The standard fully-skewed stable process $X_t \sim \text{St}(\alpha, 1, t, 0)$ of index $\alpha \neq 1$ has log ch.f. and Lévy measure

$$
\log \mathbb{E}[e^{itX_t}] = -t|\omega|^\alpha - i\, t\tan \frac{\pi\alpha}{2} (\omega - |\omega|^\alpha \text{sgn } \omega)
$$

$$
\nu(du) = \frac{2\alpha}{\pi} \Gamma(\alpha) \sin \frac{\pi\alpha}{2} u^{-1-\alpha} 1_{u>0} du
$$

and so the probability of no jump larger than $u > 0$ on the interval $t \in (0, T]$ is exactly

$$
\Pr \left[ \sup_{0 \leq t \leq T} (X_t - X_{t-}) \leq u \right] = 1 - e^{-T\nu(u, \infty)} = 1 - e^{-(2T/\pi)\Gamma(\alpha) \sin \frac{\pi\alpha}{2}} u^{-\alpha}, \quad u > 0.
$$

Thus the $(-\alpha)^{th}$ power $U_1^{-\alpha}$ of the largest jump $U_1$ has an exponential distribution with rate parameter

$$
\lambda \equiv (2T/\pi)\Gamma(\alpha) \sin \frac{\pi\alpha}{2}.
$$

A limiting argument shows the same result holds for the skewed Cauchy case $\alpha = 1$, with $\lambda = 2T/\pi$. More generally, the $J$ largest jumps $\{U_j\}$ on the interval $(0, T]$ may be written in the form

$$
U_j = (\tau_j/\lambda)^{-1/\alpha}
$$

for jumps $\{0 < \tau_1 < \tau_2 < \ldots < \tau_J\}$ of a standard unit-rate Poisson process (this is the Inverse Lévy Measure construction of Wolpert and Ickstadt, 1998). In particular, the $p^{th}$ moment of the $j^{th}$-largest jump is

$$
\mathbb{E}[U_j] = \int_0^\infty (\tau/\lambda)^{-p/\alpha} \frac{\tau^{j-1} e^{-\tau}}{\Gamma(j)} d\tau = \frac{\chi^{p/\alpha}}{\Gamma(j)} \frac{\Gamma(j - p/\alpha)}{\Gamma(j)},
$$
finite for $j > p/\alpha$; in the Cauchy case with $p = 1$, the $J$th-largest jump $U_J = 2T/(\pi \tau_J)$ has finite mean $E[U_J] = \frac{2T}{\pi (J-1)}$ for $J \geq 2$.

**Compensation**

This suggests one possible approach to compensation, particularly for the case of $1 < \alpha < 2$ where compensation is required to attain convergence as $J \to \infty$. Earlier we saw that the approximate $\alpha$-stable construction that includes only jumps of size $|U_j| \geq \epsilon$ for some $\epsilon > 0$ requires a “drift” term $X_t = \delta\epsilon t + \sum_{|U_j| \geq \epsilon, \sigma_j \leq t} U_j$ (here $\{\sigma_j\} \overset{iid}{\sim} Un(0, T)$) with drift rate

$$
\delta\epsilon = \begin{cases} 
\frac{2\Gamma(\alpha)\sin \frac{\pi\alpha}{2}}{\pi (1-\alpha)} \left[ \alpha \epsilon^{1-\alpha} - \Gamma(2-\alpha) \sin \frac{\pi \alpha}{2} \right] & \alpha \neq 1 \\
\frac{2}{\pi} \left[ \log \epsilon + \gamma - 1 \right] & \alpha = 1
\end{cases}
$$

For the fixed-$J$ approach we may try the same thing, using $E[U_J]$ in place of $\epsilon$; in the Cauchy case, this would be

$$
X_t^J = \delta_J t + \sum_{j \leq J, \sigma_j \leq t} U_j, \quad \text{where}
$$

$$
\delta_J \equiv \frac{2}{\pi} \left[ \log \frac{2T}{\pi (J-1)} + \gamma - 1 \right] \quad \text{and} \quad U_j \equiv \frac{2T}{\pi \tau_j}
$$

with (again) $\{\tau_j\}$ the event times of a standard Poisson process and $\{\sigma_j\} \overset{iid}{\sim} Un(0, T)$. Let’s show that this converges as $J \to \infty$; of course it’s enough (by scaling and independence of increments) to consider $t = T = 1$ and show that

$$
\sum_{j=1}^{J} \frac{1}{\tau_j} - \log J
$$

converges (an interesting result in its own right). By adding and subtracting the harmonic series it’s enough to show that the mean-zero random variables $\frac{1}{\tau_j} - \frac{1}{j-1}$ are summable. These have variance and covariances that are easy to compute from the relation

$$
E \left[ \frac{1}{\tau_i \tau_j} \right] = \frac{1}{(i-1)(j-2)}, \quad 1 < i \leq j < \infty
$$
(obtained by writing \( \tau_i = Z \tau_j \) for \( Z \sim \text{Be}(i,j) \) independent of \( \tau_j \), whence \( \frac{1}{\tau_i \tau_j} = Z^{-1} \tau_j^{-2} \) so the expectation is routine),

\[
\text{Cov} \left[ \frac{1}{\tau_i}, \frac{1}{\tau_j} \right] = \frac{1}{(i-1)(j-1)(j-2)}, \quad 1 < i \leq j < \infty
\]

and hence

\[
\mathbb{E} \left[ \left( \sum_{j=3}^{J} \left( \frac{1}{\tau_j} - \frac{1}{j-1} \right) \right)^2 \right] = \sum_{i=3}^{J} \frac{1}{(i-1)^2(i-2)} + \sum_{3 \leq i < j \leq J} \frac{2}{(i-1)(j-1)(j-2)}
\]

\[
= \sum_{i=3}^{J} \left\{ \frac{1}{(i-1)^2(i-2)} + \frac{2(J-i)}{(i-1)^2(J-1)} \right\}
\]

\[
\leq \sum_{i=3}^{\infty} \frac{3}{(i-1)^2} = \frac{\pi^2 - 6}{2} < \infty,
\]

so the series converges in \( L_2 \) and hence in distribution and, along some subsequence, almost-surely as \( J \to \infty \). The same approach may work for \( \alpha \neq 1 \), too.

**Another Route (probably not useful)**

Another entirely different approach would be to construct a sequence \( \{ \Delta_j \} \) with the property that \( M_J \equiv \left[ \sum_{j=2}^{J} U_j + \Delta_j \right] \) is a discrete martingale in \( J \) (the term \( j = 1 \) was omitted to make \( M_J \) integrable; this doesn’t affect the convergence of the sequence). The conditional expectation of \( U_{j+1} \) given the history up to time \( j \) is easily seen to be

\[
\mathbb{E}[U_{j+1} | \mathcal{F}_j] = \int_{0}^{\infty} \left( \frac{U_j + \tau}{\lambda} \right)^{-1/\alpha} e^{-\tau} d\tau
\]

\[
= \lambda^{1/\alpha} e^{U_j} \Gamma(1 - \alpha^{-1}, U_j)
\]

for \( \alpha \neq 1 \) or, for \( \alpha = 1 \),

\[
= \frac{2T}{\pi} e^{U_j} E_1(U_j)
\]

(here \( \Gamma(\alpha, z) \) and \( E_1(z) \) are the incomplete Gamma function and exponential integral function, respectively; see Abramowitz and Stegun, 1964, §6.5.3 and
§5.1.1). It follows that

\[ M_J = \sum_{j=2}^{J} U_j + \Delta_J T \]

is a discrete martingale in \( J \) if we set

\[ \Delta_J = -\frac{2}{\pi} \sum_{j=1}^{J-1} e^{U_j} E_1(U_j). \]

Since \( \frac{1}{2} \log(1 + \frac{4}{z}) < e^z E_1(z) < \frac{1}{2} \) for \( z > 0 \) (op. cit., §5.1.19–20), this is approximately \( \Delta_J \approx c - (2/\pi) \log J \), similar to \( \delta_J \) before. Now we can use the martingale convergence theorem to prove a.s. convergence as \( J \to \infty \) of the sequence Eqn. (1) defining \( X^J_t \) if we can show that the expectations

\[ E \left[ U_j - (2/\pi) e^{U_j} E_1(U_j) \right] \]

are absolutely summable. I’m not sure how to do that.

References
