Supplementary material for 'Gaussian process modulated renewal processes'

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We first prove equation (4) of the main text for a general nonstationary hazard function $h(\tau, t)$.

Proposition S.1 For a renewal process with nonstationary hazard function $h(\tau, t)$, the waiting time τ given that the last event occured at time t_{prev} is given by

$$g(\tau|t_{prev}) = h(\tau, t_{prev} + \tau) \exp\left(-\int_0^\tau h(u, t_{prev} + u) du\right) \tag{1}$$

Proof. By definition (see equation (2) in the main text),

$$h(\tau, t_{prev} + \tau) = \frac{g(\tau|t_{prev})}{1 - \int_0^\tau g(u|t_{prev}) du} \tag{2}$$

Let $y = 1 - \int_0^\tau g(u|t_{prev}) du$. It follows that

$$h(\tau,t_{prev} + \tau) = \frac{-dy/d\tau}{y}, \text{ so that} \tag{3}$$

$$y = \exp\left(-\int_0^\tau h(u, t_{prev} + u) du\right) \tag{4}$$

Substituting back for y and differentiating w.r.t. τ, we get equation (1). □

We now prove proposition 2 from the main text.

Proposition 2 For any $\Omega \geq \max_{t,\tau} h(\tau)\lambda(t)$, F is a sample from a modulated renewal process with hazard $h(\cdot)$ and modulating intensity $\lambda(\cdot)$.

Proof. We need to show that $F_i - F_{i-1} \sim g$.
Denote by E_i^* the restriction of E to the interval (F_{i-1}, F_i), not including boundaries. Note that

$$P(F_i, E_i^*|F_{i-1}) = \left(\prod_{e \in E_i^*} 1 - \frac{\lambda(e)h(e - F_{i-1})}{\Omega}\right) \frac{\lambda(F_i)h(F_i - F_{i-1})}{\Omega} \tag{5}$$

Defining $n = |E_i^*|$ and $t_0 = F_{i-1}$, we have

$$P(F_i, n|F_{i-1}) = \frac{\lambda(F_i)h(F_i - F_{i-1})}{\Omega}$$

$$\int_{F_{i-1}}^{F_i} \int_{t_1}^{F_i} ... \int_{t_{n-1}}^{F_i} dt_1 dt_2 ... dt_n \left(\prod_{j=1}^{n} \Omega \exp -\Omega(t_j - t_{j-1}) \right) \left(\prod_{j=1}^{n} 1 - \frac{\lambda(t_j)h(t_j - F_{i-1})}{\Omega} \right) (\Omega \exp -(\Omega(F_i - t_n)))$$

$$= \lambda(F_i)h(F_i - F_{i-1}) \exp\left(-\Omega(F_i - F_{i-1})\right) \int_{F_{i-1}}^{F_i} \int_{t_1}^{F_i} ... \int_{t_n}^{F_i} dt_1 dt_2 ... dt_n \left(\prod_{j=1}^{n} \left(\Omega - \lambda(t_j)h(t_j - F_{i-1})\right) \right) \tag{6}$$

$$= \lambda(F_i)h(F_i - F_{i-1}) \exp\left(-\Omega(F_i - F_{i-1})\right) \frac{1}{n!} \left(\int_{F_{i-1}}^{F_i} dt \left(\Omega - \lambda(t)h(t - F_{i-1})\right) \right)^n \tag{7}$$

Marginalizing out n, we then have

$$\begin{aligned} P(F_i|F_{i-1}) &= \lambda(t)h(F_i - F_{i-1}) \exp\left(-\Omega(F_i - F_{i-1})\right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{F_{i-1}}^{F_i} dt \left(\Omega - \lambda(t)h(t - F_{i-1})\right) \right)^n \right) \\ &= \lambda(F_i)h(F_i - F_{i-1}) \exp\left(- \int_{F_{i-1}}^{F_i} \lambda(t)h(t - F_{i-1}) dt \right) \end{aligned} \tag{8}$$

Comparing equation (4) of the main text, we have the desired result.

□