STA 250: Statistics
Lab 3

In this lab, we will compare ML test against a median based test for the normal model.

The setting

Consider the NSW study where difference in earnings $X_1, \cdots, X_n$ are recorded for $n = 185$ individuals enrolled in a job training program. We model the data as $X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$, $\mu \in (-\infty, \infty)$, $\sigma \in (0, \infty)$, both unknown. We want to test $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

We saw in Tuesday's lecture that the MLE for μ and σ^2 are $\hat{\mu}_{\text{MLE}}(x) = \bar{x}$ and $\hat{\sigma}^2_{\text{MLE}}(x) = \frac{n-1}{n} s_x^2$. In fact for any $\alpha \in (0, 1)$, the size α ML test simplifies to our familiar test:

$$\text{“reject } H_0 \text{ if } \frac{\sqrt{n}|\bar{x}|}{s_x} > z_{n-1}(\alpha)\text{”.}$$

On the other hand, a size α median based test is given by

$$\text{“reject } H_0 \text{ if } \frac{\sqrt{2n}|x_{\text{med}}|}{s_x\sqrt{\pi}} > z(\alpha)\text{”.}$$

[In HW2 #5, we used the test statistic $\sqrt{2n}|x_{\text{med}}|/(\sigma\sqrt{\pi})$ when σ was known. The above statistic retains that form, only substitutes s_x for σ.]

With $\alpha = 5\%$, we will look at the size 5% ML test and the size 5% median test as given above, and first confirm that they indeed have size 5% and then compare their power across chosen values of (μ, σ) from the alternative. From Neyman-Pearson lemma, as well as our calculations in HW2 #5, we will expect the ML based test to be better.

Test rules

The following function takes as input the data vector `x`, number of data points `n` and the desired size `alpha`, and performs the size `alpha` ML test on the data. A `TRUE/FALSE` output is provided to indicate whether to reject H_0 (`TRUE`) or not (`FALSE`).

```
test.ML <- function(x, n = 185, alpha = 0.05){
  cut.off <- qt(1 - alpha / 2, df = n - 1)
  test.stat <- sqrt(n) * abs(mean(x)) / sd(x)
  return(test.stat > cut.off)
}
```

Task 1. Write an analogous function `test.Med()` that will perform the median test.

Experiments to determine size

The following piece of code has two functions. The first function `expt.ml()` encodes a random experiment with our ML test at a given value of (μ, σ): we simulate $n = 185$ random numbers from the corresponding $\mathsf{Normal}(\mu, \sigma^2)$ and apply our ML test to take a decision on H_0 (for a given α). The second function `power()` repeats this experiment 5000 times to approximate the power of the test at the specified (μ, σ) pair. It returns the power value rounded up to two decimal places.

```
expt.ML <- function(mu, sigma, n = 185, alpha = 0.05){
  x <- rnorm(n, mu, sigma)
  return(test.ML(x, n, alpha))
}

power.ML <- function(mu, sigma, n = 185, alpha = 0.05){
  power <- mean(replicate(5e3, expt.ML(mu, sigma, n, alpha)))
  return(round(power, 2))
}
```

To calculate the size of the ML test, we will need to calculate its power at all (μ, σ) matching H_0, i.e., we must pick $\mu = 0$ but consider all $\sigma > 0$. The code below performs this for $\sigma = 1, 2, 4$.

```
> sigma.list <- c(1, 2, 4)
> pow.ML.null <- rep(NA, 3)
> for(i in 1:3){
+   pow.ML.null[i] <- power.ML(mu = 0, sigma = sigma.list[i], 185, 0.05)
+ }
> print(power.ML.null)
[1] 0.05 0.05 0.05
```

TASK 2. Repeat the above for the median based test with $\alpha = 0.05$. Would you be reasonably confident in claiming the size of this test is 5%?

Power comparison

To compare the two tests on their power at the alternative (μ, σ) values, we consider the following candidate μ values: $-0.75, -0.25, 0.25, 0.75$. These μ values, combined with the three σ values above, give 12 pairs of (μ, σ). The following piece calculates and prints the power of the ML test at these 12 pairs:

```
> mu.list <- c(-0.75, -0.25, 0.25, 0.75)
> sigma.list <- c(1, 2, 4)
> pow.ML.alt <- matrix(NA, 4, 3)
> for(j in 1:3){
+   for(i in 1:4){
+     pow.ML.alt[i, j] <- power.ML(mu = mu.list[i], sigma = sigma.list[j])
+   }
> }
> print(pow.ML.alt)
     [,1] [,2] [,3]
[1,] 1.00 1.00 0.72
[2,] 0.92 0.39 0.13
[3,] 0.92 0.39 0.14
[4,] 1.00 1.00 0.72
```

TASK 3. In the matrix printed above, the second row, third column element represents the power of the ML test at which (μ, σ) pair?

TASK 4. Repeat the above power calculations for the median test. Which test has better power at the alternative?