-measurable for every \( A \in \mathcal{F} \).

**Definition 2.3.** A kernel \( T \) is called strongly (respectively weakly) continuous if the mapping
\[
x \mapsto \int T(x, dy) f(y)
\]
is continuous for every bounded measurable (respectively bounded continuous) \( f: X \to \mathbb{R} \).

**Definition 2.4.** We will call a non-negative kernel \( T \) a component of a function \( K: X \times \mathcal{F} \to \mathbb{R}^+ \) if \( T(x, A) \leq K(x, A) \) for every \( x \in X \) and \( A \in \mathcal{F} \). The component \( T \) is called non-trivial at \( x \) if \( T(x, X) > 0 \), and continuous at \( x \) if \( T(\cdot, A) \) is lower semicontinuous at \( x \) for every \( A \in \mathcal{F} \). If \( T \) is continuous at every \( x \in X \) it is called a continuous component of \( K \). If \( K(x, A) > 0 \) implies \( T(x, A) > 0 \), then \( T \) is called equivalent to \( K \) at \( x \). If \( T \) is everywhere equivalent to \( K \), then \( T \) is called an equivalent component of \( K \).

In the sequel we shall consider components of the hitting probability function \( L(x, A) \) or a kernel
\[
K_\theta(x, A) = \sum_{n=1}^{\infty} \theta_n P^n(x, A),
\]
where \( \theta = (\theta_1, \theta_2, \ldots) \) is a probability on \( \mathbb{N}^+ \), that is,
\[
K_\theta(x, A) = P_x \{ X_r \in A \},
\]
where \( \tau \) is a random time independent of the chain and with the distribution \( \theta \). From the first entrance decomposition

\[
P^n(x, A) = A P^n(x, A) + \sum_{k=1}^{n-1} \int_A P^k(x, dy) P^{n-k}(y, A),
\]

one can immediately see that if \( K_\theta \) has a component \( T \) (for any \( \theta \)) then also \( L \) has a component with the same non-triviality properties. It is obvious that the converse need not hold true.

**Remarks.** (i) If \( T \) is a continuous component in the above sense and \( T(\cdot, X) \) is continuous then \( T \) is a strongly continuous kernel.

(ii) If the transition probability kernel \( P \) itself is strongly continuous, that is, the Markov chain is a strongly Feller chain, then \( L \) and \( K_\theta \) (for any \( \theta \)) are their own equivalent continuous components; hence all our results are true for strongly Feller chains.

(iii) If \( X \) is a locally compact Hausdorff space with a countable basis for the topology and \( \mathcal{F} \) is the Borel \( \sigma \)-field of \( X \), then, by the inner approximation property by compact sets, \( T(\cdot, A) \) is lower semicontinuous for every \( A \in \mathcal{F} \) if \( T(\cdot, C) \) is lower semicontinuous for every compact \( C \).

(iv) Suppose that \( T(\cdot, S) \) is continuous for every \( S \) belonging to a class \( \mathcal{S} \) which is closed under finite intersections, contains \( X \), and generates \( \mathcal{F} \). If in addition the following equicontinuity condition

\[
A_n \downarrow \emptyset \Rightarrow \sup_{x \in X} T(x, A_n) \to 0
\]

holds, then a straightforward application of the monotone class theorem shows that \( T \) is a strongly continuous kernel. Notice that the above equicontinuity condition is always satisfied if \( T \) is majorized by a finite measure \( \varphi \); this idea is utilized by Tweedie in [18] in order to construct a continuous component starting from measure-theoretic non-singularity conditions.

The idea of continuous components in this sense is not entirely new. In a paper by Pollard and Tweedie [10] it is mentioned that if \( P \) has a suitably non-trivial continuous component then it is possible to go in the opposite direction to that considered in this paper; that is, to connect the \( \varphi \)-solidarity properties of the chain with the topological properties of the space. In fact the following extension of those results is possible.

**Proposition 2.1.** Suppose that a Markov chain \( (X_n) \) is \( \varphi \)-irreducible and that \( K_\theta \) has a continuous component \( T \) (for some \( \theta \)) such that \( \varphi(A) > 0 \) implies \( T(\cdot, A) > 0 \) for every \( x \in X \). Then

(i) if the chain is not recurrent, there exists a sequence \( (U_n) \) of open sets with \( U_n \uparrow X \) such that \( G(x, U_n) < \infty \) for every \( x \in X \) and \( n \in \mathbb{N}^+ \).
(ii) if the chain is null recurrent, there exists a sequence \((U_n)\) of open sets with \(U_n \uparrow X\) such that \(\pi(U_n) < \infty\) for every \(n \in \mathbb{N}^+\), where \(\pi\) is the invariant measure of the chain.

**Proof.** We omit the proof which is similar to those in [10].

### 3. Recurrent and non-recurrent points

In this section we prove results for chains satisfying a variety of topological and ‘continuous component’ conditions. These results are of independent interest, although many of them are basic to the work in the sequel.

**Proposition 3.1.** If \(X\) is a \(T_1\)-space (i.e. every singleton \(\{x\}\) is closed in \(X\)) and has a countable neighbourhood basis at each point, then a point of sure return is recurrent.

**Proof.** The proof of Lemma 1 of Rosenblatt [14] can be directly adopted since it is based only on the existence of a countable decreasing system of neighbourhoods of \(x\) whose intersection is \(\{x\}\); this condition obviously follows from our assumption.

If \(P\) is weakly continuous then a conservative point need not be recurrent; see Rosenblatt [13]. In the proposition below we show that the existence of a suitably non-trivial continuous component implies that a conservative point is recurrent. Notice that here we do not need any topological conditions on the space.

**Proposition 3.2.** (i) If \(x_0\) is conservative, and \(K_\theta\), for some \(\theta\), has a component \(T\) continuous at \(x_0\) and satisfying

\[
P(x_0, A) > 0 \quad \text{implies} \quad T(x_0, A) > 0,
\]

then \(x_0\) is recurrent.

(ii) If \(x_0\) is conservative and \(K_\theta\) has a component \(T\), non-trivial and continuous at \(x_0\), then \(Q(x_0, N) > 0\) for every neighbourhood \(N\) of \(x_0\).

**Proof.** (i) Suppose that a set \(A \in \mathcal{F}\) satisfies, for some fixed \(n \in \mathbb{N}^+\) and \(\epsilon > 0\),

\[
P_x(\eta_A < n) \geq \epsilon \quad \text{for all} \quad x \in A.
\]

Then iteration gives, for every \(k \in \mathbb{N}^+\),

\[
P_x(\eta_A \geq kn) \leq (1 - \epsilon)^k \quad \text{for all} \quad x \in A,
\]

and hence

\[
G(x, A) = \mathbb{E}(\eta_A) \leq n/\epsilon < \infty.
\]

Now, assume that \(x_0\) is not recurrent, i.e. there exists a neighbourhood \(N\) of \(x_0\) with \(Q(x_0, N) < 1\). If \(N^*\) denotes the set \(\{y \in X \mid Q(y, N) = 1\}\) then
$P(x_0, N_\ast) < 1$, for if $P(x_0, N_\ast) = 1$ then also

$$Q(x_0, N) \geq \int_{N_\ast} P(x_0, dy)Q(y, N) = 1.$$  

Hence from our assumption on $T$, $T(x_0, N_\ast) > 0$ and so there exist $\varepsilon > 0$ and $n \in \mathbb{N}^+$ such that, writing

$$A = \{ y \in X \mid P_\mu(\eta_N < n) \geq \varepsilon \},$$

we have $T(x_0, A) > 0$. By the lower semicontinuity of $T(\cdot, A)$ at $x_0$, there exists $\delta > 0$ and a neighbourhood $N_\delta \subset N$ of $x_0$ such that

$$T(x, A) \geq \delta \quad \text{for all } x \in N_\delta.$$

Taking $m$ large enough to have $\sum_{k=m+1}^{\infty} \theta_n < \delta/2$ we get

$$\max_{1 \leq k \leq m} P_\mu(x, A) \geq \delta/2m \quad \text{for all } x \in N_\delta,$$

and so

$$P_\mu(\tau_A \leq m) \geq \delta/2m \quad \text{for all } x \in N_\delta.$$

Therefore we obtain

$$P_\mu(\eta_{N_\delta} < m+n) \geq P_\mu(\eta_N < m+n) \geq \sum_{k=1}^{m} \int_A P_\mu(x, dy)P_\mu(\eta_N < n) \geq \varepsilon P_\mu(\tau_A \leq m) \geq \varepsilon \delta/2m$$

for every $x \in N_\delta$, and hence by (3.1)

$$G(x_0, N_\delta) \leq 2m(m+n)/\varepsilon \delta < \infty,$$

and so $x_0$ cannot be conservative.

(ii) Assume that there exists a neighbourhood $N$ of $x_0$ with $Q(x_0, N) = 0$. Since $Q(x_0, N) \geq L(x_0, N_\ast)$ (where $N_\ast$ is as in part (i) of the proof), $L(x_0, N_\ast) = 0$ and hence also $T(x_0, N_\ast) = 0$. The non-triviality of $T(x_0, \cdot)$ implies that $T(x_0, N_\ast) > 0$ and so the remainder of the proof is exactly the same as in part (i).

The following example shows that the existence of a merely non-trivial continuous component does not imply that conservative points are recurrent.

**Example.** Let $X = [0, 1] \cup \{2\}$ equipped with the relative topology of the real line, and let $\mathcal{F}$ be the Borel $\sigma$-field of $X$. Further let $\mu$ denote the uniform distribution on the unit interval $[0, 1]$, and define

$$P(0, \cdot) = \frac{1}{2}(\mu + \varepsilon_2),$$

$$P(x, \cdot) = \mu, \quad \text{if } x \in (0, 1],$$

$$P(2, \cdot) = \varepsilon_2,$$
and

\[ T(x, \cdot) = \frac{1}{2} \mu, \quad \text{if} \ x \in [0, 1], \quad T(2, \cdot) = \epsilon_2. \]

Then \( T \) is a continuous component of \( K_{\theta} \), with \( \theta = (1, 0, 0, \ldots) \), which is non-trivial everywhere. Here \( x_0 = 0 \) is conservative but not recurrent.

**Proposition 3.3.** If \( X \) is \( T_1 \) and has a countable basis for the topology then the set of non-recurrent points is not properly essential.

**Proof.** Let \( \{N_i\} \) be a countable basis for the topology. If \( x \) is not recurrent then it is, by Proposition 3.1, not of sure return and hence there exists \( i \in \mathbb{N} \) such that

\[ x \in N_i \quad \text{and} \quad L(x, N_i) < 1. \]

Therefore the sets

\[ N_i^* = \{ y \in N_i \mid L(y, N_i) < 1 \}, \quad \text{for} \ i \in \mathbb{N}, \]

cover the set of non-recurrent points. Each \( N_i^* \) can be partitioned into

\[ N_i^* = \bigcup_j \{ y \in N_i^* \mid L(y, N_i) \leq 1 - j^{-1} \} = \bigcup_j N_{ij}. \]

If \( y \in N_{ij} \) then \( L(y, N_{ij}) \leq L(y, N_i^*) \leq L(y, N_i) \leq 1 - j^{-1} \) and hence, by (1.1), \( G(x, N_{ij}) \leq j \) for every \( x \in X \) and \( N_{ij} \) is inessential.

**Proposition 3.4.** If \( K_{\theta} \) has an everywhere non-trivial continuous component \( T \) then every \( \sigma \)-finite subinvariant measure \( \mu \) is finite on compact sets.

**Proof.** Write \( \mathcal{F}_\mu = \{ A \in \mathcal{F} \mid \mu(A) < \infty \} \). Since \( \mu \) is \( \sigma \)-finite and \( T \) is non-trivial, for any \( y \in X \) there exists \( A_y \in \mathcal{F}_\mu \) with \( T(y, A_y) > 0 \). Hence the open sets

\[ A' = \{ y \in X \mid T(y, A) > 0 \}, \quad \text{where} \ A \in \mathcal{F}_\mu, \]

cover \( X \). Let \( C \) be compact; so it is covered by finitely many sets \( A'_1, \ldots, A'_n \), corresponding to \( A_1, \ldots, A_n \) in \( \mathcal{F}_\mu \). Let

\[ B = \bigcup_{i=1}^n A_i. \]

If \( y \in C \) then \( T(y, B) > 0 \) and, since the lower semicontinuous function \( T(\cdot, B) \) attains its minimum on compact sets, there exists \( \delta > 0 \) such that

\[ T(y, B) \geq \delta \]

for every \( y \in C \). Moreover \( \mu(B) < \infty \); and by subinvariance

\[ \mu(B) \geq \int \mu(dx) K_\theta(x, B) \geq \int_C \mu(dx) T(x, B) \geq \delta \mu(C). \]
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REMARK. From Proposition 3.4 we can deduce that if the chain is \( \varphi \)-irreducible for some \( \varphi \) and \( K_\varphi \) has an everywhere non-trivial continuous component \( T \), then all relatively compact sets are status sets (see [17]): i.e.

(i) if \( A \in \mathcal{F} \) is relatively compact and \( G(x_\circ,A) = \infty \) for some \( x_\circ \), then the chain is recurrent,

(ii) if \( A \in \mathcal{F} \) is relatively compact and \( \lim \sup P^n(x,A) > 0 \) for all \( x \) in some \( \varphi \)-positive set, then the chain is positive recurrent.

PROPOSITION 3.5. (i) Suppose that \( X \) is compact and \( K_\varphi \) has an everywhere non-trivial continuous component \( T \). If the chain is \( \varphi \)-irreducible for some \( \varphi \), then it is positive recurrent.

(ii) Suppose that \( X \) is a compact \( T^1 \)-space with a countable basis for the topology and \( L \) has an everywhere non-trivial continuous component \( T \). Then there is no stochastically closed set of non-recurrent points.

Proof. (i) If the chain is not positive recurrent, then it admits at least one \( \sigma \)-finite subinvariant measure of infinite total mass: \( G(x_\circ, \cdot) \) for any fixed \( x_\circ \in X \) in the transient case (see Proposition 3.2 of [16]) and the unique invariant measure \( \pi \) in the null recurrent case. But this is impossible from Proposition 3.4 above.

(ii) Let \( B \) be a stochastically closed set of non-recurrent points, and put

\[ C = \{ x \in X | T(x,B^c) = 0 \}. \]

Then \( C \) is compact and \( B \subset C \). From Proposition 3.3 the set \( B \) is not properly essential, i.e.

\[ B = \bigcup_{i=1}^{\infty} B_i \]

where each \( B_i \) is inessential. If \( x \in C \) then, by the non-triviality of \( T(x, \cdot), T(x,B) > 0 \) and hence there exists \( i \) such that \( T(x,B_i) > 0 \). Therefore the open sets

\[ C_i = \{ x \in X | T(x,B_i) > 0 \} \]

cover the set \( C \), so \( C \) is covered by finitely many sets \( C_1, ..., C_n \), corresponding to \( B_1, ..., B_n \). Let

\[ A = \bigcup_{i=1}^{n} B_i. \]

If \( x \in C \) then \( T(x,A) > 0 \) and therefore \( T(\cdot, A) \) is bounded away from zero on \( C \). Hence also \( L(\cdot, A) \) is bounded away from zero on \( C \) and from (1.4)

\[ Q(x,A) \geq Q(x,C) \]

for every \( x \in X \). If \( x_\circ \in B \) then, since \( B \) is stochastically closed, \( Q(x_\circ,B) = 1 \) and so \( Q(x_\circ,A) \geq Q(x_\circ,C) \geq Q(x_\circ,B) = 1 \). However, \( A \) is
inessential as a finite union of inessential sets and thus \( Q(x, A) = 0 \) for every \( x \in X \).

**Remark.** If a strongly Feller chain \((X_n)\) on a compact space is \( \varphi \)-recurrent then it is, not only positive recurrent, but also uniformly \( \varphi \)-recurrent (see Orey [8, § 1.6]). For if \( \varphi(A) > 0 \) then

\[
\sum_{k=1}^{n} P_k(x, A) \uparrow 1 \quad \text{(as } n \to \infty) \]

for every \( x \in X \); and since the functions \( x \mapsto P_k(x, A) \) are continuous, the convergence is uniform on the compact space \( X \), cf. Proposition 4.1 of Cogburn [1].

### 4. Chains with some topological solidarity properties

In this section we look for topological conditions which will imply \( \varphi \)-solidarity properties for the chain, for some \( \varphi \). Under weak continuity of \( P \) there seems to be no reasonable connection between the topological irreducibility or recurrence concepts and the corresponding 'measure-theoretic' properties. As an example consider a random walk on the real line whose increment distribution \( \mu \) is concentrated on the set of rationals, \( \mathbb{Q} \). If \( \mu(\{r\}) > 0 \) for every \( r \in \mathbb{Q} \) and in addition \( \mu \) has finite expectation equal to zero then the random walk is topologically recurrent, i.e. every non-empty open set can be reached from every point infinitely often with probability 1. But since the space contains \( \mathbb{Q} \) and \( \mathbb{Q}^c \) as disjoint stochastically closed subsets the chain cannot be even \( \varphi \)-irreducible for any \( \varphi \).

However, we can show that the existence of a suitably non-trivial continuous component does allow us to connect topological and \( \varphi \)-solidarity concepts.

**Theorem 4.1.** Suppose that \( X \) is a topological space and that \( \mathcal{F} \) is a \( \sigma \)-field containing all the open sets. Assume that \((X_n)\) is a Markov chain on \((X, \mathcal{F})\) such that

(a) there is a point \( x_0 \in X \) such that \( L(x, N) > 0 \) for every \( x \in X \) and every neighbourhood \( N \) of \( x_0 \),

(b) for some \( \theta \), \( K_\theta \) has a component \( T \) which is non-trivial and continuous at \( x_0 \).

Then

(i) the chain is \( \varphi \)-irreducible for some \( \varphi \),

(ii) the chain is recurrent if and only if \( x_0 \) is conservative,

(iii) the chain is positive recurrent if and only if \( x_0 \) is positive.
Proof. (i) Define \( \varphi = T(x_0, \cdot) \). From (b) \( \varphi \) is a non-trivial (finite) measure on \((X, \mathcal{F})\), and \( \varphi(A) > 0 \) implies, by the lower semicontinuity of \( T(\cdot, A) \) at \( x_0 \), the existence of a neighbourhood \( N \) of \( x_0 \) such that

\[ K_\varphi(x, A) \geq T(x, A) > 0 \quad \text{for every} \quad x \in N. \]

Now it is obvious by (a) and (1.2) that \( L(y, A) > 0 \) for every \( y \in X \), proving the \( \varphi \)-irreducibility of the chain.

(ii) Suppose that \( x_0 \) is conservative. If the chain is not recurrent it is, by Theorem 1 of [16], 1-transient and there exists a set \( A \) with \( \varphi(A) = T(x_0, A) > 0 \) and \( G(x, A) < \infty \) for every \( x \in X \). Let \( \delta > 0 \) and let \( N \) be a neighbourhood of \( x_0 \) such that \( T(y, A) \geq \delta \) for all \( y \in N \); then by the Chapman–Kolmogorov identity

\[ \infty > G(x, A) \geq \int G(x, dy)K_\varphi(y, A) \geq \int_N G(x, dy)T(y, A) \geq \delta G(x, N) \]

for every \( x \in X \), contradicting the conservativity of \( x_0 \).

Conversely, if the chain is recurrent and \( \pi \) is its invariant measure, then from (a) \( \pi(N) > 0 \) and hence \( G(x, N) \equiv \infty \) for every neighbourhood of \( x_0 \) (see [16, Theorem 1]).

(iii) Suppose that \( x_0 \) is positive. Since positivity obviously implies conservativity, the chain is, by (ii), recurrent. Assume that it is null recurrent and choose \( A \in \mathcal{F} \) such that \( \varphi(A) = T(x_0, A) > 0 \) and \( \pi(A) < \infty \), \( \pi \)-invariant. Put

\[ B = \{ y \in A | \lim_{n \to \infty} P^n(y, A) = 0 \}. \]

Since \( P^n(y, A) \to 0 \) \( \pi \)-almost everywhere (and since \( \pi \geq \varphi \), also \( \varphi \)-almost everywhere), \( \varphi(B) > 0 \) and

\[ P^n(y, B) \leq P^n(y, A) \to 0 \]

as \( n \to \infty \) for every \( y \in B \). Using a bounded convergence argument in the first entrance decomposition we can easily see that in fact

\[ P^n(x, B) \to 0 \]

for every \( x \in X \).

Since \( \varphi(B) = T(x_0, B) > 0 \) there exist \( \delta > 0 \) and a neighbourhood \( N \) of \( x_0 \) such that

\[ K_\varphi(y, B) \geq T(y, B) \geq \delta \]

for each \( y \in N \). Further there exists \( m \) (depending only on \( \theta \) and \( \delta \)) such that

\[ \sum_{k=1}^m \theta_k P^k(y, B) \geq \frac{1}{2} \delta. \]
for every \( y \in N \). By the Chapman–Kolmogorov identity

\[
\sum_{k=1}^{m} \theta_k P^{n+k}(x, B) \geq \int_N P^n(x, dy) \sum_{k=1}^{m} \theta_k P^k(y, B) \geq \frac{1}{2} \delta P^n(x, N).
\]

Let \( n \to \infty \); the finite sum on the left in (4.1) tends to zero and hence also

\[ P^n(x, N) \to 0 \]

for every \( x \in X \), which obviously contradicts the positivity of \( x_0 \).

Conversely, if the chain is positive recurrent then, from (a), every neighbourhood of \( x_0 \) has positive invariant measure which immediately implies the positivity of \( x_0 \).

**Remark.** It is obvious that part (i) in Theorem 4.1 holds true if \( L \) has a corresponding component. The following example shows that part (ii) requires a \( K_\theta \)-component rather than an \( L \)-component.

**Example.** Let \( X = \{0\} \cup \{-n| n \in N^+\} \cup \{n^{-1}| n \in N^+\} \), and let

\[
P(-n, -n-1) = 1 - P(-n, 0) = \alpha_n, \quad \text{where } \alpha_n < 1, \quad \prod_{n=1}^{\infty} \alpha_n > 0,
\]

\[
P(n^{-1}, (n+1)^{-1}) = 1 \quad \text{if } n \in N^+ \text{ and } n \neq 2^k,
\]

\[
P(n^{-1}, (n+1)^{-1}) = P(n^{-1}, 0) = \frac{1}{2} \quad \text{if } n \in N^+ \text{ and } n = 2^k.
\]

The neighbourhoods of \( x_0 = 0 \) can be reached from every \( x \in X \). Also, it can easily be seen that 0 is conservative but the whole chain is not recurrent. By defining

\[
T(x, 0) = \frac{1}{2} \quad \text{if } x > 0,
\]

\[
T(x, A) = 0 \quad \text{otherwise},
\]

we have a continuous component for \( L \) which is non-trivial in particular at 0. Notice that \( Q(0, N) = 0 \) for every sufficiently small neighbourhood of 0. In fact, it is easy to see that if the conservativity of \( x_0 \) in Theorem 4.1(ii) is replaced by the condition \( Q(x_0, N) > 0 \) for every neighbourhood \( N \) of \( x_0 \), then the existence of an \( L \)-component, non-trivial at \( x_0 \), suffices to guarantee the recurrence of the chain.

The following example shows that part (iii) of Theorem 4.1 and also Propositions 3.4 and 3.5(i) need not hold when only an \( L \)-component exists.

**Example.** Consider a null recurrent Markov chain on the set of integers. Add a point \( \partial \) to make the space compact and define

\[ P(\partial, 0) = 1. \]
Now, for every set $A$ with $A \setminus \{\emptyset\} \neq \emptyset$,
\[ L(x, A) = 1 \]
for every $x \in X$ and hence, if $\mu$ is any probability measure on the integers with $\mu(\{n\}) > 0$ for every $n$, then
\[ T(x, A) = \mu(A \setminus \{\emptyset\}) \]
is an equivalent continuous component of $L$. The whole chain is a Harris chain but it is null recurrent.

Next we give some topological conditions from which the $\varphi$-recurrence of the chain can be derived.

**THEOREM 4.2.** Assume that $(X_n)$ is a Markov chain on $(X, \mathcal{F})$ such that
(a) there is a point $x_0$ such that $\inf_{x \in X} L(x, N) > 0$ for every neighbourhood $N$ of $x_0$,
(b) $L$ has a component $T$ which is non-trivial and continuous at $x_0$.
Then the chain is a Harris chain.

**Proof.** Assumption (a) implies by (1.4) that $Q(x, N) = 1$ for every $x \in X$ and every neighbourhood $N$ of $x_0$. Choose again $\varphi = T(x_0, \cdot)$. If $\varphi(A) > 0$ then there exists a neighbourhood $N$ of $x_0$ such that $T(\cdot, A)$, and hence also $L(\cdot, A)$, is bounded away from zero in $N$. Therefore, by (1.4), $Q(x, A) \geq Q(x, N) = 1$ for every $x \in X$.

The following simple example shows that assumption (a) cannot be replaced by the conditions: $x_0$ recurrent and the neighbourhoods of $x_0$ can be reached from every point $x \in X$, even in the case of a strongly Feller chain.

**Example.** Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ equipped with the natural (discrete) topology and $P(0, 0) = 1$ while
\[ P(n, n + 1) = \alpha_n = 1 - P(n, 0) \quad (n = 1, 2, \ldots), \]
where $\alpha_n < 1$ and $\prod_{n=1}^{\infty} \alpha_n > 0$. Here $P$ is trivially strongly continuous, $0$ is recurrent, and $L(x, 0) > 0$ for every $x \in X$. Condition (a) is not satisfied since
\[ L(n, 0) = 1 - \prod_{k=n}^{\infty} \alpha_k \to 0 \]
as $n \to \infty$. The only irreducibility measure for this chain is $\varphi = \varepsilon_0$ but the chain is not $\varphi$-recurrent and hence is not a Harris chain.

**5. Continuous components for random walks**

Now we look at the meaning of our continuous component conditions in the special case when $(X_n)$ is a random walk and see that our results
in §4 are essentially generalizations of the properties of spread-out random walks (for which see Revuz [1]).

Let \( G \) be a locally compact Hausdorff topological group with a countable basis for the topology and let \( \mathcal{F} \) be the Borel \( \sigma \)-field of \( G \). Let \((X_n)\) be a (right) random walk on \((G, \mathcal{F})\) with distribution \( \mu \), that is,

\[
P(x, A) = \mu(x^{-1}A)
\]

for every \( x \in G \) and \( A \in \mathcal{F} \) where \( \mu \) is a probability measure on \((G, \mathcal{F})\). Notice that \( P \) is always weakly continuous. A random walk \((X_n)\) is called spread-out if some convolution power \( \mu^n \) of \( \mu \) is not singular with respect to the (right) Haar measure \( m \).

**Theorem 5.1.** (i) If the random walk is spread-out then there exists \( \theta \) such that \( K_\theta \) has an everywhere non-trivial continuous component \( T \) and hence also \( L \) has such a component. Conversely, if \( L \) has a component continuous and non-trivial at a point \( x_0 \in G \) then the random walk is spread-out.

(ii) \( L \) has an equivalent continuous component if and only if \( \mu \) is absolutely continuous with respect to \( m \).

**Proof.** To prove the sufficiency in (i) and (ii) it suffices to show that a kernel \( K(x, A) = \nu(x^{-1}A) \) induced by any finite measure \( \nu \) on \((G, \mathcal{F})\) is strongly continuous if and only if \( \nu \) is absolutely continuous with respect to \( m \). If \( \nu = \lambda m \) then we can write

\[
K(x, A) = \int_{x^{-1}A} f(y)m(dy) = \Delta(x) \int_{1_A(y)} f(x^{-1}y)m(dy),
\]

where \( \Delta \) denotes the modular function of \( m \) and \( 1_A \) is the indicator of \( A \). Now \( \Delta \) is known to be continuous (see Husain [3, Proposition 36.7, p.119]) and the integral is actually uniformly continuous on \( G \). This follows from the fact that the mapping \( x \mapsto f_x, f_x(y) = f(xy) \), for a fixed \( f \in L^1(m) \), is a uniformly continuous mapping from \( G \) into \( L^1(m) \), see, for example, the proof of Proposition 47.4 in [3, p. 173]. Conversely, if \( K \) is strongly continuous and \( \nu(A) = K(e, A) > 0 \) then there exist \( \delta > 0 \) and a neighbourhood \( N \) of \( e \) such that \( K(x, A) \geq \delta \) for \( x \in N \) and hence

\[
\nu(G)m(A) = \int m(dx)K(x, A) \geq \int_N m(dx)K(x, A) \geq \delta m(N) > 0.
\]

For the necessity, suppose that \( L \) has a component \( T \) continuous and non-trivial at a point \( x_0 \) and \( \mu \) is not spread-out. Without loss of generality, we may assume \( x_0 = e \). The assumption is then that there exists a set \( A \) with

\[
m(A) = 0 \quad \text{and} \quad L(e, A^c) = 0.
\]
Hence by the non-triviality of $T$, $T(e, A) > 0$, and by the lower semi-continuity there exist $\delta > 0$ and a neighbourhood $N$ of $e$ such that $T(x, A) \geq \delta$ for $x \in N$. Hence

$$L(x, A) = \sum_{A} P^n(x, A) \geq \delta$$

for $x \in N$, and, since $m(N) > 0$, there exists $k$ such that the set

$$N_k = \{ x \in N | \sum_{n=1}^{k} P^n(x, A) \geq \frac{1}{2} \delta \}$$

has a positive $m$-measure. Using again the invariance of $m$ we get

$$km(A) = \int m(dx) \sum_{n=1}^{k} P^n(x, A) \geq \int_{N_k} m(dx) \sum_{n=1}^{k} P^n(x, A) \geq \frac{1}{2} \delta m(N_k) > 0,$$

contradicting $m(A) = 0$. The proof of necessity in (ii) is similar.

**Remark.** As a corollary of our Theorem 4.2 we get the following (cf. Theorem 3.4.11 of Revuz [11]): if a spread-out random walk is topologically recurrent then it is a Harris chain.

### 6. Decomposition theorems

The main contribution of this section is a ‘Doeblin decomposition’ theorem for chains which do not a priori have any topological solidarity structure. We shall show that under our continuous component assumption, together with an extremely mild topological condition for the space, the chain is in fact a countable union of disjoint Harris chains together with a transient part which is not properly essential. By considering a Markov chain on the real line which keeps every point fixed one sees that such a decomposition is not necessarily available when $P$ is weakly continuous; whilst the same chain with the discrete topology shows that the strong continuity of $P$ does not suffice without the topological condition we impose.

**Theorem 6.1.** Suppose that $X$ is a topological space and $\mathcal{F}$ is a $\sigma$-field containing all the open sets. Assume that $(X_n)$ is a Markov chain on $(X, \mathcal{F})$ and

(a) $X$ is $T_1$ and has a countable basis for the topology,

(b) $L$ has a continuous component $T$ which is non-trivial at every recurrent point $x$ of $X$.

(i) Then $X$ can be decomposed into

$$X = \sum_{i \in I} H_i + E$$

where $I$ is countable, each $H_i$ is a maximal Harris set, and $E$ is not properly essential. The set $R$ of recurrent points is contained in the Harris part $\sum H_i$. 
of the space. The sets $R \cap H_i$ are both open and closed in the relative topology of $R$ and hence $\text{card}(I)$ is at most the number of topological components of $R$.

(ii) If (b) is replaced by

(c) $L$ has an equivalent continuous component, then each $H_i$ is also topologically closed, and $E$ can be further represented as

$$E = E' + E''$$

where $E'$ is open and $L(x, \sum H_i) > 0$ for every $x \in E'$, while $E''$ is both stochastically and topologically closed (or empty).

REMARK. Jain and Jamison [4], Jamison [6], and Winkler [19] have proved a variety of similar decomposition theorems for chains satisfying certain measure-theoretic non-singularity or countability assumptions. Connections between these conditions are explored in [18] where it is also shown that for a 'Doeblin decomposition' it is necessary to have some topology in which (a) and (b) hold true.

The idea of our proof is to construct for every recurrent point $x$ a local irreducible set $V_x$ (Lemma 6.1) and a local recurrent set $H_x$ (Lemma 6.2) using Theorems 4.1 and 4.2 as patterns. Then it will be shown that distinct sets $H_x$ are disjoint (Lemma 6.3) and finally that a countable number of them will suffice.

**Lemma 6.1.** Suppose that the assumptions (a) and (b) hold true. Let $x$ be recurrent and let

$$V_x = \{y \in X \mid L(y, N(x)) > 0 \text{ for every neighbourhood } N(x) \text{ of } x\},$$

$$U_x = \{y \in X \mid T(y, V_x) > 0\}.$$

Then $U_x$ is open with $x \in U_x \subset V_x$; and $V_x$ is $T(x, \cdot)$-irreducible. If also (c) holds then $U_x = V_x$.

**Proof.** First we notice that $V_x$ can be represented as a countable intersection

$$V_x = \bigcap_{i \in \mathbb{N}} \{y \in X \mid L(y, N_i(x)) > 0\},$$

where $\{N_i(x)\}$ is a neighbourhood basis of $x$. Hence $V_x \in \mathcal{F}$ and $U_x$ is open by the lower semicontinuity of $T(\cdot, V_x)$. If $y \in U_x$ then

$$L(y, V_x) \geq T(y, V_x) > 0$$

and, by (1.2), $y \in V_x$. So $U_x \subset V_x$.

Next we shall show that $x \in \bar{U}_x$. Suppose the contrary, i.e. $T(x, V_x) = 0$. Then by assumption (b), $L(x, V_x^c) \geq T(x, V_x^c) > 0$. Since

$$V_x^c = \bigcup_{i \in \mathbb{N}} M_i, \quad M_i = \{y \in X \mid L(y, N_i(x)) = 0\},$$
there exists $i \in \mathbb{N}$ such that $L(x, M_i) > 0$. Then, by (1.3), $Q(x, N_i(x)) < 1$, contradicting the recurrence of $x$.

Now we proceed to the $T(x, \cdot \cdot)$-irreducibility. We have already observed that $T(x, V_z) > 0$ (in fact that $T(x, V^+_z) = 0$). If $A \subset V_z$ and $T(x, A) > 0$ then there exists a neighbourhood $N$ of $x$ such that

$$L(y, A) \geq T(y, A) > 0$$

for every $y \in N$. If $z \in V_z$ then $L(z, N) > 0$ and then also $L(z, A) > 0$ by (1.2).

Finally suppose that (c) holds. If $y \in V_z$ then, since $U_z$ is a neighbourhood of $x$,

$$L(y, V_z) \geq L(y, U_z) > 0$$

and by (c) also $T(y, V_z) > 0$, that is, $V_z \subset U_z$.

**Lemma 6.2.** Suppose that (a) and (b) hold true. Let $x$ be recurrent and suppose $V_z$ as in Lemma 6.1. Define

$$H_x = \{y \in X \mid Q(y, N(x)) = 1 \text{ for every neighbourhood } N(x) \text{ of } x\}.$$

Then $H_x$ is a Harris set, $x \in H_x \subset V_z$, and $E_x = V_z \setminus H_x$ is not properly essential. If also (c) holds $H_x$ is also topologically closed and $E_x$ is open.

**Proof.** Let $\{N_i(x)\}$ be a neighbourhood basis of $x$. Then

$$H_x = \bigcap_{i \in \mathbb{N}} \{y \in X \mid Q(y, N_i(x)) = 1\},$$

which shows that $H_x \in \mathcal{F}$ and further that $H_x$ is stochastically closed as a countable intersection of stochastically closed sets. Obviously $x \in H_x \subset V_z$.

Write $\varphi_x = T(x, \cdot \cdot)$. The first step to prove the $\varphi_x$-recurrence of $H_x$ is to show that $\varphi_x(H_x) > 0$. Assume the contrary, that is, $T(x, H_x) = 0$, which implies by (b) that $L(x, H^c_x) \geq T(x, H^c_x) > 0$. Since

$$H^c_x = \bigcup_{i \in \mathbb{N}} M'_i, \quad M'_i = \{y \in X \mid Q(y, N_i(x)) < 1\}$$

there exists $i \in \mathbb{N}$ such that $L(x, M'_i) > 0$. Then by (1.3) $Q(x, N_i(x)) < 1$, contradicting the recurrence of $x$. Suppose that $A \subset H_x$ with $\varphi_x(A) = T(x, A) > 0$. Then there exists a neighbourhood $N(x)$ of $x$ in which $L(\cdot, A) \geq T(\cdot, A)$ is bounded away from zero and again an application of (1.4) shows that

$$Q(y, N(x)) \leq Q(y, A).$$

Hence if $y \in H_x$ then $Q(y, A) = 1$, which completes the proof of $\varphi_x$-recurrence.
By the definitions of \( H_x \) and \( V_x \),
\[
E_x = V_x \setminus H_x \subset \bigcup_{t \in \mathbb{N}} \{ y \in X \mid 0 < L(y, N_t(x)), Q(y, N_t(x)) < 1 \}
\]
\[
= \bigcup_{t \in \mathbb{N}} \bigcup_{k \in \mathbb{N}^+} \{ y \in X \mid k^{-1} \leq L(y, N_t(x)), Q(y, N_t(x)) \leq 1 - k^{-1} \}.
\]

Hence \( E_x \) is a countable union of inessential sets (see Proposition 1.5.1 of Orey [8]).

For the rest of the proof suppose that (c) holds.

To prove \( H_x \) topologically closed we in fact show that
\[
H_x^c = \{ y \in X \mid T(y, H_x^c) > 0 \},
\]
which is an open set. If \( T(y, H_x^c) > 0 \) then also \( L(y, H_x^c) > 0 \) and so \( y \in H_x^c \) since \( H_x \) is stochastically closed. Conversely, if \( T(y, H_x^c) = 0 \) then, by (c), \( L(y, H_x^c) = 0 \) and hence \( P(y, H_x) = 1 \). Let \( N(x) \) be a neighbourhood of \( x \). Then
\[
Q(y, N(x)) \geq \int_{H_x} P(y, dz)Q(z, N(x)) = P(y, H_x) = 1.
\]

Since this is true for every neighbourhood \( N(x), y \in H_x \). Finally, \( E_x \) is open because now \( V_x = U_x \) is open (Lemma 6.1), \( H_x \) is closed, and \( E_x = V_x \cap H_x^c \).

**Lemma 6.3.** Suppose that (a) and (b) hold true. Let \( x \) be recurrent and let \( H_x \) be as in Lemma 6.2. Then \( H_x \) is a maximal Harris set and hence \( H_x = H_y \) or \( H_x \cap H_y = \emptyset \) for \( x, y \in R \). If also (c) holds and \( H_x \neq H_y \) then \( E_x \cap H_y = \emptyset \).

**Proof.** The maximal Harris set containing \( H_x \) is given by
\[
\bar{H}_x = \{ y \in X \mid Q(y, H_x) = 1 \}
\]
and hence from the definition of \( H_x \) it is obvious that \( \bar{H}_x = H_x \). Distinct maximal Harris sets are known to be disjoint.

Suppose now that (c) holds and \( z \in E_x \cap H_y \). By the definition of \( V_x \), \( z \in E_x \) implies \( L(z, N(x)) > 0 \) for every neighbourhood \( N(x) \) of \( x \). Since \( z \in H_y \) and \( H_y \) is stochastically closed, this implies that \( N(x) \cap H_y \neq \emptyset \) for every such neighbourhood. But from Lemma 6.2, \( H_y \) is also topologically closed and so \( x \in H_y \) and hence \( H_x = H_y \).

**Proof of Theorem 6.1.** Let \( U_x \) be as in Lemma 6.1 and \( U = \bigcup_{x \in R} U_x \). Since each \( U_x \) is open and the topology of \( X \) has a countable basis, Lindelöf's theorem (Kelley [7, p. 49]) implies that \( U \) can be represented as a countable union \( U = \bigcup_{i \in I} U_i \) where \( U_i = U_{x_i} \) for some recurrent \( x_i \). All the points in \( U^c \) are non-recurrent and therefore \( U^c \) is not properly
essential by Proposition 3.3. Let $V_i$ be the set corresponding to $U_i$, for $i \in I$. Then by Lemma 6.2,

$$V_i = H_i + E_i,$$

where $H_i$ is a Harris set and $E_i$ is not properly essential. Distinct sets $H_i$ are disjoint by Lemma 6.3. Hence we have the representation

$$X = \left( \sum_{i \in I} H_i \right) \cup \left( \bigcup_{i \in I} E_i \right) \cup U^c,$$

which shows that $E = X \setminus \sum H_i$ is not properly essential. Let $x$ be recurrent and $H_x$ the corresponding Harris set. If $H_x$ is not contained in $\sum H_i$ then $H_x \cap H_i = \emptyset$ for every $i \in I$, by Lemma 6.3, which implies $H_x \subset E$. But this is impossible because $H_x$ is properly essential but $E$ is not. So we have proved that all the recurrent points belong to the Harris part $\sum H_i$.

To prove that the sets $H_i \cap R$ are both open and closed in the relative topology of $R$ we show that each $H_i \cap R$ can be represented as

$$H_i \cap R = \{ y \in R \mid T(y, H_i) > 0 \} = \{ y \in R \mid T(y, H^c_i) = 0 \},$$

that is, as an intersection with $R$ of an open and a closed set, which is the result. Suppose that $x \in H_i \cap R$. Since $H_i$ is stochastically closed, $T(x, H^c_i) = 0$ and further, since $T(x, \cdot)$ is non-trivial, $T(x, H_i) > 0$. Conversely, if $x \in R$ then $x \in H_i$ for exactly one $i \in I$, $T(x, H_i) > 0$, and $T(x, H_j) = 0$ for every $j \in I$, $j \neq i$. So, $x \in R$ and $T(x, H_i) > 0$ implies that $x \in H_i \cap R$. Finally $x \in R$ and $T(x, H^c_i) = 0$ implies $T(x, H_i) > 0$ by the non-triviality of $T(x, \cdot)$.

For the rest of the proof suppose that (c) holds true. Then each $H_i$ is topologically closed by Lemma 6.2. Put

$$E'' = U^c = X \setminus \bigcup_{i \in I} V_i,$$

since $U_i = V_i$. The set $E''$ is topologically closed as a complement of the open set $U$ but it is also stochastically closed since each $V_i$ is stochastically closed by definition. Let

$$E' = \bigcup_{i \in I} E_i.$$

Then $E = E' + E''$, from Lemma 6.3, and $E'$ is open as a union of open sets $E_i$. If $y \in E_i = V_i \setminus H_i$, for $i \in I$, then $L(y, H_i) > 0$ by the irreducibility of $V_i$, proving the ultimate assertion of Theorem 6.1.

The following example shows that we need an equivalent rather than merely a non-trivial component to identify the topological nature of the sets in the decomposition of Theorem 6.1.
EXAMPLE. Let \( X = [0, 1] \cup \{1 + n^{-1} | n \in \mathbb{N}^+ \} \cup \{-n | n \in \mathbb{N}^+ \} \) equipped with the real line topology, and put

\[
P(x, 2) = P(x, -1) = \frac{1}{2}, \quad \text{if } x \in [\frac{1}{2}, 1],
\]

\[
P(x, -1) = 1, \quad \text{if } x \in [0, \frac{1}{2}],
\]

\[
P(1 + n^{-1}, 1 + (n + 1)^{-1}) = P(1 + n^{-1}, 2) = \frac{1}{2} \quad (n \in \mathbb{N}^+),
\]

\[
P(-n, -n - 1) = 1 \quad (n \in \mathbb{N}^+).
\]

Then \( H = \{1 + n^{-1} | n \in \mathbb{N}^+ \} \) is a single \( \varphi \)-recurrent (and positive recurrent) set which is not closed. The set \( E' \) of points in \( H^c \) which lead to \( H \) is \( [\frac{1}{2}, 1] \) and hence not open; and the stochastically closed set \( E'' \) of non-recurrent points is \( [0, \frac{1}{2}) \cup \{-n | n \in \mathbb{N}^+ \} \) which is not closed.

This chain has an everywhere non-trivial continuous component \( T \) of \( P \) (and hence also such a component of \( L \)) given by

\[
T(x, 2) = \frac{1}{2}(2 - x^{-1}), \quad \text{if } x \in [\frac{1}{2}, 1],
\]

\[
T(x, -1) = \frac{1}{2}(1 - x), \quad \text{if } x \in [0, 1],
\]

\[
T(1 + n^{-1}, 2) = \frac{1}{2} \quad (n \in \mathbb{N}^+),
\]

\[
T(-n, -n - 1) = 1 \quad (n \in \mathbb{N}^+),
\]

and by additivity otherwise.

Combining the result of Theorem 6.1 with various additional conditions for the chain, we easily deduce the following sequence of corollaries.

**Corollary 6.1.** Suppose that (a) and (b) in Theorem 6.1 hold true. If all the points are recurrent then

\[
X = \sum_{i \in I} H_i,
\]

where each \( H_i \) is a Harris set and both open and closed. Hence \( \text{card}(I) \) is at most the number of topological components of \( X \).

**Proof.** The proof follows directly by taking \( R = X \) in Theorem 6.1.

**Corollary 6.2.** Suppose that (a) holds true. The space \( X \) is properly essential if there exists a recurrent point \( x_0 \) and \( L \) has a continuous component \( T \) non-trivial at \( x_0 \). Conversely, if the space is properly essential then there exists at least one recurrent point.

**Proof.** If \( x_0 \) is recurrent and \( T \) is non-trivial at \( x_0 \) then the corresponding Harris set \( H_{x_0} \) of Lemma 6.2 is properly essential. The converse follows directly from Proposition 3.3.
Corollary 6.3. Suppose that (a) holds and \( X \) is indecomposable.

(i) If there exists a recurrent point \( x_0 \) and \( L \) has a continuous component \( T \) non-trivial at \( x_0 \) then the chain is recurrent. Conversely, if the chain is recurrent then there exists at least one recurrent point.

(ii) If there exists a recurrent point \( x_0 \) and \( L \) has an everywhere non-trivial continuous component \( T \) then the chain is recurrent and \( X \) can be decomposed into

\[
X = H + E,
\]

where \( H \) is a Harris set and \( E \) is not properly essential.

Proof. (i) If \( X \) is indecomposable then the chain is recurrent if and only if \( X \) is properly essential, see Theorem 1.5 of [15]. Hence the statement follows directly from Corollary 6.2.

(ii) There exists at least one Harris set \( H = H_{x_0} \). Since \( X \) is indecomposable the number of Harris sets in the decomposition of Theorem 6.1 is at most one.

Remark. If (a) holds true, \( X \) is indecomposable and \( L \) has a continuous component, non-trivial at a recurrent point, then the chain is also normal; see, for example, Jain and Jamison [4].

Corollary 6.4. Suppose that (a) and (b) hold true. If all the points are recurrent then each of the irreducibility conditions

(i) there exists \( x_0 \) such that \( L(x_0, N) > 0 \) for every non-empty open set \( N \),
(ii) there exists \( x_0 \) such that \( L(x, N(x_0)) > 0 \) for every \( x \in X \) and every neighbourhood \( N(x_0) \) of \( x_0 \),

implies that the chain is a Harris chain.

Proof. From Corollary 6.1, \( X = \sum_{i \in I} H_i \) where each \( H_i \) is both an open and a closed Harris set. Now it is easy to see that both (i) and (ii) imply that the number of Harris sets is exactly one.

Our next corollary shows that, under the assumptions (a) and (c) of Theorem 6.1, the recurrent points have a kind of solidarity property.

Corollary 6.5. Suppose that (a) and (c) hold true. If \( x \) is recurrent and \( L(x, N(y)) > 0 \) for every neighbourhood \( N(y) \) of \( y \) then also \( y \) is recurrent.

Proof. Let \( H_x \) be the \( \varphi_x \)-recurrent set of Lemma 6.2 corresponding to \( x \). Since \( H_x \) is stochastically closed, \( L(x, N(y)) > 0 \) is possible only if \( H_x \cap N(y) \neq \emptyset \) for every neighbourhood of \( y \). Since \( H_x \) is also topologically closed this further implies that \( y \in H_x \). If \( N(y) \) is any neighbourhood of \( y \) then by (c) also \( \varphi_x(N(y)) = T(x, N(y)) > 0 \) and the statement follows from the \( \varphi_x \)-recurrence of \( H_x \).
Corollary 6.6. Suppose that (a) and (c) hold true. If there exists a recurrent point \( x_0 \) such that \( L(x_0, N) > 0 \) for every non-empty open set \( N \) then all the points are recurrent and the chain is a Harris chain.

Proof. This follows directly from Corollaries 6.5 and 6.4 above.

Applying our decomposition to a spread-out random walk and taking into consideration the spatial homogeneity of such a chain we have the following conclusion, a detailed proof of which we omit.

Corollary 6.7. If \((X_n)\) is a (right) random walk on a locally compact Hausdorff topological group with a countable basis for the topology and the increment distribution \( \mu \) is spread-out, then the existence of a conservative point implies that \( G \) can be decomposed into

\[
G = \sum_{i \in I} H_i,
\]

where each \( H_i \) is a Harris set and is both open and closed. If \( G_\mu \) denotes the closed subgroup generated by the support of \( \mu \), then the sets \( H_i \) are the left cosets of \( G_\mu \) and hence \( \text{card}(I) \) is the number of left cosets of \( G_\mu \).

7. Compact state spaces

In this section we consider the special features our decomposition theorem implies for compact state spaces.

Theorem 7.1. Suppose that \( X \) is a compact \( T_1 \)-space with a countable basis for the topology, \( \mathcal{F} \) contains all the open sets of \( X \), and \((X_n)\) is a Markov chain on \((X, \mathcal{F})\).

(i) If \( L \) has an everywhere non-trivial continuous component \( T \), then there exists a positive integer \( n \) such that

\[
X = \sum_{i=1}^{n} H_i + E,
\]

where each \( H_i \) is a Harris set and \( E \) is inessential.

(ii) If \( K_\theta \), for some \( \theta \), has an everywhere non-trivial continuous component \( T \), then (7.1) holds with each \( H_i \) positive recurrent and \( E \) uniformly transient, that is, there exists \( M < \infty \) such that \( G(x, E) \leq M \) for every \( x \in X \).

(iii) If \( L \) has an equivalent continuous component \( T \), then (7.1) holds with each \( H_i \) closed and \( E \) open.

Proof. (i) By Theorem 6.1,
where each $H_i$ is a Harris set. Put

$$H'_i = \{ y \in X \mid T(y, H_i) > 0 \} \quad (i \in I).$$

If $x \in H_i$ then $L(x, H'_i) = 0$ and hence also $T(x, H'_i) = 0$. Therefore

$$H'_i = H_i \cup (E \cap H'_i) \quad (i \in I).$$

Write $H = \sum_{i \in I} H_i$; then the closed set

$$E^* = \{ y \in X \mid T(y, E) = 0 \}$$

contains $H$. If $y \in E^*$ then, by the non-triviality of $T(y, \cdot)$, $T(y, H_i) > 0$ for some $i \in I$, that is, $\{H'_i \mid i \in I\}$ is an open cover of the compact set $E^*$. Therefore

$$E^* \subset \bigcup_{i=1}^n H'_i,$$

which implies that

$$H = H \cap E^* = \bigcup_{i=1}^n H \cap H'_i = \bigcup_{i=1}^n H_i.$$

From Proposition 3.5(ii) there is a recurrent point, so $H$ is non-empty and $n$ is in fact a positive integer.

Let $E'' = \{ y \in X \mid L(y, H) = 0 \}; E''$ consists entirely of non-recurrent points (by Theorem 6.1) and is stochastically closed. So from Proposition 3.5(ii), $E''$ is empty. Hence

$$X = \bigcup_{n=1}^\infty A_n,$$

where $A_n = \{ x \in X \mid L(x, H) > n^{-1} \}$. By the non-triviality of $T$, for any $y \in X, T(y, A_n) > 0$ for some $n$. Write

$$B_{n,m} = \{ y \in X \mid T(y, A_n) > m^{-1} \};$$

by construction

$$\inf_{y \in B_{n,m}} L(y, H) \geq (nm)^{-1}.$$

But the sets $B_{n,m}$ are an open cover of $X$ and so $X$ is covered by finitely many of them. Hence $L(\cdot, H)$ is bounded away from zero on $X$ and, by (1.4), $Q(x, H) = 1$ for every $x \in X$. Since $H$ is stochastically closed this implies $Q(x, H^c) = 0$ for every $x \in X$ and so $E = H^c$ is inessential as claimed.

(ii) If $K_\theta$ has an everywhere non-trivial continuous component then also $L$ has and hence (7.1) holds. Let $\pi_\cdot$ be the invariant measure for $H_i \ (i = 1, \ldots, n)$. Define $\mu_i = \pi_\cdot$ on $H_i$ and $\mu_i(H_i^c) = 0$. Since $H_i$ is stochastically closed, $\mu_i$ is invariant for the whole chain; hence $\mu_i(X) = \pi_\cdot(H_i) \leq \infty$ from Proposition 3.4, that is, $H_i$ is positive recurrent.
As in the proof of (i) we can deduce that $K_t(\cdot, H)$ is bounded away from zero on $X$ and hence there exists $\delta > 0$ and $m \in \mathbb{N}^+$ such that
\[ P_x(\tau_H \leq m) \geq \delta \]
for every $x \in X$, cf. the proof of Proposition 3.2(i). By iterating we obtain
\[ E_x(\tau_H) \leq m/\delta \]
for every $x \in X$. Since $H$ is stochastically closed and so the uniform transience of $E$ holds with $M = m/\delta - 1$.

(iii) If $T$ is equivalent to $L$ then by Theorem 6.1 each $H_i$ is also topologically closed, which completes the proof.

The example below shows that, even on a compact space, we need an equivalent rather than merely a non-trivial component to identify the topological nature of the sets $H_i$ and $E$.

**EXAMPLE.** Let $X = \{0, n^{-1}, 1 + n^{-1} | n \in \mathbb{N}^+\}$ equipped again with the real line topology and define
\[
P(n^{-1}, (n+1)^{-1}) = P(n^{-1}, \frac{1}{2}) = \frac{1}{3}, \quad \text{if } n \geq 2,
\]
\[
P(1, \frac{1}{2}) = P(1, 2) = \frac{1}{3},
\]
\[
P(1 + n^{-1}, 1 + (n+1)^{-1}) = P(1 + n^{-1}, 2) = \frac{1}{3} \quad (n \in \mathbb{N}^+),
\]
\[
P(0, \frac{1}{2}) = P(0, 2) = \frac{1}{3}.
\]
The kernel given by
\[
T(x, \frac{1}{2}) = \frac{1}{3}, \quad \text{if } x < 1,
\]
\[
T(x, 2) = \frac{1}{3}, \quad \text{if } x \geq 1,
\]
\[
T(x, A) = 0, \quad \text{otherwise,}
\]
is an everywhere non-trivial continuous component of $P$, but not equivalent to $L$. Here $H_1 = \{n^{-1} | n \geq 2\}$ and $H_2 = \{1 + n^{-1} | n \in \mathbb{N}^+\}$ which are not closed, whilst $E = \{0, 1\}$ which is not open.

**COROLLARY 7.1.** Suppose that the assumptions of part (ii) of Theorem 7.1 hold true. Then there exists an invariant probability measure $\pi$ such that
\[
\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=1}^{n} P^k(x, \cdot) - \pi \right\| \to 0
\]
for $\pi$-a.e. $x \in X$.

**Proof.** Theorem 7.1 implies that there is at least one Harris set $H_1 \subset X$ and that the chain on $H_1$ is positive recurrent. Let $\pi_1$ be the invariant probability measure for the chain on $H_1$. Define $\pi = \pi_1$ on $H_1$ and $\pi(H_i^c) = 0$. Since $H_1$ is stochastically closed, $\pi$ is invariant for the whole
chain and the convergence statement follows from Theorem 1.7.1 of Orey [8].

Remark. In the case (ii) of Theorem 7.1 our decomposition implies also that there exists a finite number of mutually singular invariant probability measures \( \pi_1, \ldots, \pi_n \) such that every invariant measure \( \pi \) is a linear combination of the measures \( \pi_i \). Corollary 7.1 can be compared with Theorem IV.3.1 of Rosenblatt [12] which states that if \( X \) is a compact Hausdorff space and \( P \) is weakly continuous then the chain has at least one invariant probability measure.

**Corollary 7.2.** Suppose that the assumptions of part (i) of Theorem 7.1 hold true. Then the following conditions are equivalent:

(i) the topological irreducibility condition (a) of Theorem 4.1 holds;
(ii) \( X \) is indecomposable;
(iii) \( X \) is a Harris set.

**Proof.** Both (i) and (ii) imply the existence of at most (and hence exactly) one Harris set in the decomposition (7.1). The proof of Theorem 7.1 shows that \( Q(x, H_j) = 1 \) for all \( x \in X \) and therefore \( X = H_1 \), and (iii) holds. Conversely, (iii) implies (ii) trivially, and (iii) implies (i) with the point \( x_0 \) in Theorem 4.1 taken as any one of the recurrent points of \( X \).

Note added in proof. We have found that Grigorescu, in [20], has shown several of the results of §7 under the much more restrictive assumption that the chain is strongly Feller.

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