1 Random Measure and Point Process

Let $E$ be a locally compact Hausdorff space whose topology has a countable base (abbr. LCCB), with Borel $\sigma$-algebra $\mathcal{E}$. Let $\mathcal{B}$ be the family of all bounded subsets of $E$.

Next, define functions on $E$, hence construct following functional spaces, $C$:
- $C$, bounded continuous functions on $E$.
- $C_0$, bounded, continuous functions vanishing at infinity, i.e., $f \in C_0$, if and only if $f$ is continuous, and for every $\epsilon > 0$, there is a compact set $K \subseteq E$, such that $|f(x)| < \epsilon$ for all $x \notin K$.
- $C_K$, continuous functions on $E$, with compact support.

Then, define Radon measure (NOT random measure) and the collections of Radon measures.

Firstly, a measure is a function from $\mathcal{E}$ to $[0, \infty]$ which satisfies zero empty set and countable additivity.

**Definition** A Radon measure $\mu$ on a Hausdorff topological space $E$ is defined to be a measure on the Borel $\sigma$-algebra $\mathcal{E}$ that is locally finite and regular. By locally finite, we mean every point has a neighborhood of finite measure. By regular, we mean for every $A \in \mathcal{E}$,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ is compact}\} = \inf\{\mu(G) : A \subseteq G, G \text{ is open}\}.$$

- $M$, the set of Radon measures on $E$.
- $M_p$, the set of point measures on $E$, i.e., $\mu \in M_p$ if and only if $\mu(A) \in \mathbb{N}$ for every $A \in \mathcal{B}$.
- $M_s$, the set of simple point measures on $E$, i.e., $\mu \in M_s$ if and only if $\mu \in M_p$ and $\mu(\{x\}) \leq 1$ for every $x \in E$.
- $M_a$, the set of purely atomic measures on $E$, i.e., $\mu \in M_a$ if and only if the diffuse component is zero.
- $M_d$, the set of diffuse measures on $E$, i.e., $\mu \in M_d$ if and only if $\mu(\{x\}) = 0$ for every $x \in E$.

On $M$ we define $\sigma$-algebra

$$\mathcal{M} = \sigma\{I_\mu : \mu \mapsto I_\mu(f) : f \in C_K\}, \quad \text{where } I_\mu(f) := \int f(x) \mu(dx).$$

In fact, $\mathcal{M} = \sigma\{I_\mu : \mu \mapsto \mu(A), A \in \mathcal{B}\}$. Actually, $\mu$ is equivalent as $I_\mu$, hence $\mathcal{M}$ can be viewed as a $\sigma$-algebra of $M$. Similarly, define $\mathcal{M}_p$, $\mathcal{M}_s$, $\mathcal{M}_a$, $\mathcal{M}_d$ as $\sigma$-algebras on $M_p$, $M_s$, $M_a$, $M_d$ respectively.

Use the duality of weak topology and the Rieze representation theorem, for any positive linear functional $L$ on $C_K$, there is a unique element $\mu$ in $M$ such that $L(f) = I_\mu(f)$ for all $f$.

*This is a note from book *Point Process and Their Statistical Inference*, by Alan F. Karr.
Definition Let \((\Omega, \mathcal{F}, P)\) be a probability space.

(a) A random measure \(M\) on \(E\) is a measurable mapping from \((\Omega, \mathcal{F})\) to \((M, M)\).

(b) A point process \(N\) on \(E\) a measurable mapping from \((\Omega, \mathcal{F})\) to \((M_p, M_p)\).

Denote \(M_f := M(f) := I_M(f) := \int_E f(x)M(dx)\), where \(f\) is either non-negative and \(\mathcal{E}\)-measurable (denoted as in set \(\mathcal{E}^+\)), or in \(C_K\), then \(M_f\) is a random variable on \((\Omega, \mathcal{F}, P)\). To see this, recall a random variable is a measurable function from \((\Omega, \mathcal{F})\) to \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). For \(\omega \in \Omega\), \(M_f(\omega) = \int_E f(x)M(\omega)(dx) \in \mathbb{R}\). The conditions on \(f\) ensures the measurability. Random variable \(M(A) := M(1_A)\) is called the mass of \(A\). Another view of \(M_f\) is as follows: Denote \(I\) as the integral function from \(M \times C_K\) to \(\mathbb{R}\), and \(M_f = I(\cdot, f) \circ M\) is a measurable function(random variable).

A point process \(N\) can be viewed as a random distribution of indistinguishable points in \(E\), i.e., \(N(A)\) is the random variable of number of points in the set \(A \in E\). There exists a representation that

\[
N = \sum_{i=1}^{K} \delta_{X_i}, \quad \text{hence } N_f(\omega) = \sum_{i=1}^{K} f(X_i(\omega)).
\]

where \(X_i\) are measurable mappings from \((\Omega, \mathcal{F})\) to \((E, \mathcal{E})\), termed the atoms of \(N\), and \(K\) is a random variable with values in \(\mathbb{N} = \{0, 1, 2, \ldots, \infty\}\). \(X_i\) need not to be distince, hence the order is not unique.

Definition Let \(\mu\) be an element in \(M\), and a point process \(N\) on \(E\) is called a Poisson process with mean measure \(\mu\) if

(a) Random variables \(N(A_i)\) are independent if \(A_i \in \mathcal{E}\) are disjoint.

(b) For each \(A \in \mathcal{E}\), \(N(A)\) has a Possion distribution with mean \(\mu(A)\).

Definition Let \(M\) be a random measure on \(E\), and a point process \(N\) on \(E\) is called a Cox Process if conditional on \(M\), \(N\) is a Poisson process with mean measure \(M\). We refer \((N, M)\) as a Cox pair, and \(M\) as the directing measure of \(N\).

Definition Given i.i.d. random elements \(X_i\) of \(E\), fixed integer \(n\), the point process \(N = \sum_{i=1}^{n} \epsilon_{X_i}\) is an \((n\text{-sample})\) empirical process on \(E\).

Definition Let i.i.d. random elements \(X_i\) of \(E\), and let \(K\) be a non-negative, integer-valued random variable independent of \(X_i\), then the point process \(N = \sum_{i=1}^{K} \epsilon_{X_i}\) is a mixed empirical process on \(E\).

Definition A point process \(\sum_{n=1}^{\infty} \delta_{T_n}\) on \(\mathbb{R}_+\) is a renewal process with interarrival distribution \(F\)(a probability on \(\mathbb{R}_+\) if \(T_0 = 0\) and if the interarrival times \(U_i\) are i.i.d. with distribution \(F\).

Definition A point process \(N = \sum \delta_{X_i}\) on \(\mathbb{R}^d\) is stationary, if for all \(x \in E\), the translated point process \(N(x) = \sum \delta_{X_i-x}\) has the same distribution as \(N\).

2 Distribution Descriptors