Deterministic chaos can be described by deterministic functions, such as the logistic map \( f \) or tent map \( T \) as follows (the support and range of these mappings are all \([0, 1]\))

\[
f(x) = 4x(1-x) \times 1(0 \leq x \leq 1);
T(x) = 2x \times 1(0 \leq x < .5) + (1 - 2x) \times 1(.5 \leq x \leq 1).
\]

Starting from any point between 0 and 1, the trace of the function would still lie in \([0, 1]\), which is an absorbing set. Note also each point inside \([0, 1]\) will be visited arbitrarily closely and infinitely often by any ‘typical’ solution, hence it is also an attractor.

A stationary measure \( \phi \) associated with the evolution equation \( f \) is invariant under \( f \), i.e.

\[X \sim \phi(X), \quad Y = f(X), \quad \Rightarrow \quad Y \sim \phi(Y).\]

And if \( \phi \) is stationary and noise-tolerant, then it is the natural measure. In our example, \( 1/\pi \sqrt{x(1-x)} \) and Uniform\([0,1]\) are the natural measures of the logistic map and the tent map respectively. The ergodicity means that no matter what initial distribution \( X \) has (as long as it satisfies certain loose conditions), the distribution of \( f^n(X) \) converges quickly to its natural measure, which could also be viewed as the collection of \( f^m(X_0) \) for a fixed initial value \( X_0 \) and many \( m \)s after some burn in period of \( n_0 \).

The Lyapunov exponent is defined by the average amplifier factor on the log scale, i.e.

\[
\lambda = \int \log |f'(x)| \phi(x) \, dx \approx \frac{1}{m} \sum_{i=0}^{m-1} \log |f'(x_i)|.
\]

This basically means the amplification rate is \( \exp \lambda m \) after \( m \) iterations. In our case, both logistic and tent maps have Lyapunov exponent of \( \log 2 \). A dynamic system is called chaos if and only if it admits a positive Lyapunov exponent.

The correlation dimension is based on the order of the difference of two independent random variables. Suppose \( X \) and \( X' \) are independent identically distributed with respect to \( \phi \), then the correlation dimension is \( \rho \) if

\[
C(r) = P(|X - X'|_\infty \leq r) \sim r^\rho \quad \text{(reads... of order } r^\rho).\]

In our case, both logistic and tent maps have correlation dimension of 1. In fact, denote \( h(x) = \frac{x}{\pi} \arcsin(\sqrt{x}) \), then \( T = h \circ f \circ h \), and we know the logistic map and the tent map is in the same equivalent class, hence the intrinsic measures are the same. This is called invariance of equivalent dynamic systems.

Last not the least, if we don’t measure the dynamic system directly. Say we measure \( y = g(x) \), and \( x \) follows a deterministic structure of \( x, f(x), f^2(x), \ldots \), \( g \) is called the read-out function. If \( g \) is not one to one, then \( y_t \) may not sufficient in inference on \( y_{t+1} \). However, Taken’s theory shows that, for a generic dynamic system and a generic read-out function, the delay coordinates \( z_t = (y_t, y_{t+1}, \ldots, y_{t+q-1}) \) unfolds the attractor if \( q \geq 2d + 1 \), where \( d \) is the (usual) dimension of the attractor. The minimal number of \( q \) needed for the decay coordinate to unfold an attractor is called the embedding dimension, which is 2 in our example (not 2+1=3).