Review of Statistical Arbitrage, Cointegration, and Multivariate Ornstein-Uhlenbeck

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Abstract

We introduce the multivariate Ornstein-Uhlenbeck process, solve it analytically, and discuss how it generalizes a vast class of continuous-time and discrete-time multivariate processes. Relying on the simple geometrical interpretation of the dynamics of the Ornstein-Uhlenbeck process we introduce cointegration and its relationship to statistical arbitrage. We illustrate an application to swap contract strategies. Fully documented code illustrating the theory and the applications is available at `www.symmys.com` ⇒ Teaching ⇒ MATLAB.

JEL Classification: C1, G11

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The multivariate Ornstein-Uhlenbeck process is arguably the model most utilized by academics and practitioners alike to describe the multivariate dynamics of financial variables. Indeed, the Ornstein-Uhlenbeck process is parsimonious, and yet general enough to cover a broad range of processes. Therefore, by studying the multivariate Ornstein-Uhlenbeck process we gain insight into the properties of the main multivariate features used daily by econometricians.

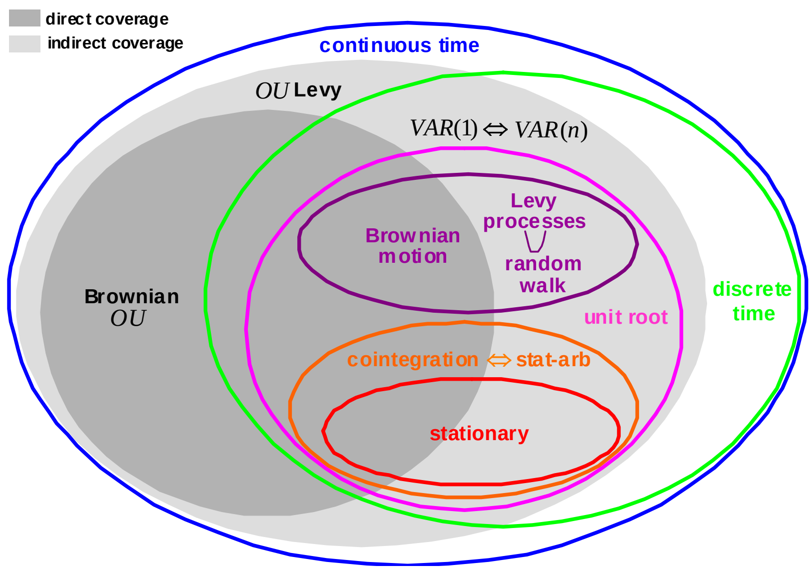

Figure 1: Multivariate processes and coverage by OU

Indeed, the following relationships hold, refer to Figure 1 and see below for a proof. The Ornstein-Uhlenbeck is a continuous time process. When sampled in discrete time, the Ornstein-Uhlenbeck process gives rise to a vector autoregression of order one, commonly denoted by VAR(1). More general VAR(n) processes can be represented in VAR(1) format, and therefore they are also covered by the Ornstein-Uhlenbeck process. VAR(n) processes include unit root processes, which in turn include the random walk, the discrete-time counterpart of Levy processes. VAR(n) processes also include cointegrated dynamics, which are the foundation of statistical arbitrage. Finally, stationary processes are a special case of cointegration.

In Section 1 we derive the analytical solution of the Ornstein-Uhlenbeck process. In Section 2 we discuss the geometrical interpretation of the solution. Building on the solution and its geometrical interpretation, in Section 3 we introduce naturally the concept of cointegration and we study its properties. In Section 4 we discuss a simple model-independent estimation technique for cointegration and we apply this technique to the detection of mean-reverting trades, which is the foundation of statistical arbitrage. Fully documented code that illustrates the theory and the empirical aspects of the models in this article is available at `www.symmys.com` ⇒ Teaching ⇒ MATLAB.

1 The multivariate Ornstein-Uhlenbeck process

The multivariate Ornstein-Uhlenbeck process is defined by the following stochastic differential equation

$$d\mathbf{X}_t = -\boldsymbol{\Theta}\left(\mathbf{X}_t - \boldsymbol{\mu}\right) dt + \mathbf{S} d\mathbf{B}_t. \tag{1}$$

In this expression $\boldsymbol{\Theta}$ is the transition matrix, namely a fully generic square matrix that defines the deterministic portion of the evolution of the process; $\boldsymbol{\mu}$ is a fully generic vector, which represents the unconditional expectation when this is defined, see below; $\mathbf{S}$ is the scatter generator, namely a fully generic square matrix that induces the dispersion of the process; $\mathbf{B}_t$ is typically assumed to be a vector of independent Brownian motions, although more in general it is a vector of independent Levy processes, see Barndorff-Nielsen and Shephard (2001) and refer to Figure 1.

To integrate the process (1) we introduce the integrator

$$\mathbf{Y}_t \equiv e^{\boldsymbol{\Theta} t}\left(\mathbf{X}_t - \boldsymbol{\mu}\right). \tag{2}$$

Using Ito's lemma we obtain

$$d\mathbf{Y}_t = e^{\boldsymbol{\Theta} t} \mathbf{S} d\mathbf{B}_t. \tag{3}$$

Integrating both sides and substituting the definition (2) we obtain

$$\mathbf{X}_{t+\tau} = \left(\mathbf{I} - e^{-\boldsymbol{\Theta}\tau}\right)\boldsymbol{\mu} + e^{-\boldsymbol{\Theta}\tau}\mathbf{X}_t + \boldsymbol{\epsilon}_{t,\tau}, \tag{4}$$

where the invariants are mixed integrals of the Brownian motion and are thus normally distributed

$$\boldsymbol{\epsilon}_{t,\tau} \equiv \int_t^{t+\tau} e^{\boldsymbol{\Theta}(u-\tau)} \mathbf{S} d\mathbf{B}_u \sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_\tau\right). \tag{5}$$

The solution (4) is a vector autoregressive process of order one VAR(1), which reads

$$\mathbf{X}_{t+\tau} = \mathbf{c} + \mathbf{C}\mathbf{X}_t + \boldsymbol{\epsilon}_{t,\tau}, \tag{6}$$

for a conformable vector and matrix $\mathbf{c}$ and $\mathbf{C}$ respectively. A comparison between the integral solution (4) and the generic VAR(1) formulation (6) provides the relationship between the continuous-time coefficients and their discrete-time counterparts.

The conditional distribution of the Ornstein-Uhlenbeck process (4) is normal at all times

$$\mathbf{X}_{t+\tau} \sim \mathrm{N}\left(\mathbf{x}_{t+\tau}, \boldsymbol{\Sigma}_\tau\right). \tag{7}$$

The deterministic drift reads

$$\mathbf{x}_{t+\tau} \equiv \left(\mathbf{I} - e^{-\boldsymbol{\Theta}\tau}\right)\boldsymbol{\mu} + e^{-\boldsymbol{\Theta}\tau}\mathbf{x}_t \tag{8}$$

and the covariance can be expressed as in Van der Werf (2007) in terms of the stack operator vec and the Kronecker sum $\oplus$ as

$$\operatorname{vec}(\boldsymbol{\Sigma}_\tau) \equiv (\boldsymbol{\Theta} \oplus \boldsymbol{\Theta})^{-1}\left(\mathbf{I} - e^{-(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta})\tau}\right)\operatorname{vec}(\boldsymbol{\Sigma}), \tag{9}$$

where $\boldsymbol{\Sigma} \equiv \mathbf{S}\mathbf{S}'$, see the proof in Appendix A.2. Formulas (8)-(9) describe the propagation law of risk associated with the Ornstein-Uhlenbeck process: the location-dispersion ellipsoid defined by these parameters provides an indication of the uncertainly in the realization of the next step in the process.

Notice that (8) and (9) are defined for any specification of the input parameters $\boldsymbol{\Theta}$, $\boldsymbol{\mu}$, and $\mathbf{S}$ in (1). For small values of τ, a Taylor expansion of these formulas shows that

$$\mathbf{X}_{t+\tau} \approx \mathbf{X}_t + \boldsymbol{\epsilon}_{t,\tau}, \tag{10}$$

where $\boldsymbol{\epsilon}_{t,\tau}$ is a normal invariant:

$$\boldsymbol{\epsilon}_{t,\tau} \sim \mathrm{N}(\tau\boldsymbol{\Theta}\boldsymbol{\mu}, \tau\boldsymbol{\Sigma}). \tag{11}$$

In other words, for small values of the time step τ the Ornstein-Uhlenbeck process is indistinguishable from a Brownian motion, where the risk of the invariants (11), as represented by the standard deviation of any linear combination of its entries, displays the classical "square-root of τ" propagation law.

As the step horizon τ grows to infinity, so do the expectation (8) and the covariance (9), unless all the eigenvalues of $\boldsymbol{\Theta}$ have positive real part. In that case the distribution of the process stabilizes to a normal whose unconditional expectation and covariance read

$$\mathbf{x}_\infty = \boldsymbol{\mu} \tag{12}$$

$$\operatorname{vec}(\boldsymbol{\Sigma}_\infty) = (\boldsymbol{\Theta} \oplus \boldsymbol{\Theta})^{-1}\operatorname{vec}(\boldsymbol{\Sigma}) \tag{13}$$

To illustrate, we consider the bivariate case of the two-year and the ten-year par swap rates. The benchmark assumption among buy-side practitioners is that par swap rates evolve as the random walk (10), see Figure 3.5 and related discussion in Meucci (2005).

However, rates cannot diffuse indefinitely. Therefore, they cannot evolve as a random walk for any size of the time step τ: for steps of the order of a month or larger, mean-reverting effects must become apparent.

The Ornstein-Uhlenbeck process is suited to model this behavior. We fit this process for different values of the time step τ and we display in Figure 2 the location-dispersion ellipsoid defined by the expectation (8) and the covariance

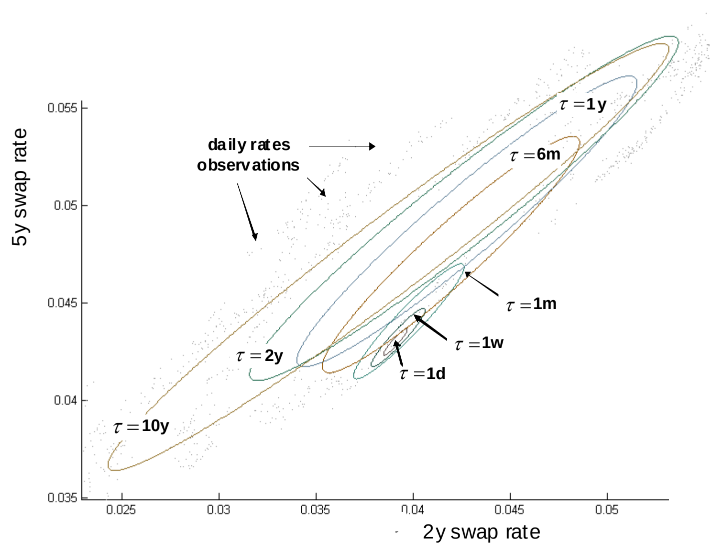

Figure 2: Propagation law of risk for OU process fitted to swap rates

(9), refer to `www.symmys.com` ⇒ Teaching ⇒ MATLAB for the fully documented code.

For values of τ of the order of a few days, the drift is linear in the step and the size increases as the square root of the step, as in (11). As the step increases and mean-reversion kicks in, the ellipsoid stabilizes to its unconditional values (12)-(13).

2 The geometry of the Ornstein-Uhlenbeck dynamics

The integral (4) contains all the information on the joint dynamics of the Ornstein-Uhlenbeck process (1). However, that solution does not provide any intuition on the dynamics of this process. In order to understand this dynamics we need to observe the Ornstein-Uhlenbeck process in a different set of coordinates.

Consider the eigenvalues of the transition matrix $\boldsymbol{\Theta}$: since this matrix has real entries, its eigenvalues are either real or complex conjugate: we denote them respectively by $(\lambda_1, \ldots, \lambda_K)$ and $(\gamma_1 \pm i\omega_1), \ldots, (\gamma_J \pm i\omega_J)$, where $K + 2J = N$. Now consider the matrix $\mathbf{B}$ whose columns are the respective, possibly complex, eigenvectors and define the real matrix $\mathbf{A} \equiv \mathrm{Re}(\mathbf{B}) - \mathrm{Im}(\mathbf{B})$. Then the transition matrix can be decomposed in terms of eigenvalues and eigenvectors as follows

$$\mathbf{\Theta} \equiv \mathbf{A}\mathbf{\Gamma}\mathbf{A}^{-1}, \tag{14}$$

where $\mathbf{\Gamma}$ is a block-diagonal matrix

$$\mathbf{\Gamma} \equiv \operatorname{diag}\left(\lambda_1, \ldots, \lambda_K, \mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_J\right), \tag{15}$$

and the generic j-th matrix $\mathbf{\Gamma}_j$ is defined as

$$\mathbf{\Gamma}_j \equiv \begin{pmatrix} \gamma_j & \omega_j \\ -\omega_j & \gamma_j \end{pmatrix}. \tag{16}$$

With the eigenvector matrix $\mathbf{A}$ we can introduce a new set of coordinates

$$\mathbf{z} \equiv \mathbf{A}^{-1}\left(\mathbf{x} - \boldsymbol{\mu}\right). \tag{17}$$

The original Ornstein-Uhlenbeck process (1) in these coordinates follows from Ito's lemma and reads

$$d\mathbf{Z}_t = -\mathbf{\Gamma}\mathbf{Z}_t dt + \mathbf{V} d\mathbf{B}_t, \tag{18}$$

where $\mathbf{V} \equiv \mathbf{A}^{-1}\mathbf{S}$. Since this is another Ornstein-Uhlenbeck process, its solution is normal

$$\mathbf{Z}_t \sim \mathrm{N}\left(\mathbf{z}_t, \mathbf{\Phi}_t\right), \tag{19}$$

for a suitable deterministic drift $\mathbf{z}_t$ and covariance $\mathbf{\Phi}_t$. The deterministic drift $\mathbf{z}_t$ is the solution of the ordinary differential equation

$$d\mathbf{z}_t = -\mathbf{\Gamma}\mathbf{z}_t dt, \tag{20}$$

Given the block-diagonal structure of (15), the deterministic drift splits into separate sub-problems. Indeed, let us partition the N-dimensional vector $\mathbf{z}_t$ into K entries which correspond to the real eigenvalues in (15), and J pairs of entries which correspond to the complex-conjugate eigenvalues summarized by (16):

$$\mathbf{z}_t \equiv \left(z_{1,t}, \ldots, z_{K,t}, z_{1,t}^{(1)}, z_{1,t}^{(2)}, \ldots, z_{J,t}^{(1)}, z_{J,t}^{(2)}\right)'. \tag{21}$$

For the variables corresponding to the real eigenvalues, (20) simplifies to:

$$dz_{k,t} = -\lambda_k z_{k,t} dt. \tag{22}$$

For each real eigenvalue indexed by $k = 1, \ldots, K$ the solution reads

$$z_{k;t} \equiv e^{-\lambda_k t} z_{k,0}. \tag{23}$$

This is an exponential shrinkage at the rate λ_k. Note that (23) is properly defined also for negative values of λ_k, in which case the trajectory is an exponential explosion. If $\lambda_k > 0$ we can compute the half-life of the deterministic trend, namely the time required for the trajectory (23) to progress half way toward the long term expectation, which is zero:

$$\widetilde{t} \equiv \frac{\ln 2}{\lambda_k}. \tag{24}$$

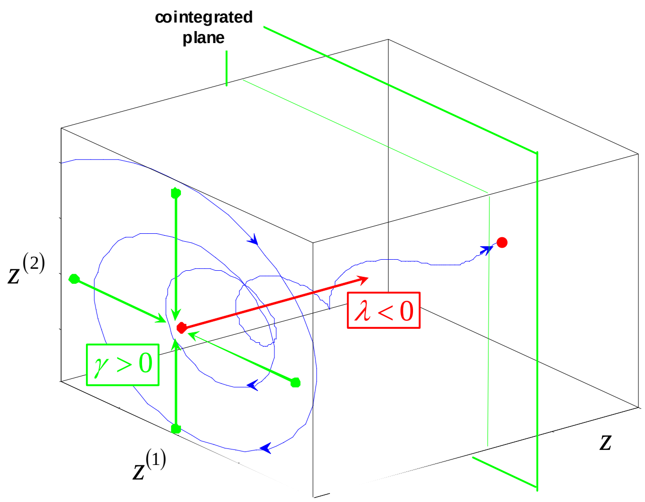

Figure 3: Deterministic drift of OU process

As for the variables among (21) corresponding to the complex eigenvalue pairs (20) simplifies to

$$d\mathbf{z}_{j,t} = -\boldsymbol{\Gamma}_j \mathbf{z}_{j,t} dt, \quad j = 1, \ldots, J. \tag{25}$$

For each complex eigenvalue, the solution reads formally

$$\mathbf{z}_{j,t} \equiv e^{-\boldsymbol{\Gamma}_j t} \mathbf{z}_{j,0}. \tag{26}$$

This formal bivariate solution can be made more explicit component-wise. First, we write the matrix (16) as follows

$$\boldsymbol{\Gamma}_j = \gamma_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \omega_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{27}$$

As we show in Appendix A.1, the identity matrix on the right hand side generates an exponential explosion in the solution (26), where the scalar γ_j determines the rate of the explosion. On the other hand, the second matrix on the right hand side in (27) generates a clockwise rotation, where the scalar ω_j determines the frequency of the rotation. Given the minus sign in the exponential in (26) we obtain an exponential shrinkage at the rate γ_j, coupled with a counterclockwise rotation with frequency ω_j

$$z_{j;t}^{(1)} \equiv e^{-\gamma_j t} \left(z_{j,0}^{(1)} \cos \omega_j t - z_{j,0}^{(2)} \sin \omega_j t \right) \tag{28}$$

$$z_{j;t}^{(2)} \equiv e^{-\gamma_j t} \left(z_{j,0}^{(1)} \sin \omega_j t + z_{j,0}^{(2)} \cos \omega_j t \right). \tag{29}$$

Again, (28)-(29) are properly defined also for negative values of γ_j, in which case the trajectory is an exponential explosion.

To illustrate, we consider a tri-variate case with one real eigenvalue λ and two conjugate eigenvalues $\gamma + i\omega$ and $\gamma - i\omega$, where $\lambda < 0$ and $\gamma > 0$. In Figure 3 we display the ensuing dynamics for the deterministic drift (21), which in this context reads $\left(z_t, z_t^{(1)}, z_t^{(2)}\right)$: the motion (23) escapes toward infinity at an exponential rate, whereas the motion (28)-(29) whirls toward zero, shrinking at an exponential rate. The animation that generates Figure 3 can be downloaded from `www.symmys.com` $\Rightarrow$ Teaching $\Rightarrow$ MATLAB.

Once the deterministic drift in the special coordinates $\mathbf{z}_t$ is known, the deterministic drift of the original Ornstein-Uhlenbeck process (4) is obtained by inverting (17). This is an affine transformation, which maps lines in lines and planes in planes

$$\mathbf{x}_t \equiv \boldsymbol{\mu} + \mathbf{A}\mathbf{z}_t \tag{30}$$

Therefore the qualitative behavior of the solutions (23) and (28)-(29) sketched In Figure 3 is preserved.

Although the deterministic part of the process in diagonal coordinates (18) splits into separate dynamics within each eigenspace, these dynamics are not independent. In Appendix A.3 we derive the quite lengthy explicit formulas for the evolution of all the entries of the covariance $\boldsymbol{\Phi}_t$ in (19). For instance, the covariances between entries relative to two real eigenvalues reads:

$$\Phi_{k,\widetilde{k};t} = \frac{\Phi_{k,\widetilde{k}}}{\lambda_k + \lambda_{\widetilde{k}}} \left(1 - e^{-\left(\lambda_k + \lambda_{\widetilde{k}}\right)t}\right), \tag{31}$$

where $\boldsymbol{\Phi} \equiv \mathbf{V}\mathbf{V}'$. More in general, the following observations apply. First, as time evolves, the relative volatilities change: this is due both to the different speed of divergence/shrinkage induced by the real parts λ's and γ's of the eigenvalues, and to different speed of rotation, induced by the imaginary parts ω's of the eigenvalues. Second, the correlations only vary if rotations occur: if the imaginary parts ω's are null, the correlations remain unvaried.

Once the covariance $\boldsymbol{\Phi}_t$ in the diagonal coordinates is known, the covariance of the original Ornstein-Uhlenbeck process (4) is obtained by inverting (17) and using the affine equivariance of the covariance, which leads to $\boldsymbol{\Sigma}_t = \mathbf{A}\boldsymbol{\Phi}_t\mathbf{A}'$.

3 Cointegration

The solution (4) of the Ornstein-Uhlenbeck dynamics (1) holds for *any* choice of the input parameters $\boldsymbol{\mu}$, $\boldsymbol{\Theta}$ and $\mathbf{S}$. However, from the formulas for the covariances (31), and similar formulas in Appendix A.3, we verify that if some $\lambda_k \leq 0$, or some $\gamma_j \leq 0$, i.e. if any eigenvalues of the transition matrix $\boldsymbol{\Theta}$ are null or negative, then the overall covariance of the Ornstein-Uhlenbeck $\mathbf{X}_t$

does not converge: therefore $\mathbf{X}_t$ is stationary only if the real parts of all the eigenvalues of $\boldsymbol{\Theta}$ are strictly positive.

Nevertheless, as long as some eigenvalues have strictly positive real part, the covariances of the respective entries in the transformed process (18) stabilizes in the long run. Therefore these processes and any linear combination thereof are stationary. Such combinations are called *cointegrated*: since from (17) they span a hyperplane, that hyperplane is called the *cointegrated space*, see Figure 3.

To better discuss cointegration, we write the Ornstein-Uhlenbeck process (4) as

$$\Delta_\tau \mathbf{X}_t = \boldsymbol{\Psi}_\tau \left(\mathbf{X}_t - \boldsymbol{\mu}\right) + \boldsymbol{\epsilon}_{t,\tau}, \tag{32}$$

where Δ_τ is the difference operator $\Delta_\tau \mathbf{X}_t \equiv \mathbf{X}_{t+\tau} - \mathbf{X}_t$ and $\boldsymbol{\Psi}_\tau$ is the transition matrix

$$\boldsymbol{\Psi}_\tau \equiv e^{-\boldsymbol{\Theta}\tau} - \mathbf{I}. \tag{33}$$

If some eigenvalues in $\boldsymbol{\Theta}$ are null, the matrix $e^{-\boldsymbol{\Theta}\tau}$ has unit eigenvalues: this follows from

$$e^{-\boldsymbol{\Theta}\tau} = \mathbf{A} e^{-\boldsymbol{\Gamma}\tau} \mathbf{A}^{-1}, \tag{34}$$

which in turn follows from (14). Processes with this characteristic are known as *unit-root* processes. A very special case arises when all the entries in $\boldsymbol{\Theta}$ are null. In this circumstance, $e^{-\boldsymbol{\Theta}\tau}$ is the identity matrix, the transition matrix $\boldsymbol{\Psi}_\tau$ is null and the Ornstein-Uhlenbeck process becomes a multivariate random walk.

More in general, suppose that L eigenvalues are null. Then $\boldsymbol{\Psi}_\tau$ has rank $N - L$ and therefore it can be expressed as

$$\boldsymbol{\Psi}_\tau \equiv \boldsymbol{\Phi}_\tau' \boldsymbol{\Upsilon}_\tau, \tag{35}$$

where both matrices $\boldsymbol{\Phi}_\tau$ and $\boldsymbol{\Upsilon}_\tau$ are full-rank and have dimension $(N - L) \times N$. The representation (35), known as the *error correction* representation of the process (32), is not unique: indeed any pair $\widetilde{\boldsymbol{\Phi}}_\tau \equiv \mathbf{P}' \boldsymbol{\Phi}_\tau$ and $\widetilde{\boldsymbol{\Upsilon}}_\tau \equiv \mathbf{P}^{-1} \boldsymbol{\Upsilon}_\tau$ gives rise to the same transition matrix $\boldsymbol{\Psi}_\tau$ for fully arbitrary invertible matrices $\mathbf{P}$.

The L-dimensional hyperplane spanned by the rows of $\boldsymbol{\Upsilon}_\tau$ does not depend on the horizon τ. This follows from (33) and (34) and the fact that since $\boldsymbol{\Gamma}$ generates rotations (imaginary part of the eigenvalues) and/or contractions (real part of the eigenvalues), the matrix $e^{-\boldsymbol{\Gamma}\tau}$ maps the eigenspaces of $\boldsymbol{\Theta}$ into themselves. In particular, any eigenspace of $\boldsymbol{\Theta}$ relative to a null eigenvalue is mapped into itself at any horizon.

Assuming that the non-null eigenvalues of $\boldsymbol{\Theta}$ have positive real part[2], the rows of $\boldsymbol{\Upsilon}_\tau$, or of any alternative representation, span the contraction hyperplanes and the process $\boldsymbol{\Upsilon}_\tau \mathbf{X}_t$ is stationary and would converge exponentially fast to the the unconditional expectation $\boldsymbol{\Upsilon}_\tau \boldsymbol{\mu}$, if it were not for the shocks $\boldsymbol{\epsilon}_{t,\tau}$ in (32).

[2] One can take differences in the time series of $\mathbf{X}_t$ until the fitted transition matrix does not display negative eigenvalues. However, the case of eigenvalues with pure imaginary part is not accounted for.

4 Statistical arbitrage

Cointegration, along with its geometric interpretation, was introduced above building on the multivariate Ornstein-Uhlenbeck dynamics. However, cointegration is a model-independent concept. Consider a generic multivariate process $\mathbf{X}_t$. This process is cointegrated if there exists a linear combination of its entries which is stationary. Let us denote such combination as follows

$$Y_t^{\mathbf{w}} \equiv \mathbf{X}_t' \mathbf{w}, \tag{36}$$

where $\mathbf{w}$ is normalized to have unit length for future convenience.

If $\mathbf{w}$ belongs to the cointegration space, i.e. $Y_t^{\mathbf{w}}$ is cointegrated, its variance stabilizes as $t \to \infty$. Otherwise, its variance diverges to infinity. Therefore, the combination that minimizes the conditional variance among all the possible combinations is the best candidate for cointegration:

$$\widetilde{\mathbf{w}} \equiv \underset{\|\mathbf{w}\|=1}{\operatorname{argmin}} \left[\operatorname{Var} \left\{ Y_\infty^{\mathbf{w}} | \mathbf{x}_0 \right\} \right]. \tag{37}$$

Based on this intuition, we consider formally the conditional covariance of the process

$$\boldsymbol{\Sigma}_\infty \equiv \operatorname{Cov} \left\{ \mathbf{X}_\infty | \mathbf{x}_0 \right\}, \tag{38}$$

although we understand that this might not be defined. Then we consider the formal principal component factorization of the covariance

$$\boldsymbol{\Sigma}_\infty \equiv \mathbf{E} \boldsymbol{\Lambda} \mathbf{E}, \tag{39}$$

where $\mathbf{E}$ is the orthogonal matrix whose columns are the eigenvectors

$$\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(N)} \right); \tag{40}$$

and $\boldsymbol{\Lambda}$ is the diagonal matrix of the respective eigenvalues, sorted in decreasing order

$$\boldsymbol{\Lambda} \equiv \operatorname{diag} \left(\lambda^{(1)}, \ldots, \lambda^{(N)} \right). \tag{41}$$

Note that some eigenvalues might be infinite.

The formal solution to (37) is $\widetilde{\mathbf{w}} \equiv \mathbf{e}^{(N)}$, the eigenvector relative to the smallest eigenvalue $\lambda^{(N)}$. If $\mathbf{e}^{(N)}$ gives rise to cointegration, the process $Y_t^{\mathbf{e}^{(N)}}$ is stationary and therefore the eigenvalue $\lambda^{(N)}$ is not infinite, but rather it represents the unconditional variance of $Y_t^{\mathbf{e}^{(N)}}$.

If cointegration is found with $\mathbf{e}^{(N)}$, the next natural candidate for another possible cointegrated relationship is $\mathbf{e}^{(N-1)}$. Again, if $\mathbf{e}^{(N-1)}$ gives rise to cointegration, the eigenvalue $\lambda_t^{(N-1)}$ converges to the unconditional variance of $Y_t^{\mathbf{e}^{(N-1)}}$.

In other words, the PCA decomposition (39) partitions the space into two portions: the directions of infinite variance, namely the eigenvectors relative to the infinite eigenvalues, which are not cointegrated, and the directions of finite variance, namely the eigenvectors relative to the finite eigenvalues, which are cointegrated.

The above approach assumes knowledge of the true covariance (38), which in reality is not known. However, the sample covariance of the process $\mathbf{X}_t$ along the cointegrated directions approximates the true asymptotic covariance. Therefore, the above approach can be implemented in practice by replacing the true, unknown covariance (38) with its sample counterpart.

To summarize, the above rationale yields a practical routine to detect the cointegrated relationships in a vector autoregressive process (1). Without analyzing the eigenvalues of the transition matrix fitted to an autoregressive dynamics, we consider the sample counterpart of the covariance (38); then we extract the eigenvectors (40); finally we explore the stationarity of the combinations $Y_t^{\mathbf{e}^{(n)}}$ for $n = N, \ldots, 1$.

To illustrate, we consider a trading strategy with swap contracts. First, we note that the p&l generated by a swap contract is faithfully approximated by the change in the respective swap rate times a constant, known among practitioners as "dv01". Therefore, we analyze linear combinations of swap rates, which map into portfolios p&l's, hoping to detect cointegrated patterns.

In particular, we consider the time series of the 1y, 2y, 5y, 7y, 10y, 15y, and 30y rates. We compute the sample covariance and we perform its PCA decomposition. In Figure 4 we plot the time series corresponding with the first, second, fourth and seventh eigenvectors, refer to `www.symmys.com` ⇒ Teaching ⇒ MATLAB for the fully documented code.

In particular, to test for the stationarity of the potentially cointegrated series it is convenient to fit to each of them a AR(1) process, i.e. the univariate version of (4), to the cointegrated combinations:

$$y_{t+\tau} \equiv \left(1 - e^{-\theta\tau}\right)\mu + e^{-\theta\tau} y_t + \epsilon_{t,\tau}. \tag{42}$$

In the univariate case the transition matrix $\boldsymbol{\Theta}$ becomes the scalar θ. Consistently with (23) cointegration corresponds to the condition that the mean-reversion parameter θ be larger than zero.

By specializing (12) and (13) to the one-dimensional case, we can compute the expected long-term gain

$$\alpha \equiv |y_t - \mathrm{E}\{y_\infty\}| = |y_t - \mu|\,;$$

the z-score

$$Z_t \equiv \frac{|y_t - \mathrm{E}\{y_\infty\}|}{\mathrm{Sd}\{y_\infty\}} = \frac{|y_t - \mu|}{\sqrt{\sigma^2/2\theta}}; \tag{43}$$

and the half-life (24) of the deterministic trend

$$\widetilde{\tau} \propto \frac{1}{\theta}. \tag{44}$$

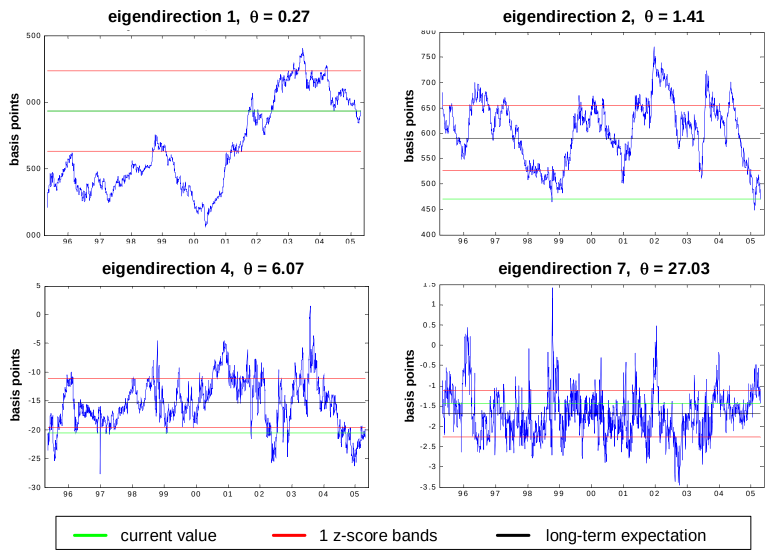

Figure 4: Cointegration search among swap rates

The expected gain (4) is also known as the "alpha" of the trade. The z-score represents the ex-ante Sharpe ratio of the trade and can be used to generate signals: when the z-score is large we enter the trade; when the z-score has narrowed we cash a profit; and if the z-score has widened further we take a loss. The half-life represents the order of magnitude of the time required before we can hope to cash in any profit: the higher the mean-reversion θ, i.e. the more cointegrated the series, the lesser the wait.

A caveat is due at this point: based on the above recipe, one would be tempted to set up trades that react to signals from the most mean-reverting combinations. However, such combinations face two major problems. First, in-sample cointegration does not necessarily correspond to out-of-sample results: as a matter of fact, the eigenseries relative to the smallest eigenvalues, i.e. those that allow to make quicker profits, are the least robust out-of-sample. Second, the "alpha" (4) of a trade has the same order of magnitude as its volatility. In the case of cointegrated eigenseries, the volatility is the square root of the respective eigenvalue (41): this implies that the most mean-reverting series correspond to a much lesser potential return, which is easily offset by the transaction costs.

In the example in Figure 4 the AR(1) fit (42) confirms that cointegration increases with the order of the eigenseries. In the first eigenseries the mean-reversion parameter $\theta \approx 0.27$ is close to zero: indeed, its pattern is very similar to a random walk. On the other hand, the last eigenseries displays a very high mean-reversion parameter $\theta \approx 27$.

The current signal on the second eigenseries appears quite strong: one would be tempted to set up a dv01-weighted trade that mimics this series and buy it. However, the expected wait before cashing in on this trade is of the order of $\widetilde{\tau} \propto 1/1.41 \approx 0.7$ years.

The current signal on the seventh eigenseries is not strong, but the mean-reversion is very high, therefore, soon enough the series should hit the 1-z-score bands: if the series first hits the lower 1-z-score band one should buy the series, or sell it if the series first hits the upper 1-z-score band, hoping to cash in in $\widetilde{\tau} \propto 252/27 \approx 9$ days. However, the "alpha" (43) on this trade would be minuscule, of the order of the basis point: such "alpha" would not justify the transaction costs incurred by setting up and unwinding a trade that involves long-short positions in seven contracts.

The current signal on the fourth eigenseries appears strong enough to buy it and the expected wait before cashing in is of the order of $\widetilde{\tau} \propto 12/6.07 \approx 2$ months. The "alpha" is of the order of five basis points, too low for seven contracts. However, the dv01-adjusted presence of the 15y contract is almost null and the 5y, 7y, and 10y contracts appear with the same sign and can be replicated with the 7y contract only without affecting the qualitative behavior of the eigenseries. Consequently the trader might want to consider setting up this trade.

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A Appendix

In this appendix we present proofs, results and details that can be skipped at first reading.

A.1 Deterministic linear dynamical system

Consider the dynamical system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{45}$$

To compute the solution, consider the auxiliary complex problem

$$\dot{z} = \mu z, \tag{46}$$

where $z \equiv z_1 + iz_2$ and $\mu \equiv a + ib$. By isolating the real and the imaginary parts we realize that (45) and (46) coincide.

The solution of (46) is

$$z(t) = e^{\mu t} z(0). \tag{47}$$

Isolating the real and the imaginary parts of this solution we obtain:

$$\begin{aligned} z_1(t) &= \operatorname{Re}\left(e^{\mu t} z^0\right) = \operatorname{Re}\left(e^{(a+ib)t}\left(z_1(0) + iz_2(0)\right)\right) & (48) \\ &= \operatorname{Re}\left(e^{at}\left(\cos bt + i \sin bt\right)\left(z_1(0) + iz_2(0)\right)\right) \\ z_2(t) &= \operatorname{Im}\left(e^{\mu t} z^0\right) = \operatorname{Im}\left(e^{(a+ib)t}\left(z_1(0) + iz_2(0)\right)\right) & (49) \\ &= \operatorname{Im}\left(e^{at}\left(\cos bt + i \sin bt\right) z_1(0) + iz_2(0)\right) \end{aligned}$$

or

$$\begin{aligned} z_1(t) &= e^{at}\left(z_1(0) \cos bt - z_2(0) \sin bt\right) & (50) \\ z_2(t) &= e^{at}\left(z_1(0) \sin bt + z_2(0) \cos bt\right) & (51) \end{aligned}$$

The trajectories (50)-(51) depart from (shrink toward) the origin at the exponential rate a and turn counterclockwise with frequency b.

A.2 Distribution of Ornstein-Uhlenbeck process

We recall that the Ornstein-Uhlenbeck process

$$d\mathbf{X}_t = -\mathbf{\Theta}\left(\mathbf{X}_t - \mathbf{m}\right) dt + \mathbf{S} d\mathbf{B}_t. \tag{52}$$

integrates as follows

$$\mathbf{X}_t = \mathbf{m} + e^{-\mathbf{\Theta} t}\left(\mathbf{x}_0 - \mathbf{m}\right) + \int_0^t e^{\mathbf{\Theta}(u-t)} \mathbf{S} d\mathbf{B}_u. \tag{53}$$

The distribution of $\mathbf{X}_t$ conditioned on the initial value $\mathbf{x}_0$ is normal

$$\mathbf{X}_t|\mathbf{x}_0 \sim \mathrm{N}\left(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\right). \tag{54}$$

The expectation follows from (53) and reads

$$\boldsymbol{\mu}_t = \mathbf{m}+e^{-\boldsymbol{\Theta}t}\left(\mathbf{x}_0 - \mathbf{m}\right). \tag{55}$$

To compute the covariance we use Ito's isometry:

$$\boldsymbol{\Sigma}_t \equiv \mathrm{Cov}\left\{\mathbf{X}_t|\mathbf{x}_0\right\} = \int_0^t e^{\boldsymbol{\Theta}(u-t)}\boldsymbol{\Sigma}e^{\boldsymbol{\Theta}'(u-t)}du, \tag{56}$$

where $\boldsymbol{\Sigma} \equiv \mathbf{S}\mathbf{S}'$. We can simplify further this expression as in Van der Werf (2007). Using the identity

$$\mathrm{vec}\left(\mathbf{ABC}\right) \equiv \left(\mathbf{C}' \otimes \mathbf{A}\right)\mathrm{vec}\left(\mathbf{B}\right), \tag{57}$$

where vec is the stack operator and $\otimes$ is the Kronecker product. Then

$$\mathrm{vec}\left(e^{\boldsymbol{\Theta}(u-t)}\boldsymbol{\Sigma}e^{\boldsymbol{\Theta}'(u-t)}\right) = \left(e^{\boldsymbol{\Theta}(u-t)} \otimes e^{\boldsymbol{\Theta}(u-t)}\right)\mathrm{vec}\left(\boldsymbol{\Sigma}\right). \tag{58}$$

Now we can use the identity

$$e^{\mathbf{A}\oplus\mathbf{B}} = e^{\mathbf{A}} \otimes e^{\mathbf{B}}, \tag{59}$$

where $\oplus$ is the Kronecker sum

$$\mathbf{A}_{M\times M} \oplus \mathbf{B}_{N\times N} \equiv \mathbf{A}_{M\times M} \otimes \mathbf{I}_{N\times N} + \mathbf{I}_{M\times M} \otimes \mathbf{B}_{N\times N}. \tag{60}$$

Then we can rephrase the term in the integral (58) as

$$\mathrm{vec}\left(e^{\boldsymbol{\Theta}(u-t)}\boldsymbol{\Sigma}e^{\boldsymbol{\Theta}(u-t)}\right) = \left(e^{(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta})(u-t)}\right)\mathrm{vec}\left(\boldsymbol{\Sigma}\right). \tag{61}$$

Substituting this in (56) we obtain

$$\begin{aligned}
\mathrm{vec}\left(\boldsymbol{\Sigma}_t\right) &= \left(\int_0^t \left(e^{(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta})(u-t)}\right)du\right)\mathrm{vec}\left(\boldsymbol{\Sigma}\right) \\
&= \left(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta}\right)^{-1} e^{(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta})(u-t)}\Big|_0^t \mathrm{vec}\left(\boldsymbol{\Sigma}\right) \\
&= \left(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta}\right)^{-1}\left(\mathbf{I} - e^{-(\boldsymbol{\Theta}\oplus\boldsymbol{\Theta})t}\right)\mathrm{vec}\left(\boldsymbol{\Sigma}\right)
\end{aligned} \tag{62}$$

More in general, consider the process monitored at arbitrary times $0 \leq t_1 \leq t_2 \leq \ldots$

$$\mathbf{X}_{t_1,t_2,\ldots} \equiv \begin{pmatrix} \mathbf{X}_{t_1} \\ \mathbf{X}_{t_2} \\ \vdots \end{pmatrix}. \tag{63}$$

The distribution of $\mathbf{X}_{t_1,t_2,\ldots}$ conditioned on the initial value $\mathbf{x}_0$ is normal

$$\mathbf{X}_{t_1,t_2,\ldots}|\mathbf{x}_0 \sim \mathrm{N}\left(\boldsymbol{\mu}_{t_1,t_2,\ldots}, \boldsymbol{\Sigma}_{t_1,t_2,\ldots}\right). \tag{64}$$

The expectation follows from (53) and reads

$$\boldsymbol{\mu}_{t_1,t_2,\ldots} \equiv \begin{pmatrix} \mathbf{m}+e^{-\boldsymbol{\Theta}t_1}\left(\mathbf{x}_0-\mathbf{m}\right) \\ \mathbf{m}+e^{-\boldsymbol{\Theta}t_2}\left(\mathbf{x}_0-\mathbf{m}\right) \\ \vdots \end{pmatrix} \tag{65}$$

For the covariance, it suffices to consider two horizons. Applying Ito's isometry we obtain

$$\begin{aligned} \boldsymbol{\Sigma}_{t_1,t_2} &\equiv \mathrm{Cov}\left\{\mathbf{X}_{t_1},\mathbf{X}_{t_2}|\mathbf{x}_0\right\} = \int_0^{t_1} e^{\boldsymbol{\Theta}(u-t_1)}\boldsymbol{\Sigma}e^{\boldsymbol{\Theta}'(u-t_2)}du \\ &= \int_0^{t_1} e^{\boldsymbol{\Theta}(u-t_1)}\boldsymbol{\Sigma}e^{\boldsymbol{\Theta}'(u-t_1)-\boldsymbol{\Theta}'(t_2-t_1)}du = \boldsymbol{\Sigma}_{t_1}e^{-\boldsymbol{\Theta}'(t_2-t_1)} \end{aligned}$$

Therefore

$$\boldsymbol{\Sigma}_{t_1,t_2,\ldots} = \begin{pmatrix} \boldsymbol{\Sigma}_{t_1} & \boldsymbol{\Sigma}_{t_1}e^{-\boldsymbol{\Theta}'(t_2-t_1)} & \boldsymbol{\Sigma}_{t_1}e^{-\boldsymbol{\Theta}'(t_3-t_1)} & \cdots \\ e^{-\boldsymbol{\Theta}(t_2-t_1)}\boldsymbol{\Sigma}_{t_1} & \boldsymbol{\Sigma}_{t_2} & \boldsymbol{\Sigma}_{t_2}e^{-\boldsymbol{\Theta}'(t_3-t_2)} & \cdots \\ \vdots & e^{-\boldsymbol{\Theta}(t_3-t_2)}\boldsymbol{\Sigma}_{t_2} & \boldsymbol{\Sigma}_{t_3} & \\ & \vdots & & \ddots \end{pmatrix}, \tag{66}$$

This expression is simplified heavily by applying (62).

A.3 OU (auto)covariance: explicit solution

For $t \leq z$ the autocovariance is

$$\begin{aligned} \mathrm{Cov}\left\{\mathbf{Z}_t,\mathbf{Z}_{t+\tau}|\mathbf{Z}_0\right\} &= \mathrm{E}\left\{\left(\mathbf{Z}_t-\mathrm{E}\left\{\mathbf{Z}_t|\mathbf{Z}_0\right\}\right)\left(\mathbf{Z}_{t+\tau}-\mathrm{E}\left\{\mathbf{Z}_{t+\tau}|\mathbf{Z}_0\right\}\right)'|\mathbf{Z}_0\right\} \qquad (67) \\ &= \mathrm{E}\left\{\left(\int_0^t e^{\boldsymbol{\Gamma}(u-t)}\mathbf{V}d\mathbf{B}_u\right)\left(\int_0^{t+\tau} d\mathbf{B}_u\mathbf{V}'e^{\boldsymbol{\Gamma}'(u-t)}\right)'\right\}e^{-\boldsymbol{\Gamma}'\tau} \\ &= \mathrm{E}\left\{\left(\int_0^t e^{\boldsymbol{\Gamma}(u-t)}\mathbf{V}d\mathbf{B}_u\right)\left(\int_0^{t} d\mathbf{B}_u\mathbf{V}'e^{\boldsymbol{\Gamma}'(u-t)}\right)'\right\}e^{-\boldsymbol{\Gamma}'\tau} \\ &= \left(\int_0^t e^{\boldsymbol{\Gamma}(u-t)}\mathbf{V}\mathbf{V}'e^{\boldsymbol{\Gamma}'(u-t)}du\right)e^{-\boldsymbol{\Gamma}'\tau} \\ &= \left(\int_0^t e^{-\boldsymbol{\Gamma}s}\mathbf{V}\mathbf{V}'e^{-\boldsymbol{\Gamma}'s}du\right)e^{-\boldsymbol{\Gamma}'\tau} \\ &= \left(\int_0^t e^{-\boldsymbol{\Gamma}s}\boldsymbol{\Sigma}e^{-\boldsymbol{\Gamma}'s}ds\right)e^{-\boldsymbol{\Gamma}'\tau}, \end{aligned}$$

where

$$\boldsymbol{\Sigma} \equiv \mathbf{V}\mathbf{V}' = \mathbf{A}^{-1}\mathbf{S}\mathbf{S}\mathbf{A}'^{-1}. \tag{68}$$

In particular, the covariance reads

$$\boldsymbol{\Sigma}(t) \equiv \mathrm{Cov}\{\mathbf{Z}_t | \mathbf{Z}_0\} = \int_0^t e^{-\boldsymbol{\Gamma} s} \boldsymbol{\Sigma} e^{-\boldsymbol{\Gamma}' s} ds. \tag{69}$$

To simplify the notation, we introduce three auxiliary functions:

$$\begin{aligned} E_\gamma(t) &\equiv \frac{1}{\gamma} - \frac{e^{-\gamma t}}{\gamma} && (70) \\ D_{\gamma,\omega}(t) &\equiv \frac{\omega}{\gamma^2+\omega^2} - \frac{e^{-\gamma t}(\omega\cos\omega t + \gamma\sin\omega t)}{\gamma^2+\omega^2} && (71) \\ C_{\gamma,\omega}(t) &\equiv \frac{\gamma}{\gamma^2+\omega^2} - \frac{e^{-\gamma t}(\gamma\cos\omega t - \omega\sin\omega t)}{\gamma^2+\omega^2}. && (72) \end{aligned}$$

Using (70)-(70), the conditional covariances among the entries $k = 1, \ldots K$ relative to the real eigenvalues read:

$$\begin{aligned} \Sigma_{k,\widetilde{k}}(t) &= \int_0^t e^{-\lambda_k} \Sigma_{k,\widetilde{k}} e^{-\lambda_{\widetilde{k}} s} ds && (73) \\ &= \Sigma_{k,\widetilde{k}} E\left(t; \lambda_k + \lambda_{\widetilde{k}}\right). \end{aligned}$$

Similarly, the conditional covariances (69) of the pairs of entries $j = 1, \ldots J$ relative to the complex eigenvalues with the entries $k = 1, \ldots K$ relative to the real eigenvalues read:

$$\begin{aligned} \begin{pmatrix} \Sigma^{(1)}_{j,k}(t) \\ \Sigma^{(2)}_{j,k}(t) \end{pmatrix} &= \int_0^t e^{-\boldsymbol{\Gamma}_j s} \begin{pmatrix} \Sigma^{(1)}_{j,k} \\ \Sigma^{(1)}_{j,k} \end{pmatrix} e^{-\lambda_k s} ds && (74) \\ &= \int_0^t e^{-(\gamma_j+\lambda_k)s} \begin{pmatrix} \Sigma^{(1)}_{j,k}\cos(\omega_j s) - \Sigma^{(2)}_{j,k}\sin(\omega_j s) \\ \Sigma^{(1)}_{j,k}\sin(\omega_j s) + \Sigma^{(2)}_{j,k}\cos(\omega_j s) \end{pmatrix} ds, \end{aligned}$$

which implies

$$\begin{aligned} \Sigma^{(1)}_{j,k}(t) &\equiv \Sigma^{(1)}_{j,k} \int_0^t e^{-(\gamma_j+\lambda_k)s} \cos(\omega_j s)\, ds && (75) \\ &\quad - \Sigma^{(2)}_{j,k} \int_0^t e^{-(\gamma_j+\lambda_k)s} \sin(\omega_j s)\, ds \\ &= \Sigma^{(1)}_{j,k} C\left(t; \gamma_j + \lambda_k, \omega_j\right) - \Sigma^{(2)}_{j,k} S\left(t; \gamma_j + \lambda_k, \omega_j\right) \end{aligned}$$

and

$$\begin{aligned} \Sigma^{(2)}_{j,k}(t) &\equiv \Sigma^{(1)}_{j,k} \int_0^t e^{-(\gamma_j+\lambda_k)s} \sin(\omega_j s)\, ds && (76) \\ &\quad + \Sigma^{(2)}_{j,k} \int_0^t e^{-(\gamma_j+\lambda_k)s} \cos(\omega_j s)\, ds \\ &= \Sigma^{(1)}_{j,k} S\left(t; \gamma_j + \lambda_k, \omega_j\right) + \Sigma^{(2)}_{j,k} C\left(t; \gamma_j + \lambda_k, \omega_j\right). \end{aligned}$$

Finally, using again (70)-(70), the conditional covariances (69) among the pairs of entries $j = 1, \ldots J$ relative to the complex eigenvalues read:

$$\begin{aligned}
\begin{pmatrix} \Sigma^{(1,1)}_{j,\widetilde{j}}(t) & \Sigma^{(1,2)}_{j,\widetilde{j}}(t) \\ \Sigma^{(2,1)}_{j,\widetilde{j}}(t) & \Sigma^{(2,2)}_{j,\widetilde{j}}(t) \end{pmatrix} &= \int_0^t e^{-\boldsymbol{\Gamma}_j s} \begin{pmatrix} \Sigma^{(1,1)}_{j,\widetilde{j}} & \Sigma^{(1,2)}_{j,\widetilde{j}} \\ \Sigma^{(2,1)}_{j,\widetilde{j}} & \Sigma^{(2,2)}_{j,\widetilde{j}} \end{pmatrix} e^{-\boldsymbol{\Gamma}'_{\widetilde{j}} s} ds \qquad (77) \\
&= \int_0^t e^{-\left(\gamma_j + \gamma_{\widetilde{j}}\right)s} \begin{pmatrix} \cos\omega_j s & -\sin\omega_j s \\ \sin\omega_j s & \cos\omega_j s \end{pmatrix} \\
&\quad \begin{pmatrix} \Sigma^{(1,1)}_{j,\widetilde{j}} & \Sigma^{(1,2)}_{j,\widetilde{j}} \\ \Sigma^{(2,1)}_{j,\widetilde{j}} & \Sigma^{(2,2)}_{j,\widetilde{j}} \end{pmatrix} \begin{pmatrix} \cos\omega_{\widetilde{j}} s & \sin\omega_{\widetilde{j}} s \\ -\sin\omega_{\widetilde{j}} s & \cos\omega_{\widetilde{j}} s \end{pmatrix} ds
\end{aligned}$$

Using the identity

$$\begin{pmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{c} & \widetilde{d} \end{pmatrix} \equiv \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad (78)$$

where

$$\begin{aligned}
\widetilde{a} &\equiv \frac{1}{2}(a+d)\cos(\alpha-\omega) + \frac{1}{2}(c-b)\sin(\alpha-\omega) \qquad (79) \\
&\quad + \frac{1}{2}(a-d)\cos(\alpha+\omega) - \frac{1}{2}(b+c)\sin(\alpha+\omega) \\
\widetilde{b} &\equiv \frac{1}{2}(b-c)\cos(\alpha-\omega) + \frac{1}{2}(a+d)\sin(\alpha-\omega) \qquad (80) \\
&\quad + \frac{1}{2}(b+c)\cos(\alpha+\omega) + \frac{1}{2}(a-d)\sin(\alpha+\omega) \\
\widetilde{c} &\equiv \frac{1}{2}(c-b)\cos(\alpha-\omega) - \frac{1}{2}(a+d)\sin(\alpha-\omega) \qquad (81) \\
&\quad + \frac{1}{2}(b+c)\cos(\alpha+\omega) + \frac{1}{2}(a-d)\sin(\alpha+\omega) \\
\widetilde{d} &\equiv \frac{1}{2}(a+d)\cos(\alpha-\omega) + \frac{1}{2}(c-b)\sin(\alpha-\omega) \qquad (82) \\
&\quad + \frac{1}{2}(d-a)\cos(\alpha+\omega) + \frac{1}{2}(b+c)\sin(\alpha+\omega)
\end{aligned}$$

we can simplify the matrix product in (77). Then, the conditional covariances among the pairs of entries $j = 1, \ldots J$ relative to the complex eigenvalues read

$$\begin{aligned}
S^{(1,1)}_{j,\widetilde{j};t} &= \frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} + S^{(2,2)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}(t) \qquad (83) \\
&\quad + \frac{1}{2}\left(S^{(2,1)}_{j,\widetilde{j}} - S^{(1,2)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}(t) \\
&\quad + \frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} - S^{(2,2)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}(t) \\
&\quad - \frac{1}{2}\left(S^{(1,2)}_{j,\widetilde{j}} + S^{(2,1)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}(t);
\end{aligned}$$

$$
\begin{aligned}
S^{(1,2)}_{j,\widetilde{j};t} \equiv\; & \frac{1}{2}\left(S^{(1,2)}_{j,\widetilde{j}} - S^{(2,1)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} + S^{(2,2)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(1,2)}_{j,\widetilde{j}} + S^{(2,1)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} - S^{(2,2)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}\left(t\right);
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
S^{(2,1)}_{j,\widetilde{j};t} \equiv\; & \frac{1}{2}\left(S^{(2,1)}_{j,\widetilde{j}} - S^{(1,2)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}\left(t\right) \\
& -\frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} + S^{(2,2)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(1,2)}_{j,\widetilde{j}} + S^{(2,1)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} - S^{(2,2)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}\left(t\right);
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
S^{(2,2)}_{j,\widetilde{j};t} \equiv\; & \frac{1}{2}\left(S^{(1,1)}_{j,\widetilde{j}} + S^{(2,2)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(2,1)}_{j,\widetilde{j}} - S^{(1,2)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}-\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(2,2)}_{j,\widetilde{j}} - S^{(1,1)}_{j,\widetilde{j}}\right) C_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}\left(t\right) \\
& +\frac{1}{2}\left(S^{(1,2)}_{j,\widetilde{j}} + S^{(2,1)}_{j,\widetilde{j}}\right) D_{\gamma_j+\gamma_{\widetilde{j}},\omega_{\widetilde{j}}+\omega_j}\left(t\right).
\end{aligned}
\tag{86}
$$

Similar steps lead to the expression of the autocovariance.