A LÉVY GENERALIZATION OF COMPOUND POISSON PROCESSES IN FINANCE: THEORY AND APPLICATIONS

by

Enrique ter Horst

Institute of Statistics and Decision Sciences
Duke University

Date: ______________________________

Approved:

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Robert L. Wolpert, Supervisor

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James O. Berger

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Mark L. Huber

____
Feng Liang

Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Institute of Statistics and Decision Sciences
in the Graduate School of
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ABSTRACT

(Statistics)

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Abstract

Since Black and Scholes (1973), Mathematical Finance has grown as a branch of mathematics in its own right. The rejection of the normality of asset return distributions by Mandelbrot (1963) led to consideration of Lévy-stable stochastic processes as an interesting alternative.

Modelling asset returns through a stochastic volatility model, composed of a diffusion and general Lévy pure jump process, is described in Chapter 2. The pricing of European options is also considered in Chapter 2, leading to a derivation of conditions that allow us to the approach of Duffie *et al.* (2000), to transform a measure to a risk-neutral one. The equivalence between risk-neutrality and no-arbitrage is then guaranteed by Delbaen and Schachermayer (1994).

In Chapter 3 we perform a Bayesian analysis of a stochastic volatility model of Barndorff-Nielsen and Shephard (2001), where we treat Lévy jump times and sizes as uncertain and are interested in their posterior distributions. We also find the posterior distribution of the parameters governing the law of the Lévy subordinator (driving the stochastic volatility model of O-U type). The computations are done through the Reversible Jump Markov Chain Monte Carlo approach of Green (1995), since we deal with a Lévy pure jump process that has an uncertain number of large jumps.

Acknowledgements

The only difference between Paul Lévy and Robert L. Wolpert is that Paul Lévy never had a student. I would like to thank and express my life-long gratitute to Prof. Robert L. Wolpert, not only for working with me in this enterprise, and spending his time generously in my education as a future researcher, but also setting an example for me to follow in the future. Without his motivation and guidance, this work would not have been possible.

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I wish to dedicate this work to my father, mother and sister; especially to my mother, who taught me the joys of studying.

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Chapter 1

A Primer in Mathematical Finance

Our problem is to determine the price of an option. An option gives you the right, but not the obligation to buy an underlying asset (e.g. a stock, index, interest rate, etc.) at a future date T for a specified price K, called the strike price. The underlying asset that we use in this chapter is a stock, but it should be noted that the analysis also holds for other underlying assets. This chapter sets forth the mathematical tools used to price assets of various kinds.

1.1 Basic Theory

1.1.1 Introduction

In this section, we review stochastic integrals, where the integrators are semimartingales and the integrands are predictable processes. We start with a definition of semimartingales and an Itô formula for functions of semimartingales.

Definition 1.1 A *semimartingale* (see Shiryaev, 1999) is a stochastic process $(X_t)_{t \geqq 0}$ that can be written:

$$X_t = X_0 + A_t + M_t \tag{1.1}$$

as the sum of a constant random variable X_0, a bounded variation process A_t, and a local martingale M_t. This representation is not unique (see Jacod and Shiryaev, 1987, page 43 for more details). Sometimes it is easier to work with the characteristic function of a semimartingale than with its decomposition. For more details see Shiryaev (1999).

1.1.2 Important Theorems and Results

In this section we assume that we have a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $(\mathscr{F}_t)_{t\geq 0}$ that is right-continuous and $\mathbb{P}$-complete. We now present important theorems (without proofs) that we will need in chapter 2. The first is Itô's theorem (see Protter, 1990, page 76 for more details):

Theorem 1.1 Let X be a semimartingale and let f be a C^2 real function. Then $f(X)$ is again a semimartingale, and the following formula holds:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0+}^{t} f'(X_{s-})dX_s + \frac{1}{2}\int_{0+}^{t} f''(X_{s-})d\,[X,X]_s^c \\ &+ \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-}) \triangle X_s\} \end{aligned}$$

Another useful result, Girsanov's Theorem, allows us to change probability measure in such a way that if S is a $\mathbb{P}_0$-semimartingale, it will also be a $\mathbb{P}_1$-semimartingale, where $\mathbb{P}_0$ and $\mathbb{P}_1$ are equivalent measures (Protter, 1990). This result is useful whenever we need to have a certain process (such as a stock price) behave as a martingale under some measure.

1.2 The Economics

1.2.1 Economic Meaning

The main goal is to model returns in such a way that we reproduce stylized facts in finance such as *volatility clustering* and the *leverage effect*. A process exhibits *volatility clustering* whenever there are long periods of time when we observe higher than usual level of volatility. The *leverage effect* is the most common of all stylized financial facts, and occurs whenever falls in the equity price are associated with simultaneous rises in volatility.

Processes with stationary and independent increments (*Lévy processes*) are not the most natural processes with which to model financial returns, because returns exhibit dependence on their past values (as well as fat tails and skewness). This dependence may be introduced in models using Lévy processes through *stochastic volatility models*. Thus a stochastic volatility model enables us to relax the assumption of independence for the increments of the process.

1.2.2 From Geometric Brownian Motion to Semimartingales

Our goal in this work is to model stock returns, using the differential of the logarithm of the stock price. Since prices are positive, it is common practice to model their logarithms as semimartingale. The use of semimartingales allows returns to move continuously, but also to incorporate jumps in price associated with surprises such as hostile takeovers or poor macroeconomic indicators. These jumps allow returns to be more flexible and to exhibit violent adjustments.

It would be nice to remind ourselves about the evolution through history of the several ways that stock prices were and are currently modelled. Bachelier (1900) was the first to model the price of a stock using Brownian motion. One major drawback is that the price can take negative values, and so Black and Scholes (1973) modeled *log* prices using Brownian motion with drift. In this model the log of the stock price process is normally distributed and its increments are independent; alas it has been observed by the financial community that returns display several features:

- heavy tails
- long range dependence (through the time series), and that
- returns are negatively correlated with their stochastic volatilities (*Leverage Effect*)

that are inconsistent with geometric Brownian motion. For an interesting exposition of the above, see Rama (2001).

Currently the models for the stock price are modelled through exponentials of (time changed) Lévy processes of the form:

$$S_t = S_0 \exp\left(X_{\tau_t}\right) \text{ for } 0 \leq t \leq T \tag{1.2}$$

Here X_t is a Lévy process independent from the increasing right-continuous with left limits ("cádlág") process τ_t. This model allows us to have a much wider structure than the previous ones. The time change allows us to incorporate features such as long range time dependence, non-stationarity of increments, and the non-independence of increments.

1.2.3 Completeness, No-arbitrage and Risk Neutrality

The most significant breakthrough of Black and Scholes (1973) was not a closed form solution for the option price, but the concept of *hedging*. This concept earned them the Nobel Prize in Economics. To "hedge" means to reduce risk against market fluctuations. This risk reduction can be achieved in many ways. For example, if you sell short a stock, you can hedge by immediately buying another stock. If you buy a contingent claim such as an option, and you purchase a portfolio that is identical to this option and sell this portfolio short, then you have hedged the option, i.e., you have eliminated the market fluctuations of that option. This assumption supposes that you can replicate your option, but you can do this in general only if the market is *complete*. A market for securities is *complete* if any security can be replicated with a portfolio of other securities. Informally, a market is complete if it has a diverse range of securities.

The general framework for pricing assets and contingent claims is based on the assumption of *no-arbitrage*. One of the key features of asset pricing theory is that any gain can only come in a random way and it is not possible to make money without incurring a risk. It can be shown that the condition of no arbitrage is equivalent to the existence of a risk-neutral probability measure (see Delbaen and Schachermayer, 1994). Under this probability measure, the discounted price of an asset is a martingale. This is what in the literature is referred to as the *First Fundamental Theorem of Asset Pricing* (see Delbaen and Schachermayer, 1994). This theorem asserts the existence of such a probability measure, but not its uniqueness. However, if the market is *complete*, then the risk-neutral probability measure is unique. This last result is what is referred to as the *Second Fundamental Theorem of Asset Pricing* (see Harrison and Pliska (1981)), which asserts that the martingale measure is unique if and only if the market is complete. The only case in the literature where we have uniqueness is when we model the stock price of a stock as a geometric Brownian motion. As soon as we add processes with jumps, the market is not complete, and thus these risk-neutral measures are not unique; in fact, uncountably many can exist.

1.2.4 Esscher Transform and Doléans-Dade Stochastic Exponential

We have two ways of constructing risk-neutral measures. One is through the Doléans-Dade stochastic exponential, which we will discuss next. The second is via the Esscher Transform.

Suppose X_t is a semimartingale and we want to find Z_t so that we can solve:

$$Z_t = 1 + \int_0^t Z_{s-} dX_s \tag{1.3}$$

The Doléans-Dade stochastic exponential is the solution to (1.3), equal to:

$$Z_t = \exp\left(X_t - \frac{1}{2}[X]_t^c\right) \prod_{0<s\leq t} (1 + \triangle X_s) \exp\left(- \triangle X_s + \frac{1}{2}(\triangle X_s)^2\right) \tag{1.4}$$

(see Protter, 1990, page 82 for more details). Now if $X_t = B_t$, where B_t is a standard Brownian motion, then $Z_t = \exp\left(B_t - \frac{t}{2}\right)$ is the regular Geometric Brownian Motion. Thus, we can see from (1.3) & (1.4) that if X_t is a $\mathbb{P}$-martingale or $\mathbb{P}$-local martingale, then Z_t is also a $\mathbb{P}$-local martingale. The process Z_t is a martingale only when X_t is Brownian motion, since integrating with respect to Brownian motion preserves the martingale property, but only integrating with respect to local martingales is stable under integration.

The second way to construct equivalent probability measures is through the Esscher Transform method. Suppose you have a probability measure $\mathbb{P}$, and wish to have an equivalent measure $\mathbb{Q}$. Then define $\mathbb{Q}$ as:

$$\mathbb{Q}^t(d\omega) = \frac{\exp(\lambda X_t)}{\Phi(\lambda)} \mathbb{P}^t(d\omega) \text{ for } \lambda \in \mathbb{R} \tag{1.5}$$

where $\Phi(\lambda)$ is the characteristic function of X_t evaluated at λ, $\mathbb{Q}^t$ and $\mathbb{P}^t$ are the restrictions of $\mathbb{Q}$ and $\mathbb{P}$ to $\mathscr{F}_t$.

As an example, suppose that X_t is a one-dimensional Lévy process with generating triplet (μ, A, ν), then $\mathbb{E}_{\mathbb{P}}\exp(i\omega X_t) = \exp\{t\phi(\omega)\}$ where $\phi(\omega)$ is equal (by the Lévy-Khinchine theorem) to:

$$\phi(\omega) = i\omega\mu - \frac{\omega^2}{2}A + \int_{\mathbb{R}} (\exp(i\omega x) - 1 - i\omega x I(\|x\| \leq 1))\nu(dx) \tag{1.6}$$

Thus relation (1.5) tells us that under the probability measure $\mathbb{P}^t$ we can identify the coefficients of the new Lévy process from the original one from the following equality:

$$\phi^{\mathbb{Q}}(\omega) = \phi^{\mathbb{P}}(\omega - i\lambda) - \phi^{\mathbb{P}}(-i\lambda) \tag{1.7}$$

From this last equality we can then derive the characteristics of the new $\mathbb{Q}$-Lévy process.

We next give a nice example of how the change of measure works. This example will use the representation theorem for continuous martingales which can be represented as a predictable process integrated with respect to Brownian motion.

Suppose M_t is a continuous $\mathbb{P}$-martingale. Then, there is a predictable process H_t so that:

$$M_t = \int_0^t H_s dB_s, \tag{1.8}$$

where B_t is standard Brownian motion. Suppose that $dS_t = S_t(\mu dt + \sigma dB_t)$, and that there is a measure $\mathbb{Q} \sim \mathbb{P}$ so that $S_t \exp(-rt)$ is a $\mathbb{Q}$-martingale. Then there exist a process $Z_t = (\frac{d\mathbb{Q}}{d\mathbb{P}})_{\mathscr{F}_t} > 0$ that is a $\mathbb{P}$-martingale with $\mathbb{E}Z_t = 1$. Z_t is often called in the literature as the *density process* or the *Radon-Nikodym* derivative. Thus Z_t is a continuous martingale and so by the representation theorem we know $dZ_t = H_t dB_t$ or equivalently $dZ_t = Z_t \frac{H_t}{Z_t} dB_t$ or $dZ_t = Z_t \phi_t dB_t$ where we define $\phi_t \equiv \frac{H_t}{Z_t}$. The solution of this equation is the the Doléans stochastic exponential, thus:

$$Z_t = \exp\left(-\frac{1}{2}\int_0^t \phi_s^2 ds + \int_0^t \phi_s dB_s\right) \tag{1.9}$$

Moreover, by Girsanov's theorem, we have $dB_t^{\mathbb{Q}} = dB_t - \phi_t dt$ so then:

$$\begin{aligned} dS_t &= S_t\{(-r+\mu)dt + \sigma(\phi_t dt + dB_t^{\mathbb{Q}})\} \\ &= S_t\{(-r+\mu+\sigma\phi_t)dt + \sigma dB_t^{\mathbb{Q}}\}. \end{aligned} \tag{1.10}$$

If under $\mathbb{Q}$ we want S_t to be a martingale, then we must have $-r+\mu+\sigma\phi_t = 0$ and thus $\phi_t = \frac{r-\mu}{\sigma}$. This is the result we find for regular geometric Brownian motion in the Black & Scholes model Black and Scholes (1973) for options. Here, $\mathbb{Q}$ is unique since in this model the market is complete. In Chapter 2, we introduce jumps, and so things are going to get more complicated, since a unique risk-neutral measure will not exist. We restrict to classes of risk-neutral measures $\mathbb{Q}$ that preserve the original structure of $\mathbb{P}$.

Chapter 2

The Model

In this chapter, we introduce a class of Lévy processes, with both a diffusion part and a pure jump component as well, as a prior distribution for log prices or of volatilities in our stochastic volatility models. We consider changes of probability measure to achieve risk-neutrality.

We extend the work of Duffie *et al.* (2000) to much more general processes. Duffie *et al.* (2000) model the jump part of the Lévy process as a compound Poisson, while we use a completely general Lévy process, and also allow dependent jumps in both (log) prices and in volatility. We show how to do option pricing using the change of measure required of us by The First Fundamental Theorem of Asset pricing (see Delbaen and Schachermayer (1994)). We will present other models in the literature as particular cases of our model. The derivations are included in the appendix.

2.1 Preliminary Results

2.1.1 Conditional Fourier Transform and Relation to Option Prices

Definition 2.1 The holder of a *European call option* has the right, but not the obligation, to buy an underlying security at a specified date (*expiration date*) for a contractually specified amount (*strike price*), irrespective of the market value of the security on that date. (see Karatzas and Shreve, 1998, page 37)

Next, we construct the conditional Fourier transform as in Duffie *et al.* (2000), and explain its relation to option prices.

Let X_t be a càdlàg stochastic process in $\mathbb{R}^n$ (i.e. one with right-continuous paths with left-limits), adapted to the right-continuous $\mathbb{P}$-complete filtration $(\mathscr{F}_t)_{t\geq 0}$. Duffie *et al.* (2000) define the conditional Fourier transform as:

$$\Psi(u, X_t, t, T) \quad = \quad \mathbb{E}_t\{\exp\left(-\int_t^T R(X_s)ds\right)\exp\left(\langle u, X_T\rangle\right)\} \tag{2.1}$$

where $\mathbb{E}_t$ is the conditional expectation given $\mathscr{F}_t$ and $u \in \mathbb{C}^n$.

The underlying securities of options can be stocks, indices such as the *Standard and Poor's 500*, interest rates, etc. It is standard in the financial literature to model stock prices on the log scale, i.e. set $S_t = \exp(X_t)$ and model X_t. At the expiration date T, the value of the option is $(S_T - K, 0)^+$, the maximum of $S_T - K$ and zero.

Payoff is at later time T, so under constant discount rate r its present value would be $\exp(-r(T-t))\,(S_T - K)^+$, where K is the strike price. If the discount rate varies with time t, perhaps as a function $R(X_t)$ of the log prices, the net present value is:

$$\exp\left(-\int_t^T R(X_s)ds\right)(S_T - K)^+$$

We are interested in the distribution of discounted price S_T at time t, and can infer it from the Fourier transform given by (2.1). If we want to solve (2.1) through the method proposed in this work, then $R(X_t)$ has to be affine in X_t. See (Karatzas and Shreve, 1997, page 85) for a detailed explanation on conditional Fourier transforms.

2.1.2 Exposition of the Model

Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$ be a probability space, equipped with a filtration satisfying the *usual hypotheses* (see Protter, 1990, page 3, for definition), and let X_t be a stochastic process with state space $S \subset \mathbb{R}^n$, Markov with respect to $(\mathscr{F}_t)_{t\geq 0}$,that solves the following SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t \tag{2.2}$$

Here dW_t is an $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion in $\mathbb{R}^n$, and dJ_t is an $(\mathscr{F}_t)_{t\geq 0}$- pure jump Lévy process in $\mathbb{R}^n$ independent from dW_t, with intensity measure $\nu(dx)$ satisfying $\int_{\mathbb{R}^n}(\|x\|^2 \wedge 1)\nu(dx) < +\infty$.

The Lévy measure density $\nu(x)$ can be made affine in X_t, as in Kawazu and Watanabe (1971) in one dimension.

We use the parameterization Duffie *et al.* (2000), with drift and several other parameters of the model affine in X_t. Thus:

$$\begin{array}{rcll} \mu(x) &=& K_0 + K_1 x, & \text{where } K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n} \\ (\sigma(x)\sigma^T(x))_{ij} &=& (H_0)_{ij} + (H_1)_{ij} x, & \text{where } H = (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n} \\ R(x) &=& \rho_0 + \rho_1 x, & \text{where } \rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n \end{array}$$

The quantities (K, H, ν, ρ) given X_0, characterize completely the distribution of X through the infinitesimal generator (Sato, 1999; Ethier and Kurtz, 1986). We denote

(K, H, ν, ρ) by Z below.

As in Duffie *et al.* (2000), (but less restrictive), define

Definition 2.2 A characteristic Z is *well-behaved* at $(u, T) \in \mathbb{C}^n \times [0, +\infty)$, if the following equations:

$$\begin{aligned}
\dot{\beta} &= \rho_1 - K_1^T \beta - \frac{1}{2}\beta^T H_1 \beta \\
\dot{\alpha} &= \rho_0 - K_0^T \beta - \frac{1}{2}\beta^T H_0 \beta - \int_{\mathbb{R}} (\exp(\langle x, \beta \rangle) - 1 - \langle x, \beta \rangle I)\nu(dx)
\end{aligned}$$

are solved uniquely, and if $\mathbb{E}(|\Psi_T|) < +\infty$.

Theorem 2.1 Assume Z is well-behaved at (u, T). Then,

$$\Psi_t = \exp\{\alpha(t, T, u) + \beta(t, T, u) X_t\} \tag{2.3}$$

for $\alpha(T, T, u) = 0$ and $\beta(T, T, u) = u$.

See proof of theorem in appendix.

2.1.3 A Two-Dimensional Jump-Diffusion Model

We model $X_t \equiv (\log S_t, \log V_t)$.

$$\begin{pmatrix} d \log S_t \\ d \log V_t \end{pmatrix} = \mu(X_t)dt + \sqrt{V_t} \begin{pmatrix} 1 & 0 \\ \gamma \sigma_{V_t} & \sqrt{1 - \gamma \sigma_{V_t}} \end{pmatrix} dW_t + dJ_t \tag{2.4}$$

where σ_{V_t} is the volatility of the volatility V_t, γ is the correlation between dW_t^1 and dW_t^2, and dJ_t is a pure jump Lévy process in $\mathbb{R} \times \mathbb{R}$.

The diffusion part is identical to that of Duffie *et al.* (2000). The novelty is the introduction of a two-dimensional jump component dJ_t, with jump rate $\nu(dx)$ on $\mathbb{R} \times \mathbb{R}$ satisfying $\int_{\mathbb{R}\times\mathbb{R}}(\|x\|^2 \wedge 1)\nu(dx) < +\infty$.

2.1.4 Special Cases of the Model

Barndorff-Nielsen and Shephard's model

Barndorff-Nielsen and Shephard (2001) consider a related model, which in our notation can be described as:

$$
\begin{aligned}
X_t &= \begin{pmatrix} Y_t \\ \sigma_t^2 \end{pmatrix} \\
\mu(X_t) &= K_0 + K_1 X_t = \begin{pmatrix} \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} \log S_t \\ \sigma_t^2 \end{pmatrix} \\
\sigma(X_t) &= \sqrt{\sigma_t^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
J_t &\sim \text{Lévy}\big(\nu(dx)\big)
\end{aligned}
$$

where J_t is a Lévy process with Lévy measure

$$
\nu(dx) = \frac{\lambda\delta}{\sqrt{8\pi}}(x_2^{-1} + \gamma^2)x_2^{-\frac{1}{2}} \exp(-\frac{\gamma^2 x_2}{2})\delta_{\rho x_2}(dx_1)\, dx_2,
$$

concentrated on the line $x_1 = \rho x_2$ so that each jump in asset log price is a constant multiple ($\rho < 0$) of a corresponding jump in volatility. Barndorff-Nielsen and Shephard (2001) denote J_t^2 by Z_t and J_t^1 by ρZ_t, and note that Z_t is a real-valued subordinator, i.e., an increasing Lévy process with no Gaussian component.

In their notation our Equation 2.4 becomes:

$$\begin{aligned} dY_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t} \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dZ_{\lambda t} \end{aligned}$$

Here $\mu + \beta\sigma_t^2$ is the risk premium in time dt, decomposable as a fixed part μ, and another that based on the risk $\beta\sigma_t^2$ due to volatility.

This model has a *leverage effect* (see Barndorff-Nielsen and Shephard, 2002, page 19, for definition) in which a drop in the price will lead to higher volatility (since $\rho \leq 0$). The time change of Z_t to $Z_{\lambda t}$ is just to simplify formulas by making the invariant distribution of σ_t^2 not depend on λ.

The volatility process σ_t^2 is of the O-U type, and the driving process Z_t is not Brownian motion, but a subordinator, and admits an invariant distribution (see Sato (1999), section 17).

Z_t is called the *background driving Lévy process* (BDLP), see Barndorff-Nielsen and Shephard (2001) for more details.

One drawback in the model of Barndorff-Nielsen *et al.* (2001), is that it must create artificial jumps, even when none are observed, to keep the volatility strictly positive. Including a diffusion term in modeling the stochastic volatility σ_t^2 would allow the process to move randomly and remain positive, even when there are no jumps. A second drawback is that the model is unable to reflect positive jumps in the returns.

Duffie and Pan's model

The model of Duffie *et al.* (2000) can be expressed in our notation as:

$$
\begin{aligned}
X_t &= \begin{pmatrix} Y_t \\ V_t \end{pmatrix} \\
\mu(X_t) &= K_0 + K_1 X_t = \begin{pmatrix} r - \bar{\xi} - \bar{\lambda}\bar{\mu} \\ \kappa_v \bar{v} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa_v \end{pmatrix} \begin{pmatrix} \log S_t \\ V_t \end{pmatrix} \\
\sigma(X_t) &= \sqrt{V_t} \begin{pmatrix} 1 & 0 \\ \bar{\rho}\sigma_v & \sqrt{1-\bar{\rho}^2}\sigma_v \end{pmatrix} \\
J_t &\sim \text{Lévy}\big(\nu(dx)\big)
\end{aligned}
$$

where J_t (denoted Z_t by Duffie *et al.* (2000)) has bivariate Lévy density:

$$
\nu(x_1, x_2) = \frac{\lambda}{\sqrt{2\pi}\sigma_{x_1}\mu_{x_2}} \exp\left(-\frac{x_2}{\mu_{x_2}} - \frac{(x_1 - \mu_{x_1} - \rho_J x_2)^2}{2\sigma_{x_1}^2} \right), \qquad x \in \mathbb{R} \times \mathbb{R}_+.
$$

Duffie *et al.* (2000) use a very specific compound Poisson process for the jumps, where we use instead a general pure Lévy jump process in $\mathbb{R} \times \mathbb{R}$, possibly with infinite Lévy measure (and hence infinitly-many jumps in finite time).

Here $x_1 \in \mathbb{R}$, the size of the jump in the log price, follows a normal distribution. It is correlated (through ρ_J) with the exponentially-distributed size $x_2 > 0$ of the jump in volatility. Finally λ is the intensity of the Poisson process. See (Duffie *et al.*, 2000) for details.

2.2 Risk Neutral Pricing

We now apply the pricing method of Duffie *et al.* (2000) to our model in (2.4).

2.2.1 The Mathematics

The density process Z_t (see equation (1.9)) is used to make change of measures when we are using Brownian motion. If we use more complicated process, who have càdlàg sample paths, this method is the same, see (Baxter and Rennie, 1996, page71) for a good example.

To price options, we have to perform a change of measure in (2.4) from the original $\mathbb{P}$ to a risk-neutral $\mathbb{Q}$. The following theorem 2.2, is for a general process X_t than the one given in (2.4).

Theorem 2.2 Assume Z is *well-behaved* at (b, T), for some $b \in \mathbb{R}^n$,

and let $\xi_t = \exp\left(\int_0^t R(X_s)ds + \alpha(t,T,b) + \beta(t,T,b)X_t\right)$ be the density process, and set $\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{\xi_T}{\xi_0}$.

Then $Z^{\mathbb{Q}}$ is given by:

$$
\begin{aligned}
K_0^{\mathbb{Q}}(t) &= K_0 + H_0\beta(t,T,b) \\
K_1^{\mathbb{Q}}(t) &= K_1 + H_1\beta(t,T,b) \\
H_0^{\mathbb{Q}} &= H_0 \\
H_1^{\mathbb{Q}} &= H_1 \\
\nu_t^{\mathbb{Q}}(dx) &= \exp\left(\beta(t,T,b)x\right)\nu(dx)
\end{aligned}
$$

We should point out that what is called *Equivalent martingale measures* (EMM) in the literature, preserves the original probabilistic structure of the model. We started out with a process X_t that is the sum of a Brownian motion and a pure Lévy jump process J_t.

When changing measure, we do not know whether J_t is a pure Lévy jump process, since the measure $\nu_t^{\mathbb{Q}}(dx)$ has a time dependence through $\beta(t,T,b)$, which tells us that J_t is an additive process (see Sato (1999)). If $\beta(t, T, b)$ is independent of time, then we have a Lévy process. Depending on the initial Lévy measure $\nu(dx)$ that we specify, we have to put specific constraints on $\beta(t, T, b)$ to get back the same Lévy measure, albeit with different parameters. This idea is explored very thoroughly in Nicolato and Venardos (2002).

2.2.2 Problems, Solutions and Future Research

Model (2.2) (through Theorem 2.2) shows us that we have to be careful about the choice of the pure jump process J_t. It would be interesting to study the general conditions we would have to impose on α and β for X_t to have the same probabilistic structure under $\mathbb{P}$ as under $\mathbb{Q}$, i.e., what would be the class of EMM under which X_t is described by a model of type (2.2), albeit with a different parameterization of the original model (see Nicolato and Venardos (2002)).

Making the Lévy measure state depend on the process $(X)_{t\geq 0}$, leads to a semi-martingale with non-independent increments, since the Lévy measure would depend on $\omega \in \Omega$. This state dependence has been discussed in one dimension by Kawazu and Watanabe (1971). It would be interesting to make the intensity of the random measure depend on $(X)_{t\geq 0}$ in several dimensions, generalizing Kawazu and Watanabe (1971), and derive conditions on β and α so that expression (2.3) is a $\mathbb{P}$-martingale again.

In Chapter 3, we perform a Bayesian analysis of the model related to Barndorff-Nielsen and Shephard (2001); examining posterior distributions of parameters and of the leverage ρ to see whether the "Leverage Effect" $\rho \leq 0$ is plausible.

Chapter 3

Bayesian inference for a Stochastic Volatility model

In this chapter we implement a Bayesian method to help assess the uncertainty about model parameters. We use the model of Barndorff-Nielsen and Shephard (2001) to describe the behavior of the logarithm of stock prices. Nicolato and Venardos (2002) show how to preserve the probabilistic structure of the historical measure in order to get an equivalent risk-neutral probability measure identical to the historical one, albeit with a different parameterization. We employ the Reversible Jump Markov Chain Monte Carlo method (RJMCMC) of Green (1995) to implement a Bayesian analysis to find the posterior distribution of the model parameters, the jump structure, and of the leverage ρ to see whether changes in the log stock price affect future volatility. Ours is an "infinite activity" model (i.e., it can have infinitely many jumps in both the log stock price and the volatility), extending existing models with finitely many jumps, see Roberts *et al.* (2001), Eraker *et al.* (2001).

3.1 The general setup

3.1.1 An SDE model

The model for the historical log stock price X_t that we use to illustrate our approach is that of Barndorff-Nielsen and Shephard (2001):

$$\begin{aligned} dX_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t} \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dZ_{\lambda t}. \end{aligned}$$

The process $Z_{\lambda t}$ is called *Background Driving Lévy Process* or BDLP. The variance process σ_t^2 is of Ornstein-Uhlenbeck type, driven by the subordinator $Z_{\lambda t}$. Barndorff-Nielsen and Shephard (2001) show that σ_t^2 has an inverse Gaussian stationary distribution. The time change from Z_t to $Z_{\lambda t}$ is done in order to simplify notation by making the λ disappear from the distribution of σ_t^2. The Lévy density of an inverse Gaussian distribution $IG(\delta, \gamma)$ is given by:

$$w(x) = \frac{\delta}{\sqrt{2\pi}} x^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\gamma^2 x\right), \quad x > 0.$$

Since $\int_0^{+\infty} w(x)dx = +\infty$, we infer that σ_t^2 is an infinite activity process, i.e. has infinitely many jumps in each open time interval. From the Laplace transform of Z_t, one can show that $Z_t = Y(t) + P(t)$, where $Y(t)$ is an inverse Gaussian Lévy process with parameters $\frac{\delta}{2}$ and γ, and $P(t)$ is a compound Poisson process with rate parameter $\frac{\delta\gamma}{2}$ and *i.i.d.* jump sizes distributed as $\chi^2_{(1)}$ (see Barndorff-Nielsen and Shephard, 2002, page 189, for details).

3.1.2 A statistical model

To simplify our subsequent analysis, we neglect the volatility component of the premium, setting $\beta = 0$. Let $\psi \equiv [\mu, \rho, \delta, \gamma, \lambda, \sigma_0^2] \in \Psi \subset \mathbb{R}^6$. Conditioning on ψ, $dZ_{\lambda t}$ and σ_t^2, the distribution of dX_t is normal.

Let $\mathscr{T} = \{0{=}t_0 < t_1 < \cdots < t_n{=}T\}$ be a partition of the interval $[0, T]$, and denote $X_{\mathscr{T}} = \{X_{t_0}, X_{t_1}, \ldots, X_{t_n}\}$ for the (partial) observations of $(X_t)_{t\geq 0}$.

Upon observing $X_{\mathscr{T}}$, the likelihood is equal to:

$$L(\theta) \;=\; \prod_{j=0}^{n-1} \frac{1}{\sqrt{2\pi \int_{t_j}^{t_{j+1}} \sigma_s^2 ds}} \exp\left(-\frac{(\triangle X_{t_{j+1}} - \mu_{t_j})^2}{2\int_{t_j}^{t_{j+1}} \sigma_s^2 ds}\right)$$

where $\theta \in \Theta = \Psi \times \cup_{M=0}^{\infty}\{(\epsilon, +\infty) \times [0, T]\}^M$, $\psi = [\mu, \rho, \delta, \gamma, \lambda, \sigma_0^2] \in \Psi \subset \mathbb{R}^6$ and $\phi = [(\tau_1, u_1), \ldots, (\tau_M, u_M)]$, and $\int_{t_j}^{t_{j+1}} \sigma_s^2 ds$ is given by:

$$\int_{t_j}^{t_{j+1}} \sigma_s^2 ds \;\equiv\; \frac{1}{\lambda}\{1 - \exp\left[-\lambda(t_{j+1} - t_j)\right]\}\sigma_{t_j}^2 + \int_{t_j}^{t_{j+1}} \frac{1}{\lambda}\{1 - \exp\left[-\lambda(t_{j+1} - s)\right]\} dZ_{\lambda s}^{\star}$$

This result follows from Fubini's theorem for Lévy integrals (see Barndorff-Nielsen and Shephard, 2002, page 202).

The prior specification for the jump times and sizes comes from the Lévy measure of $Z_{\lambda t}$. The joint prior distribution of the times and sizes of those jumps of size $u_j > \epsilon$ is proportional to

$$\{\prod_{j=1}^{M} \nu(u_j|\epsilon, \lambda, \delta, \gamma)\} \exp\left(-T\int_{\epsilon}^{+\infty} \nu(u|\epsilon, \lambda, \delta, \gamma) du\right)$$

For prior distribution on our starting value parameter σ_0^2 we take an $IG(\delta, \gamma)$ density:

$$\pi(\sigma_0^2|\delta,\gamma) = \frac{\delta \exp{(\delta\gamma)}}{\sqrt{2\pi}}(\sigma_0^2)^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\delta^2(\sigma_0^2)^{-1} + \gamma^2\sigma_0^2\right)\right]$$

3.1.3 Statistical Inference through an MCMC approach

To illustrate our modelling and inference we take the Standard and Poor's (S&P) 100 for all 253 trading days of 1987, beginning 01/02/1987. We need to specify a joint prior distribution for the six parameters of the model. As in Eraker *et al.* (2001), Sivaganesan (2001) and Roberts *et al.* (2001), we put a vague normal prior centered at zero ($N(0, 1000)$) for ρ, a flat prior for μ, and vague exponentials for the remaining four parameters with $\lambda = 0.00001$ (i.e., with large expected values $= \frac{1}{\lambda}$), all independent.

This reflects a priori our lack of any information. With our large sample size, the likelihood will dominate this vague prior.

We implement the MCMC approach to Bayesian inference, obtaining dependent samples from an ergodic Markov chain whose invariant distribution is the posterior (see Robert and Casella, 2000, page 159 for more details).

3.2 Implementing the Reversible Jump method

Any Lévy process can be decomposed as a continuous martingale, a bounded pure jump process and a compound Poisson process (see Protter (1990), Theorems 40 and 41 page 28). We will approximate such a process by one with all its jumps bigger than $\epsilon \geq 0$ in absolute value and take ϵ so small that the omitted pure jump Lévy process (whose jumps are bounded by ϵ), is negligible.

Thus we truncate the Lévy measure, making it finite, and so have a compound

Poisson with mean equal to the Poisson intensity times the jump distribution (see Sato (1999), proposition 19.5, page 123). An obstacle is that it may be difficult to sample from this jump distribution, since it will not be a known in closed form.

3.2.1 Initialization of MCMC

The Lévy density of $Z_{\lambda t}$ (Barndorff-Nielsen and Shephard, 2002, page 189) is:

$$\nu(u|\lambda,\delta,\gamma) = \frac{\lambda\delta}{2\sqrt{2\pi}}(u^{-1}+\gamma^2)u^{-\frac{1}{2}}\exp(-\frac{\gamma^2 u}{2}), \quad u > 0.$$

But we approximate $Z_{\lambda t}$, through $Z^{\star}_{\lambda t}$, whose Lévy density will be:

$$\nu^{\star}(u|\epsilon,\lambda,\delta,\gamma) = \frac{\lambda\delta}{2\sqrt{2\pi}}(u^{-1}+\gamma^2)u^{-\frac{1}{2}}\exp(-\frac{\gamma^2 u}{2}), \quad u > \epsilon.$$

We use a rejection approach with a Pareto proposals $P(\epsilon,\delta)$ having density $g(u) = \delta\epsilon^{\delta}u^{-(1+\delta)}$ for $u > \epsilon$. This leads to an acceptance probability p_{acc} equal to:

$$p_{\text{acc}} = C\frac{\lambda}{2\sqrt{2\pi}\epsilon^{\delta}}(u^{-1}+\gamma^2)u^{\frac{1}{2}+\delta}\exp(-\frac{\gamma^2 u}{2})I_{u>\epsilon}$$

with C chosen to ensure $p_{\text{acc}} \leq 1$.

Furthermore:

$$C\frac{\lambda}{2\sqrt{2\pi}\epsilon^{\delta}}(u^{-1}+\gamma^2)u^{\frac{1}{2}+\delta}\exp(-\frac{\gamma^2 u}{2})I_{u>\epsilon} < h(u) \equiv$$
$$C\frac{\lambda}{2\sqrt{2\pi}\epsilon^{\delta}}(\epsilon^{-1}+\gamma^2)u^{\frac{1}{2}+\delta}\exp(-\frac{\gamma^2 u}{2})I_{u>\epsilon}$$

Differentiating $\log h(u)$, and setting it to zero, we find the global maximum:

$$u_{\max} = \frac{1+2\delta}{\gamma^2} \vee \epsilon$$

So this yields:

$$C = \frac{2\sqrt{2\pi}\epsilon^{1+\delta}}{\lambda(1+\epsilon\gamma^2)} u_{\max}^{-(\frac{1}{2}+\delta)} \exp\left(\frac{u_{\max}\gamma^2}{2}\right)$$

We initialize then all parameters in ψ, draw M from a Poisson distribution with mean $\nu((\epsilon,\infty]|\epsilon,\lambda,\delta,\gamma)$. We then draw M points (τ_i, u_i) from a uniform $U[0,T]$ and a Pareto $Pa(\epsilon,\delta)$. Our initialization of the MCMC is based on the Inverse Lévy Measure method (see Wolpert and Ickstadt (1998)).

3.2.2 Posterior Distribution for the model

Let $\mathscr{T} = \{0{=}t_0 < t_1 < \cdots < t_n{=}T\}$ be a partition of the interval $[0,T]$, and set $X_{\mathscr{T}} = \{X_{t_0}, X_{t_1}, \ldots, X_{t_n}\}$.
Define:

$$\mu_{t_j} \equiv \int_{t_j}^{t_{j+1}} \mu ds + \rho(Z^{\star}_{\lambda t_{j+1}} - Z^{\star}_{\lambda t_j})$$

One can show (see Barndorff-Nielsen and Shephard (2002))

$$\int_{t_j}^{t_{j+1}} \sigma_s^2 ds = \frac{1}{\lambda}\{1-\exp\left[-\lambda(t_{j+1}-t_j)\right]\}\sigma_{t_j}^2 + \int_{t_j}^{t_{j+1}} \frac{1}{\lambda}\{1-\exp\left[-\lambda(t_{j+1}-s)\right]\} dZ^{\star}_{\lambda s}$$

Then

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} \frac{1}{\lambda}\{1-\exp\left[-\lambda(t_{j+1}-s)\right]\}dZ_{\lambda s} &\approx \int_{t_j}^{t_{j+1}} \frac{1}{\lambda}\{1-\exp\left[-\lambda(t_{j+1}-s)\right]\}dZ^{\star}_{\lambda s} \\
\int_{t_j}^{t_{j+1}} \frac{1}{\lambda}\{1-\exp\left[-\lambda(t_{j+1}-s)\right]\}dZ^{\star}_{\lambda s} &= \sum_{t_j \le \tau_n \le t_{j+1}} \frac{1}{\lambda}\{1-\exp\left[-\lambda(t_{j+1}-\tau_n)\right]\}u_n
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(Z_{\lambda t_{j+1}} - Z_{\lambda t_j}) &\approx \rho(Z^{\star}_{\lambda t_{j+1}} - Z^{\star}_{\lambda t_j}) \\
\rho(Z^{\star}_{\lambda t_{j+1}} - Z^{\star}_{\lambda t_j}) &= \sum_{t_j < \tau_n \le t_{j+1}} \rho u_n
\end{aligned}
$$

where (τ_j, u_j) are the jump times and sizes (bigger than ϵ), for $j = 1, \ldots, M$. The model posterior is then equal to:

$$
\pi(\theta| \triangle X_{t_1}, \ldots, \triangle X_{t_n}) \propto \pi(\sigma_0^2|\delta,\gamma)\pi(\mu,\rho,\delta,\gamma,\lambda)\{\prod_{j=1}^{M} \nu^{\star}(u_j|\epsilon,\lambda,\delta,\gamma)\} \times
$$

$$
\exp\left(-T\nu^{\star}((\epsilon,\infty)|\epsilon,\lambda,\delta,\gamma)\right) \prod_{j=0}^{n-1} \frac{1}{\sqrt{2\pi \int_{t_j}^{t_{j+1}} \sigma_s^2 ds}} \exp\left(-\frac{(\triangle X_{t_{j+1}} - \mu_{t_j})^2}{2\int_{t_j}^{t_{j+1}} \sigma_s^2 ds}\right)
$$

where $\triangle X_{t_{j+1}} \equiv X_{t_{j+1}} - X_{t_j}$ and $\theta \in \Theta = \Psi \times \cup_{M=0}^{\infty}\{(\epsilon,+\infty) \times [0,T]\}^M$, $\psi = [\mu,\rho,\delta,\gamma,\lambda,\sigma_0^2] \in \Psi \subset \mathbb{R}^6$ and $\phi = [(\tau_1,u_1),\ldots,(\tau_M,u_M)]$, $\phi^{\star} = [\phi,(\tau_{M+1},u_{M+1})]$ or can be that $\phi^{\star} = [\phi \setminus (\tau_i,u_i)]$ (depending on whether we are increasing or decreasing M), $\triangle X_{\mathscr{T}} = \{\triangle X_{t_1},\ldots,\triangle X_{t_n}\}$ and where:

$$
\pi(\sigma_0^2|\delta,\gamma) = \frac{\delta \exp(\delta\gamma)}{\sqrt{2\pi}}(\sigma_0^2)^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\delta^2(\sigma_0^2)^{-1} + \gamma^2\sigma_0^2\right)\right].
$$

3.3 An RJMCMC Algorithm for sampling from the Posterior

The notations θ^t, $\theta^\star$ and θ^{t+1} denote the current iteration, a proposal for the next iteration and the actual next iteration, respectively. The proposal $\theta^\star$ is chosen among four possible moves: changing the parameters in ψ, adding a jump, removing a jump, and updating a jump (i.e. changing its jump time and jump size). Let $p_1, \ldots, p_4$ denote their respective probabilities of choosing each of these moves, $\sum_{i=1}^4 p_i = 1$. In each of these moves, we accept the proposed $\theta^\star$ with Hastings probability $\alpha(\theta^t, \theta^\star)$, setting $\theta^{t+1} = \theta^\star$, and otherwise reject the proposal and leave $\theta^{t+1} = \theta^t$. We set $\beta = 0$ in our analysis as before.

3.3.1 Change in $\psi = [\mu, \rho, \delta, \gamma, \lambda, \sigma_0^2] \in \Psi \subset \mathbb{R}^6$

With probability p_1, we select one of the six parameters $\mu, \rho, \delta, \gamma, \lambda, \sigma_0^2$. and propose for it a new value, as follows:

Updating μ

Updating μ is done through a Gibbs step (see appendix).

$$[\mu|\ldots] \sim N\left[\sum_{j=0}^{n-1} \frac{(\triangle X_{t_{j+1}} - \rho(Z_{\lambda t_{j+1}} - Z_{\lambda t_j}))}{a_1 \int_{t_j}^{t_{j+1}} \sigma_s^2 ds}, \frac{1}{a_1}\right]$$

where $a_1 \equiv \sum_{j=0}^{n-1} \frac{1}{s_{t_j}}$.

Updating ρ

Updating ρ is done through a Gibbs step (see appendix).

$$[\rho|\ldots] \sim N\left[\sum_{j=0}^{n-1} \frac{(\triangle X_{t_{j+1}} - \mu)(Z_{\lambda t_{j+1}} - Z_{\lambda t_j})}{a_2 \int_{t_j}^{t_{j+1}} \sigma_s^2 ds}, \frac{1}{a_2}\right]$$

where $a_2 \equiv \sum_{j=0}^{n-1}(\frac{(Z_{\lambda t_{j+1}}-Z_{\lambda t_j})^2}{s_{t_j}} + \frac{1}{n\sigma_\rho^2})$, and prior for ρ is a $N(0, \sigma_\rho^2)$.

Updating $\delta, \gamma, \lambda, \sigma_0^2$

Denote by x either δ, γ, λ or σ_0^2.

- The density proposal is:

$$Q(x^\star|x^t) = \frac{1}{x^\star\sqrt{2\pi\sigma_{x^\star}^2}} \exp\left(-\frac{(\log x^\star - \log x^t)^2}{2\sigma_{x^\star}^2}\right)$$

  Below, we will compute the acceptance probabilities for $\delta, \gamma, \lambda, \sigma_0^2$. Let $\psi^t = [\mu^t, \rho^t, \delta^t, \gamma^t, \lambda^t, \sigma_0^{2t}]$, $\psi^\star = [\mu^\star, \rho^\star, \delta^\star, \gamma^\star, \lambda^\star, \sigma_0^{2\star}]$ be the proposed move.
  $\psi_1^\star$ means we update only the first component, so $\psi_1^\star = [\mu^\star, \rho^t, \delta^t, \gamma^t, \lambda^t, \sigma_0^{2t}]$, and similarly for the other (blocks of) components.
  Below we compute the Metropolis-Hastings probabilities for the parameters.

- for δ:

$$\alpha(\delta^t, \delta^\star) = \frac{\pi(\delta^\star)\{\prod_{j=1}^M \nu^\star(u_j|\epsilon, \psi_3^\star)\} \exp\left(-T\nu^\star((\epsilon,\infty)|\epsilon, \psi_3^\star)\right) \delta^\star}{\pi(\delta^t)\{\prod_{j=1}^M \nu^\star(u_j|\epsilon, \psi^t)\} \exp\left(-T\nu^\star((\epsilon,\infty)|\epsilon, \lambda, \delta^t, \gamma)\right) \delta^t} \times \frac{\pi(\sigma_0^2|\delta^\star, \gamma)}{\pi(\sigma_0^2|\delta^t, \gamma)}$$

- for γ:

$$\alpha(\gamma^t, \gamma^\star) = \frac{\pi(\gamma^\star)\{\prod_{j=1}^M \nu^\star(u_j|\epsilon, \psi_4^\star)\} \exp\left(-T\nu^\star((\epsilon,\infty)|\epsilon,\psi_4^\star)\right)\gamma^\star}{\pi(\gamma^t)\{\prod_{j=1}^M \nu^\star(u_j|\epsilon, \psi^t)\} \exp\left(-T\nu^\star(u|\epsilon,\psi^t)\right)\gamma^t} \times \frac{\pi(\sigma_0^2|\delta,\gamma^\star)}{\pi(\sigma_0^2|\delta,\gamma^t)}$$

- for λ:

$$\alpha(\lambda^t, \lambda^\star) = \frac{\pi(\lambda^\star)\{\prod_{j=1}^M \nu^\star(u_j|\epsilon, \psi_5^\star)\} \exp\left(-T\nu^\star((\epsilon,\infty)|\epsilon,\psi_5^\star)\right)}{\pi(\lambda^t)\{\prod_{j=1}^M \nu^\star(u_j|\epsilon, \psi^t)\} \exp\left(-T\nu^\star((\epsilon,\infty)|\epsilon,\psi^t)\right)} \times \frac{\prod_{j=0}^N \frac{1}{\sqrt{2\pi\int_{t_j}^{t_{j+1}}\sigma_s^{2,\star}ds}} \exp\left(-\frac{(\triangle X_{t_{j+1}} - \mu_{t_j}^{t+1})^2}{2\int_{t_j}^{t_{j+1}}\sigma_s^{2,\star}ds}\right)\lambda^\star}{\prod_{j=0}^N \frac{1}{\sqrt{2\pi\int_{t_j}^{t_{j+1}}\sigma_s^{2,t}ds}} \exp\left(-\frac{(\triangle X_{t_{j+1}} - \mu_{t_j}^{t})^2}{2\int_{t_j}^{t_{j+1}}\sigma_s^{2,t}ds}\right)\lambda^t}$$

- for σ_0^2:

$$\alpha(\sigma_0^{2,t}, \sigma_0^{2,\star}) = \frac{\pi(\sigma_0^{2,\star}|\delta,\gamma)\sigma_0^{2,\star}\prod_{j=0}^n \frac{1}{\sqrt{2\pi\int_{t_j}^{t_{j+1}}\sigma_s^{2,\star}ds}} \exp\left(-\frac{(\triangle X_{t_{j+1}} - \mu_{t_j})^2}{2\int_{t_j}^{t_{j+1}}\sigma_s^{2,\star}ds}\right)}{\pi(\sigma_0^{2,t}|\delta,\gamma)\sigma_0^{2,t}\prod_{j=0}^n \frac{1}{\sqrt{2\pi\int_{t_j}^{t_{j+1}}\sigma_s^{2,t}ds}} \exp\left(-\frac{(\triangle X_{t_{j+1}} - \mu_{t_j})^2}{2\int_{t_j}^{t_{j+1}}\sigma_s^{2,t}ds}\right)}$$

3.3.2 Increasing M to $M+1$

With probability p_2, increase M to $M+1$, and generate a new point (τ_{M+1}, u_{M+1}) and see whether we accept it or not.

The rate at which the process $Z_{\lambda t}$ has jumps of size $u > \epsilon$ is:

$$d = \int_{\epsilon}^{\infty} \frac{\lambda\delta}{2\sqrt{2\pi}}(u^{-1}+\gamma^2)u^{-\frac{1}{2}}\exp(-\frac{\gamma^2 u}{2})du$$

$$d = \frac{\lambda\delta}{\sqrt{2\pi\epsilon}}\exp\left(-\frac{\epsilon\gamma^2}{2}\right);$$

these jumps are independent with density function:

$$\nu^\star(u|\epsilon,\lambda,\delta,\gamma) = \frac{\lambda\delta}{2\sqrt{2\pi}}(u^{-1}+\gamma^2)u^{-\frac{1}{2}}\exp(-\frac{\gamma^2 u}{2}), \quad u > \epsilon$$

Let $\psi = [\mu, \rho, \delta, \gamma, \lambda, \sigma_0^2]$, $\phi^t = [(\tau_1, u_1), \ldots, (\tau_M, u_M)]$ and $\phi^\star = [\phi, (\tau_{M+1}, u_{M+1})]$ and let $L(\theta)$ be the likelihood.

The probability of going from $\phi^\star$ to ϕ^t is moving from $M+1$ to M with probability p_3, times the probability of deleting one of the $M+1$ points (τ_j, u_j), with probability $\frac{1}{M+1}$.

The probability of going from ϕ^t to $\phi^\star$ is when we move from M to $M+1$ with probability p_2, times the Uniform density Unif$[0,T]$ (from generating τ_{M+1}), times the Pareto density $g(u_{M+1})$.

The Metropolis-Hastings probability for adding (τ_{M+1}, u_{M+1}) is equal to:

$$\begin{aligned}
\alpha(\phi^t, \phi^\star) &= \frac{\pi(\phi^\star|\triangle X_{\mathscr{T}})Q(\phi^t|\phi^\star)}{\pi(\phi^t|\triangle X_{\mathscr{T}})Q(\phi^\star|\phi^t)} \\
&= \frac{L(\phi^\star)\{\prod_{j=1}^{M+1}\nu^\star(\tau_j, u_j)\}p_3\frac{1}{M+1}}{L(\phi^t)\{\prod_{j=1}^{M}\nu^\star(\tau_j, u_j)\}p_2\frac{1}{T}g(u_{M+1})} \\
&= \frac{L(\phi^\star)\nu^\star(\tau_{M+1}, u_{M+1})p_3 T}{L(\phi^t)(M+1)p_2 g(u_{M+1})}
\end{aligned}$$

where $Q(\phi^t|\phi^\star) = p_3\frac{1}{M+1}$, and $Q(\phi^\star|\phi^t) = p_2\frac{1}{T}g(u_{M+1})$

3.3.3 Decrement number M of mass points

With probability p_3, decrease M to $M-1$, and so choose and delete (with uniform probability) one of the M points (τ_j, u_j).

The Metropolis-Hastings probability for deleting this point (τ_j, u_j), is:

$$\begin{aligned}\alpha(\phi^t, \phi^\star) &= \frac{\pi(\phi^\star| \triangle X_{\mathscr{T}})Q(\phi^t|\phi^\star)}{\pi(\phi^t| \triangle X_{\mathscr{T}})Q(\phi^\star|\phi^t)} \\ &= \frac{L(\phi^\star)\{\prod_{j\neq i}^{M} \nu^\star(\tau_j, u_j)\}p_2\frac{1}{T}g(u_i)}{L(\phi^t)\{\prod_{j=1}^{M} \nu^\star(\tau_j, u_j)\}p_3\frac{1}{M}} \\ &= \frac{L(\phi^\star)Mp_2 g(u_i)}{L(\phi^t)\nu^\star(u_i)Tp_3}\end{aligned}$$

where $Q(\phi^t|\phi^\star) = p_2\frac{1}{T}g(u_M)$, $Q(\phi^\star|\phi^t) = p_3\frac{1}{M}$, $\phi^\star \equiv [\phi^t \setminus (\tau_i, u_i)]$ and $\phi^t \equiv [(\tau_1, u_1), \ldots, (\tau_M, u_M)]$.

3.3.4 Moving one of the mass points

With probability p_4, leave M unchanged, and only propose a new move for one of the M points. When choosing one (τ_j^t, u_j^t) out of the M points uniformly, we will propose a move to $(\tau_j^\star, u_j^\star)$.

Jump Sizes

Update the jump size, using a lognormal as proposal, (where σ^2 is the step size in the algorithm), with a boundary reflection at ϵ.

- The density proposal is:

$$Q(u^\star|u^t) = \frac{1}{u^\star\sqrt{2\pi\sigma^2}}\{\exp\left(-\frac{(\log u^\star - \log u^t)^2}{2\sigma^2}\right) + \exp\left(-\frac{(\log u^\star + \log u^t - 2\log\epsilon)^2}{2\sigma^2}\right)\}$$

- The Metropolis-Hastings probability is equal to:

$$\begin{aligned}\alpha(u^t, u^\star) &= \frac{\pi(\theta^\star|\triangle X_{\mathscr{T}})Q(u^t|u^\star)}{\pi(\theta^t|\triangle X_{\mathscr{T}})Q(u^\star|u^t)} \\ &= \frac{\pi(\theta^\star|\triangle X_{\mathscr{T}})u^\star}{\pi(\theta^t|\triangle X_{\mathscr{T}})u^t} \\ &= \frac{\pi(u^\star|\triangle X_{\mathscr{T}})u^\star}{\pi(u^t|\triangle X_{\mathscr{T}})u^t}\end{aligned}$$

Jump Times

Here we will have to have a boundary reflection for times smaller than zero and bigger than T, the terminal value. Again, here σ^2 is the step size in the algorithm.

- Given the boundary restrictions, the density proposal is:

$$Q(\tau^\star|\tau^t) = \frac{1}{\sqrt{2\pi\sigma^2}}\sum_{k=-\infty}^{+\infty}\exp\left(-\frac{(\tau^\star - t_k)^2}{2\sigma^2}\right)$$

  where $t_{-1} = -2T - \tau^t, t_0 = \tau^t, t_1 = 2T - \tau^t, t_2 = 2T + \tau^t$, and $t_3 = 4T - \tau^t$. We can show that this proposal is symmetric.

- The Metropolis-Hastings probability is equal to:

$$\begin{aligned}
\alpha(\tau^t, \tau^\star) &= \frac{\pi(\theta^\star| \bigtriangleup X_{\mathscr{T}})Q(\tau^t|\tau^\star)}{\pi(\theta^t| \bigtriangleup X_{\mathscr{T}})Q(\tau^\star|\tau^t)} \\
&= \frac{\pi(\theta^\star| \bigtriangleup X_{\mathscr{T}})}{\pi(\theta^t| \bigtriangleup X_{\mathscr{T}})} \\
&= \frac{\pi(\tau^\star| \bigtriangleup X_{\mathscr{T}})}{\pi(\tau^t| \bigtriangleup X_{\mathscr{T}})}
\end{aligned}$$

The Metropolis-Hastings probability for updating the i^{th} point (τ_j^t, u_j^t) to $(\tau_j^\star, u_j^\star)$, is:

$$\begin{aligned}
\alpha(\phi^t, \phi^\star) &= \frac{\pi(\theta^\star| \bigtriangleup X_{\mathscr{T}})Q(u_i^t|u_i^\star)}{\pi(\theta^t| \bigtriangleup X_{\mathscr{T}})Q(u_i^\star|u_i^t)} \\
&= \frac{L(u_i^\star)\nu^\star(u_i^\star)u_i^\star}{L(u_i^t)\nu^\star(u_i^t)u_i^t}
\end{aligned}$$

3.4 MCMC results and conclusion

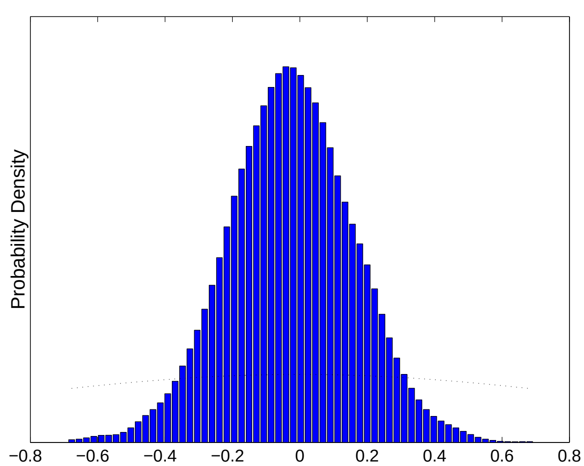

Figure 3.1: Prior (dots) and Posterior (bars) Distribution of ρ.

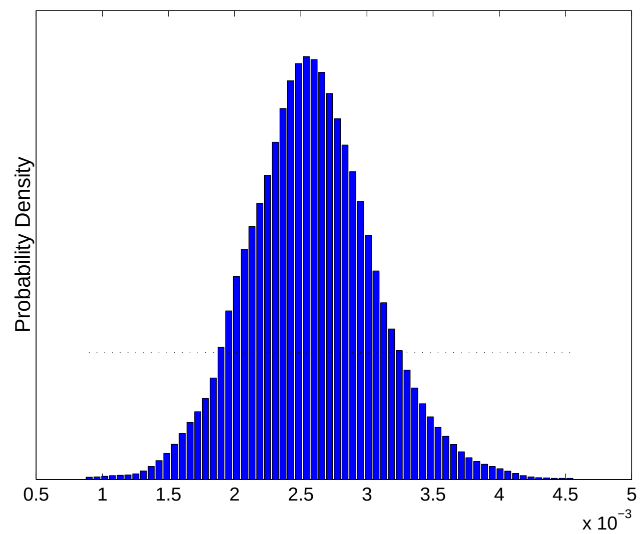

Figure 3.2: Prior (dots) and Posterior (bars) Distribution of μ.

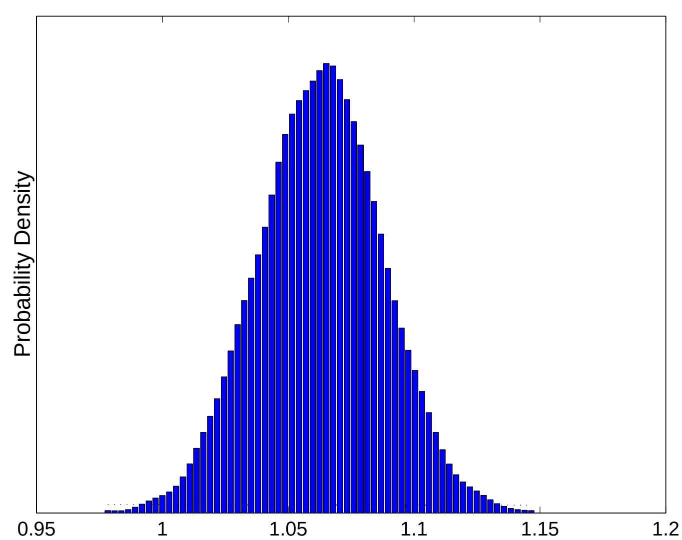

Figure 3.3: Prior (dots) and Posterior (bars) Distribution of volatility return λ.

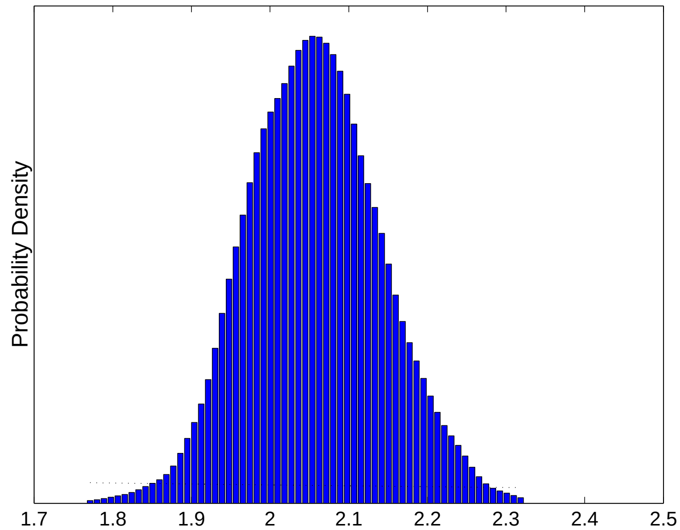

Figure 3.4: Prior (dots) and Posterior (bars) Distribution of δ.

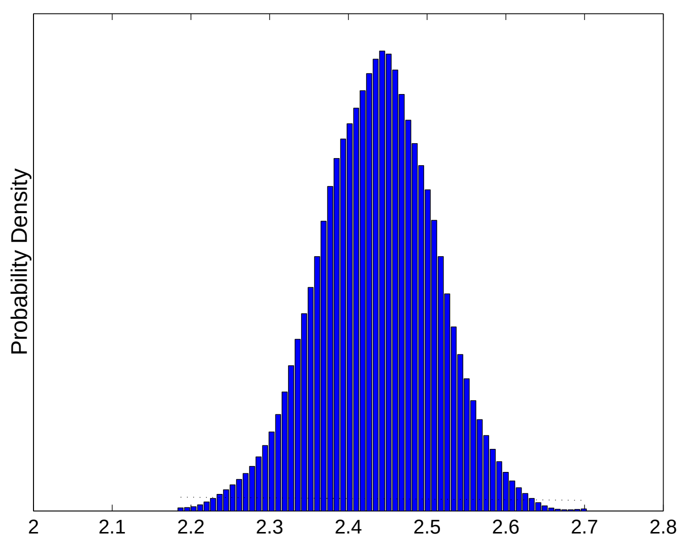

Figure 3.5: Prior (dots) and Posterior (bars) Distribution of γ.

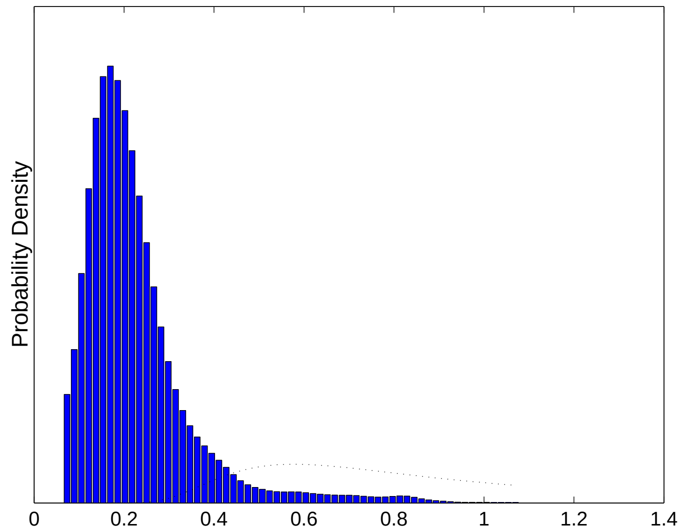

Figure 3.6: Prior (dots) and Posterior (bars) Distribution of σ_0^2.

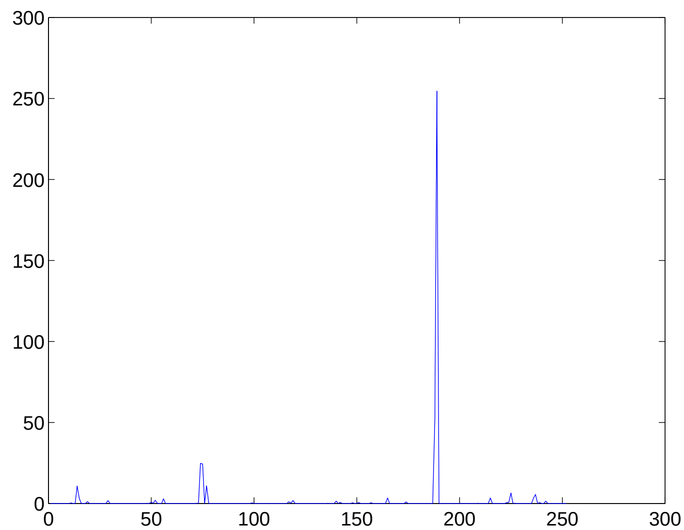

Figure 3.7: Posterior mean of jump sizes in 1987.

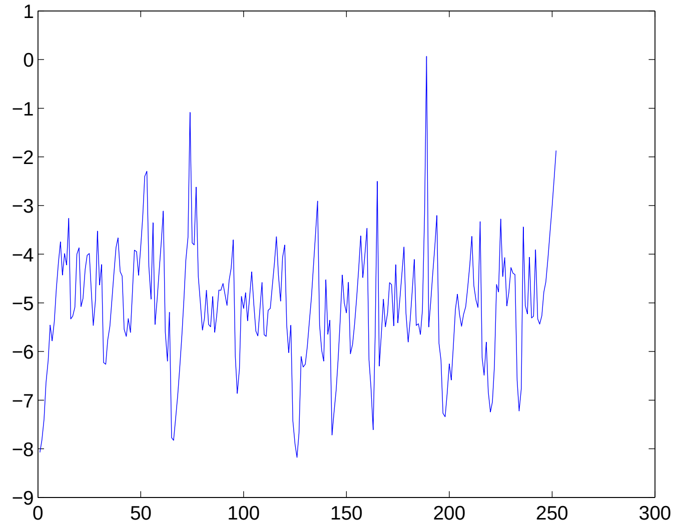

Figure 3.8: Posterior mean of log integrated volatility.

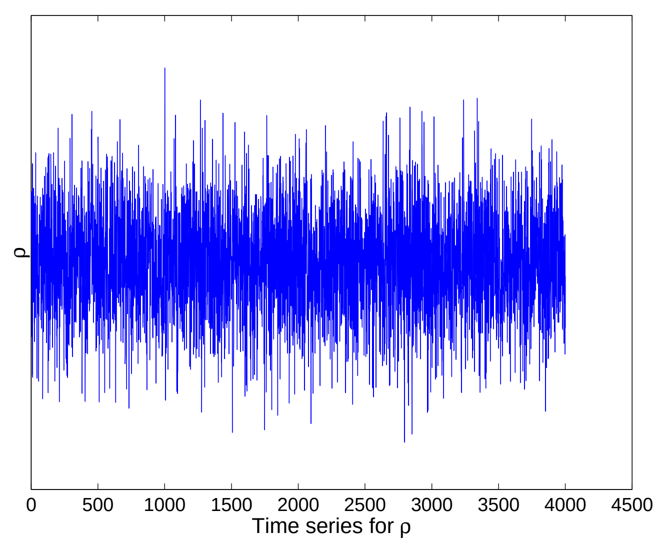

Figure 3.9: MCMC time series for ρ.

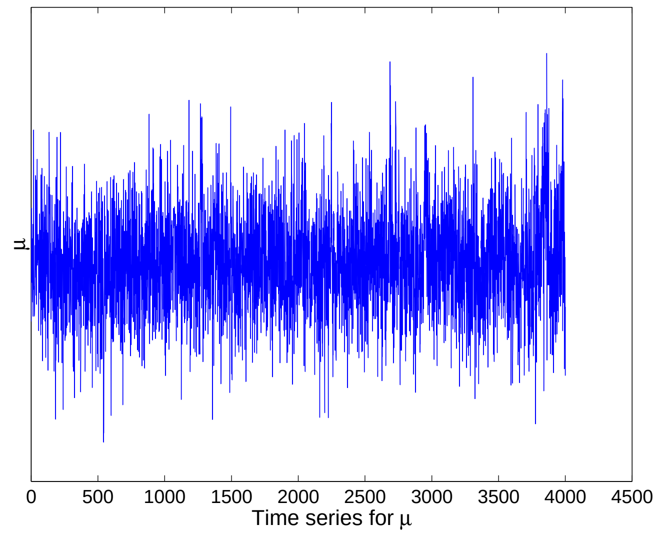

Figure 3.10: MCMC time series for μ.

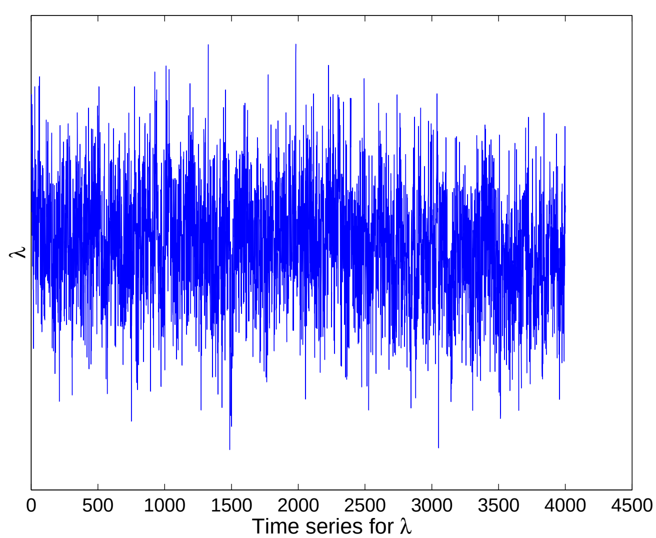

Figure 3.11: MCMC time series for λ.

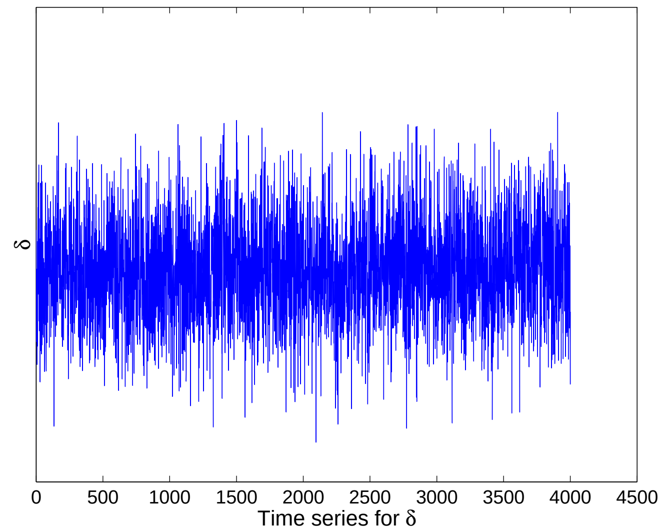

Figure 3.12: MCMC time series for δ.

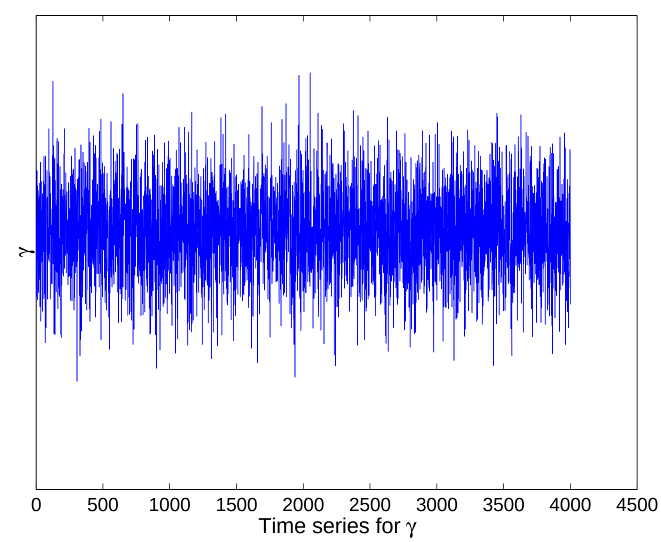

Figure 3.13: MCMC time series for γ.

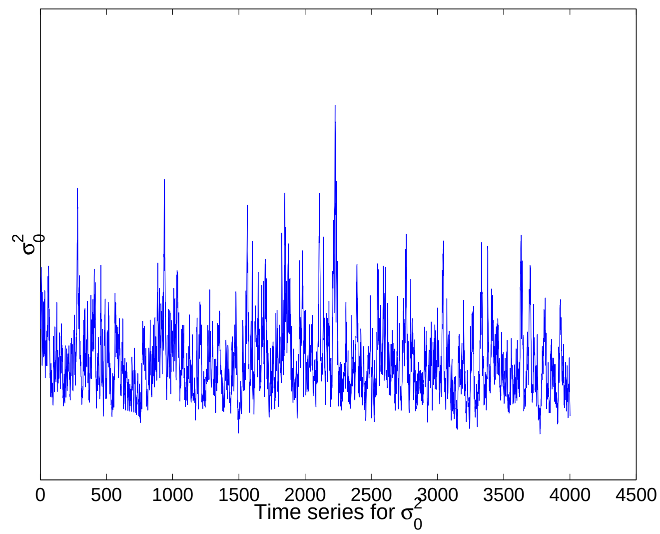

Figure 3.14: MCMC time series for σ_0^2.

We ran the MCMC for 4 million iterations and kept every 200^{th} from the last million. The parameters do not have large autocorrelations, and trace plots suggest that MCMC convergence has been obtained. The acceptance probabilities were between 20 and 35 percent, whenever we used Metropolis-Hastings updating. We display the posteriors as well as the priors to see how much information the data gives .
We can see that ρ is not symmetric around 0 (see figure 3.1) and gives more mass to negative values, but the data are not conclusive. The intuition might tell us that when ρ is positive, it might be because positive jumps are not taken into account.
We note the following observations:

- Some parameters like λ and δ are correlated. We performed block updating in such cases to improve convergence.
- For a wide range of starting values, including starting number of jumps, we found that M always stabilizes at about 14.
- Data suggest that $\rho < 0$ but are not conclusive.

- There is no sensitivity to the cut-off $\epsilon \leq 0.1$ for the Lévy prior. We used ϵ such that we have almost the whole mass of the expected value.

3.5 Future Research

The approach we present in Chapter 2 (see especially Equation (2.4)) models jumps in log asset-price and in volatility as components of a vector in $\mathbb{R}^2$ that can, in general, take any value in the plane. In contrast, all jump vectors in the model of Barndorff-Nielsen and Shephard (2002) lie on a one-dimensional subspace.

The methods we developed in Chapter 3 can also be applied in our more general setting, using an arbitrary bivariate Lévy process to model the prior distribution of jumps in log asset-price and volatility. Constructing a likelihood function from the data as in Chapter 3 leads to a joint posterior distribution for all the jump vectors in $\mathbb{R}^2$, along with any uncertain model parameters.

Posterior distributions of related quantities of interest, such as the prices for European call options, are available simply by evaluating these quantities at each iteration (by using theorems from chapter 2 and the Inverse Fourier Transform, as in Duffie *et al.* (2000)), allowing us to evaluate the posterior means (or even the entire distributions) of such quantities.

Appendix A

Details of Chapter 2

A.1 appendix Chapter 2

Proof for theorem 2.1:

Here the proof is when X_t is a one dimensional stochastic process but the analysis for X_t being multidimensional can be carried out without a problem.

Let $\Psi_t = \exp(V_t)$ with:

$$V_t = -\int_0^t R(X_s)ds + \alpha(t) + \beta(t)X_t \tag{A.1}$$

We also know that $dX_t = \mu(X_t) + \sigma(X_t)dW_t + dJ_t$ where dJ_t is a pure jump Lévy process with $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(dx) < +\infty$

Thus by Itô's formula we get:

$$\Psi_t = \Psi_0 + \int_0^t \exp(V_{s-})\, dV_s + \frac{1}{2}\int_0^t \exp(V_{s-})\, d[V,V]_s^c + \sum_{0<s\le t}\{\Psi_s - \Psi_{s-} - \Psi'_{s-} \triangle V_s\}$$

$$\Psi_t = \Psi_0 + \int_0^t \Psi_{s-}\left[-R(X_s)ds + \dot{\alpha} ds + \dot{\beta} X_s ds + \beta dX_s\right] + \frac{1}{2}\int_0^t \Psi_{s-} d[\beta\sigma W, \beta\sigma W]_s$$

$$+ \sum_{0<s\le t}\{\Psi_s - \Psi_{s-} - \Psi'_{s-}\beta \triangle X_s\}$$

$$\Psi_t = \Psi_0 + \int_0^t \Psi_{s-}\left[-R(X_s) + \dot{\alpha} + \dot{\beta} X_s + \frac{1}{2}\beta^2\sigma^2(X_s) + \beta\mu(X_s)\right] ds + \int_0^t \Psi_{s-}\beta dJ_s$$

$$+ \sum_{0<s\le t}\Psi_{s-}\{\exp(\beta dJ_s) - 1 - \beta dJ_s\}$$

We will write β for $\beta(t)$ and α for $\alpha(t)$ to simplify the notation.

Now, $\sum_{0<s\le t}\Psi_{s-}\{\exp(\beta dJ_s) - 1 - \beta dJ_s\} = \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta) - 1 - x\beta)\, N(ds,dx)$

Let us choose $I(\|x\| \le 1))$ as the compensator function. We will not be doing any weak convergence here, so we will not need this truncation function to be continuous. For ease of notation let us write I for $I(\|x\| \le 1))$. Then,

$$\begin{aligned} A &= \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta) - 1 - x\beta)\, N(ds,dx) \\ &= \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta) - 1 - Ix\beta + Ix\beta - x\beta)\, N(ds,dx) \\ &= \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta) - 1 - Ix\beta)\, N(ds,dx) - \int_{(0,t]\times\mathbb{R}} \Psi_{s-}x\beta(1-I)N(ds,dx) \end{aligned}$$

Now, $B = \int_0^t \Psi_{s-}\beta dJ_s = \int_{(0,t]\times\|x\|\le 1} \Psi_{s-}x\beta\tilde{N} + \int_{(0,t]\times\|x\|>1} \Psi_{s-}x\beta N(ds,dx)$ by the Lévy-Itô decomposition of the process J_t. We should point out that here J_t depends on the truncation function I. Let us point out that for less arrogant notation, we will write $\tilde{N}$ for $(N(ds,dx)-\nu(ds,dx))$ and this is called the compensated poisson random measure.

Since $\int_{(0,t]\times||x||\leq 1)} x\tilde{N}$ is a $\mathbb{P}$-martingale as well as $\int_{(0,t]\times||x||\leq 1)} \Psi_{s-}x\beta\tilde{N}$, so the only term that is not random and that goes into the drift part is

$\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta(s))-1-Ix\beta)\nu(ds,dx)$.

We thus have for Ψ_t the following expression:

$$\begin{aligned}\Psi_t = \Psi_0 \quad &+ \quad \int_0^t \Psi_{s-}\left[-R(X_s)+\dot{\alpha}+\dot{\beta}X_s+\frac{1}{2}\beta^2\sigma^2(X_s)+\beta\mu(X_s)\right]ds \\ &+ \quad A+B\end{aligned}$$

which simplifies to:

$$\begin{aligned}\Psi_t = \Psi_0 \quad &+ \quad \int_0^t \Psi_{s-}\left[-R(X_s)+\dot{\alpha}+\dot{\beta}X_s+\frac{1}{2}\beta^2\sigma^2(X_s)+\beta\mu(X_s)\right]ds \\ &+ \quad \int_{(0,t]\times||x||\leq 1)} \Psi_{s-}x\beta\tilde{N}+\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta(s))-1-Ix\beta)\tilde{N} \\ &+ \quad \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta(s))-1-Ix\beta)\nu(ds,dx)\end{aligned}$$

Thus now we need to put $\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(x\beta(s))-1-Ix\beta)\nu(ds,dx)$ into the drift and note that since we are dealing with a Lévy process then we have $\nu(dt,dx)=dt\nu(dx)$. One could allow for a more general Lévy measure of the form $\nu_s(dx)$, that would depend on time just as the case of an additive process (Sato (1999)). Coming back to our problem, the drift D would thus be finally equal to:

$$
\begin{aligned}
D &= \Psi_{t-}\left[-R(X_t)+\dot{\alpha}+\dot{\beta}X_t+\frac{1}{2}\beta^2\sigma^2(X_t)+\beta\mu(X_t)\right]dt \\
&+ \Psi_{t-}\left[\int_{\mathbb{R}}(\exp{(x\beta)}-1-xI\beta)\nu(dx)\right]dt
\end{aligned}
$$

From this equation we get:

$$
\begin{aligned}
\Psi_t = \Psi_0 &+ \int_0^t \Psi_{s-}\left[-\rho_0+\dot{\alpha}+\frac{1}{2}\beta^2 H_0+\beta K_0+\int_{\mathbb{R}}(\exp{(x\beta)}-1-xI\beta)\nu(dx)\right]ds \\
&+ \int_0^t \Psi_{s-}X_s\left[-\rho_1+\frac{1}{2}\beta^2 H_1+\dot{\beta}+\beta K_1\right]ds+\int_0^t \Psi_{s-}\beta\sigma(X_s)dW_s \\
&+ \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp{(x\beta(s))}-1-Ix\beta)\tilde{N}
\end{aligned}
$$

And since $\int_0^t \Psi_{s-}\beta\sigma(X_s)dW_s$ and $\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp{(x\beta(s))}-1-Ix\beta)\tilde{N}$ are $\mathbb{P}$-local-martingales (actualy they are $\mathbb{P}$-martingales), so if we want Ψ_t to be a $\mathbb{P}$-martingale, then we need the following two equations to be true:

$$
\begin{aligned}
\dot{\beta} &= \rho_1-\beta K_1-\frac{1}{2}\beta^2 H_1 \\
\dot{\alpha} &= \rho_0-\beta K_0-\frac{1}{2}\beta^2 H_0-\int_{\mathbb{R}}(\exp{(x\beta)}-1-xI\beta)\nu(dx)
\end{aligned}
$$

Doing the same analysis but for a process X_t in $\mathbb{R}^n$, we get:

$$
\dot{\beta} = \rho_1-K_1^T\beta-\frac{1}{2}\beta^T H_1\beta \tag{A.2}
$$

$$
\dot{\alpha} = \rho_0-K_0^T\beta-\frac{1}{2}\beta^T H_0\beta-\int_{\mathbb{R}}(\exp{(\langle x,\beta\rangle)}-1-\langle x,\beta\rangle I)\nu(dx) \tag{A.3}
$$

where $\langle , \rangle$ is just the inner-product in $\mathbb{C}^n$.

Proof for theorem 2.2:

We will be dealing with the case where X_t is in $\mathbb{R}$ and the extension when X_t is in $\mathbb{R}^n$ is the same idea.

We are using the state density process which we recall is equal to (see Duffie *et al.* (2000)):

$$\xi_t \;=\; \exp\left(\int_0^t R(X_s)ds + \alpha(t,T,b) + \beta(t,T,b)X_t\right)$$

where $\alpha(T,T,b) = 0$ and $\beta(T,T,b) = b$ at the terminal date T. Here of course this b is in $\mathbb{R}^n$ and is such that the conditional characteristic Fourier transform is well-defined (as we defined what being well-defined was back in chapter two). We already know that we need to normalize this ξ_t by ξ_0 so that we can define a new probability measure $d\mathbb{Q}$ as $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\xi_T}{\xi_0}$.

We do not care about the Brownian motion part under $\mathbb{Q}$ since it is shown in Duffie *et al.* (2000) how to obtain it through Girsanov's theorem. We will see how to get the new intensity measure under $\mathbb{Q}$ for $dJ_t^{\mathbb{Q}}$.

The goal here is to make also the process $\exp\left(-\int_0^T R(X_s)ds + uX_T\right)$

, i.e. $\exp\left(-\int_0^t R(X_s)ds + \alpha_2(t,T,u) + \beta_2(t,T,u)X_T\right)$ a $\mathbb{Q}$-martingale (where α_2 and β_2 are to be found). The spirit of the proof is similar to that of theorem 2.1. Let us denote $\mathbb{E}_t$ as being the conditional expectation conditional on the filtration $\mathscr{F}_t$.

$$
\begin{aligned}
&= \mathbb{E}_t^{\mathbb{Q}}\{\exp\left(-\int_0^T R(X_s)ds + uX_T\right)\} \\
&= \frac{1}{\xi_t}\,\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_0^T R(X_s)ds + uX_T + \alpha(T,T,b) + \beta(T,T,b)X_T\right)\right\} \\
&= \exp\left(-\int_0^t R(X_s)ds - \alpha(t,T,b) - \beta(t,T,b)X_t\right)\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left((b+u)X_T\right)\right\} \\
&= \exp\left(-\int_0^t R(X_s)ds - \alpha(t,T,b) - \beta(t,T,b)X_t\right) \\
&\quad \mathbb{E}_t^{\mathbb{P}}\{\exp\left(-\int_0^T 0ds + \alpha_1(T,T,b+u) + \beta_1(T,T,b+u)X_T\right)\}
\end{aligned}
$$

where, to simplify notation, we will write α_1 for $\alpha(t,T,b+u)$, β_1 for $\beta(t,T,b+u)$, α for $\alpha(t,T,b)$ and β for $\beta(t,T,b)$.

$$
\begin{aligned}
\exp\left(-\int_0^t R(X_s)ds - \alpha - \beta X_t + \alpha_1 + \beta_1 X_t\right) &= \exp\left(-\int_0^t R(X_s)ds + \alpha_2 + \beta_2 X_t\right) \\
&= \exp\left(M_t\right) \\
&= \Psi_t
\end{aligned}
$$

where $\alpha_2 = \alpha(t,T,b+u) - \alpha(t,T,b)$, $\beta_2 = \beta(t,T,b+u) - \beta(t,T,b)$,

$\alpha_2(T,T) = 0 - 0 = 0$,

$\beta_2(T,T) = b + u - b = u$ and under $\mathbb{P}$ we have:

$$
\begin{aligned}
dM_t &= -R(X_t)dt + \dot{\alpha}_2 dt + \dot{\beta}_2 X_t dt + \beta_2 dX_t \\
dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t
\end{aligned}
$$

Thus, by Itô's lemma:

$$\Psi_t = \Psi_0 + \int_0^t \Psi_{s-}\left[-R(X_s) + \dot{\alpha_2} + \dot{\beta_2}X_s + \frac{1}{2}\beta_2^2\sigma_{\mathbb{Q}}^2(X_s^{\mathbb{Q}}) + \beta_2\mu_{\mathbb{Q}}(X_s)\right]ds$$

$$\int_0^t \Psi_{s-}\beta_2 dJ_s + \int_0^t \Psi_{s-}\beta_2\sigma_{\mathbb{Q}}(X_s)dW_s + \sum_{0<s\leq t}\Psi_{s-}\{\exp(\beta_2 dJ_s) - 1 - \beta_2 dJ_s\}$$

and we want again to make Ψ_t a $\mathbb{Q}$-martingale as we did under $\mathbb{P}$. Notice that here the parameters are not the same as they are under $\mathbb{P}$. This is why we put a subscript near each one, and the goal is to recover the current parameters as functions from the previous ones under $\mathbb{P}$. When we deal with Brownian motion and perform a change of measure equivalent to the original one, we get Brownian motion with drift. In this case, therefore, μ should not be equal to $\mu_{\mathbb{Q}}$ a priori. We will not perform a full proof of the coefficients $H_0^{\mathbb{Q}}$, $H_1^{\mathbb{Q}}$, $K_0^{\mathbb{Q}}$, $K_1^{\mathbb{Q}}$ and $W_t^{\mathbb{Q}}$ since they are derived in Duffie *et al.* (2000). Instead, we will give a quick proof of them at the end. The question of the existence of $J_t^{\mathbb{Q}}$ and $W_t^{\mathbb{Q}}$ is guaranteed by the application of Girsanov's theorem together with Lévy's theorem to $W_t^{\mathbb{Q}}$, and the canonical decomposition of semimartingales (Jacod and Shiryaev (1987)) to guarantee the existence of $J_t^{\mathbb{Q}}$. We are using the result that if X_t is a $\mathbb{P}$-semimartingale and $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, then X_t is also a $\mathbb{Q}$-semimartingale, and using the Bichteler-Dellacherie theorem, then X_t is a classical $\mathbb{Q}$-semimartingale (see Protter (1990)). Our problem will then be to see what J_t looks like under $\mathbb{Q}$.

Let $C = \sum_{0<s\leq t}\Psi_{s-}\{\exp(\beta_2 dJ_s) - 1 - \beta_2 dJ_s\} = \int_{(0,t]\times\mathbb{R}}\Psi_{s-}(\exp(\beta_2 x) - 1 - \beta_2 x)N(ds,dx)$

$$\begin{aligned} C &= \int_{(0,t]\times\mathbb{R}}\Psi_{s-}(\exp(\beta_2 x) - 1 - \beta_2 x)N(ds,dx) \\ &= \int_{(0,t]\times\mathbb{R}}\Psi_{s-}(\exp(\beta_2 x) - 1 - I\beta_2 x)N(ds,dx) - \int_{(0,t]\times\mathbb{R}}\Psi_{s-}\beta_2 x(1-I)N(ds,dx) \end{aligned}$$

set $D = \int_{(0,t]\times\mathbb{R}}\Psi_{s-}(\exp(\beta_2 x) - 1 - I\beta_2 x)N(ds,dx)$, and as we saw in last theorem we have:

$$\int_0^t \Psi_{s-} x\beta_2 dJ_s \;\; = \;\; \int_{(0,t]\times||x||\leq 1} \Psi_{s-}\beta_2 x\tilde{N} + \int_{(0,t]\times||x||>1} \Psi_{s-}\beta_2 x N(ds,dx)$$

set $E = \int_{(0,t]\times||x||\leq 1} \Psi_{s-}\beta_2 x\tilde{N}$, and we have that:

$$D \;\; = \;\; \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(\beta_2 x) - 1 - I\beta_2 x)\tilde{N} + \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(\beta_2 x) - 1 - I\beta_2 x)\nu(ds,dx)$$

Now we need to find $\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(\beta_2 x) - 1 - I\beta_2 x)\nu^{\mathbb{Q}}(ds,dx)$ and so we need to compute

$\mathbb{E}^{\mathbb{Q}}\{\sum_{0<s\leq t} \Psi_{s-}(\exp(\beta_2 dJ_s - 1 - I\beta_2 dJ_s))\}$ but:

$$\mathbb{E}^{\mathbb{Q}}\{\sum_{0<s\leq t} \Psi_{s-}(\exp(\beta_2 dJ_s) - 1 - I\beta_2 dJ_s)\} \;\; = \;\; \mathbb{E}^{\mathbb{P}}\{\sum_{0<s\leq t} \xi_s \Psi_{s-}(\exp(\beta_2 dJ_s) - 1 - Ix\beta_2 dJ_s)\}$$

which is equal to:

$\mathbb{E}^{\mathbb{P}}\{\sum_{0<s\leq t} \xi_{s-}\Psi_{s-}(\exp((\beta_2+\beta)dJ_s) - \exp(\beta dJ_s) - I\beta_2 dJ_s \exp(\beta dJ_s)\}$ which equals:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}} \xi_{s-}\Psi_{s-}(\exp((\beta+\beta_2)x) - \exp(\beta x) - Ix\beta_2\exp(\beta x))N(ds,dx)\} &= \\
\mathbb{E}^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}} \xi_{s-}\Psi_{s-}(\exp((\beta+\beta_2)x) - \exp(\beta x) - Ix\beta_2\exp(\beta x))\nu(ds,dx)\} &= \\
\mathbb{E}^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}} \xi_{s-}\Psi_{s-}(\exp(\beta_2 x) - 1 - Ix\beta_2)\exp(\beta x)\,\nu(ds,dx)\}&
\end{aligned}$$

by the predictable projection of the Poisson random measure $N(ds,dx)$ and its compensator $\nu(ds,dx)$ (see Jacod and Shiryaev, 1987, page 71 for more details), from proposition 1.21.

Set $Z_{t-} = \xi_{t-}$, $X(w,t,x) = \Psi_{t-}(\exp(\beta_2 x) - 1 - Ix\beta_2)$, $A_t = X * N$, $B_t = YX * \nu$ apply equality (5.6) (see Liptser and Shiryayev, 1989, page 223 for more details) to both $X^+(w,t,x) \equiv \max[0, X(w,t,x)]$ and $X^-(w,t,x) \equiv \min[0, X(w,t,x)]$ and we get:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\{\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(\beta_2 x) - 1 - Ix\beta_2)N(ds,dx)\} &= \\
\mathbb{E}^{\mathbb{Q}}\{\int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp(\beta_2 x) - 1 - Ix\beta_2)\nu^{\mathbb{Q}}(ds,dx)\} &= \\
\mathbb{E}^{\mathbb{P}}\{\int_{(0,t]\times\mathbb{R}} \xi_{s-}\Psi_{s-}(\exp(\beta_2 x) - 1 - Ix\beta_2)\exp(\beta x)\,\nu(ds,dx)\} &
\end{aligned}
$$

Therefore, equating $Y(w,t,x)$ to $\exp(\beta x)$ (see Liptser and Shiryayev, 1989, page 223 for more details), we get that:

$$
\nu^{\mathbb{Q}}(dt,dx) = \exp(\beta x)\,\nu(dx)dt \tag{A.4}
$$

Before we pursue this line of reasoning, notice that the process $J_t^{\mathbb{Q}}$ might not be anymore a Lévy process and so we will talk about how to overcome this in chapter 2. The fact is that here (unless β does not depend on time), we get an additive process and not anymore a Lévy process (unless β does not depend on time, i.e. $\dot{\beta} = 0$). Thus if β is independent of time, we get that J_t is again a Lévy process under $\mathbb{Q}$.

To see what happens with the parameters $H_0^{\mathbb{Q}}$, $H_1^{\mathbb{Q}}$, $K_0^{\mathbb{Q}}$, $K_1^{\mathbb{Q}}$ under the risk neutral measure $\mathbb{Q}$, recall that:

$$
\begin{aligned}
\Psi_t = \Psi_0 \quad &+ \quad \int_0^t \Psi_{s-}\left[\int_{\mathbb{R}} \Psi_{s-}(\exp{(\beta_2 x)} - 1 - Ix\beta_2)\exp{(\beta x)}\,\nu(dx)\right] ds \\
&+ \quad \int_0^t \Psi_{s-}\left[-\rho_0 + \dot{\alpha_2} + \frac{1}{2}\beta_2^2 H_0^{\mathbb{Q}} + \beta_2 K_0^{\mathbb{Q}} + \right] ds \\
&+ \quad \int_0^t \Psi_{s-}X_s\left[-\rho_1 + \frac{1}{2}\beta_2^2 H_1^{\mathbb{Q}} + \dot{\beta_2} + \beta_2 K_1^{\mathbb{Q}}\right] ds + \int_0^t \Psi_{s-}\beta_2\sigma^{\mathbb{Q}}(X_s)dW_s \\
&+ \quad \int_{(0,t]\times\mathbb{R}} \Psi_{s-}(\exp{(x\beta_2)} - 1 - Ix\beta_2)\tilde{N}
\end{aligned}
$$

We therefore get the equations:

$$
\begin{aligned}
\dot{\alpha_2} \quad &= \quad \rho_0 - \int_{\mathbb{R}} \Psi_{s-}(\exp{(\beta_2 x)} - 1 - Ix\beta_2)\nu^{\mathbb{Q}}(dx) - \beta_2 K_0^{\mathbb{Q}} - \frac{1}{2}\beta_2^2 H_0^{\mathbb{Q}} \\
\dot{\alpha_1} \quad &= \quad \rho_0 - \int_{\mathbb{R}} \Psi_{s-}(\exp{(\beta_2 x)} - 1 - Ix\beta_2)\nu^{\mathbb{Q}}(dx) + \dot{\alpha} + \beta K_0^{\mathbb{Q}} - \frac{1}{2}\beta_1^2 H_0^{\mathbb{Q}} - \beta_1(K_0^{\mathbb{Q}} - \beta H_0^{\mathbb{Q}}) \\
&\quad -\frac{1}{2}\beta_1^2 H_0^{\mathbb{Q}}
\end{aligned}
$$

and so we identify the coefficients multiplying β_1 and β_1^2 from this last equation with those of:

$$
\dot{\alpha_1} \quad = \quad 0 - \beta_1 K_0 - \frac{1}{2}\beta_1^2 H_0 - \int_{\mathbb{R}} \Psi_{s-}(\exp{(\beta_1 x)} - 1 - Ix\beta_1)\nu(dx)
$$

and so we have that:

$$
\begin{aligned}
H_0^{\mathbb{Q}} \quad &= \quad H_0 \\
K_0^{\mathbb{Q}} \quad &= \quad \beta H_0^{\mathbb{Q}} + K_0 \\
K_0^{\mathbb{Q}} \quad &= \quad \beta H_0 + K_0
\end{aligned}
$$

Similarily we get for the equation of $\dot{\beta}_1$ that:

$$\begin{aligned} H_1^{\mathbb{Q}} &= H_1 \\ K_1^{\mathbb{Q}} &= \beta H_1^{\mathbb{Q}} + K_1 \\ K_1^{\mathbb{Q}} &= \beta H_1 + K_1 \end{aligned}$$

These results are similar to those of Duffie *et al.* (2000), except for the measure $\nu^{\mathbb{Q}}$.

Appendix B

Details of Chapter 3

B.1 Posteriors for μ and ρ

- The Posterior for μ is:

$$\pi(\mu|\triangle X_{\mathscr{T}}) \quad \propto \quad \pi(\mu)\exp\left(-\sum_{j=0}^{n-1}\frac{(\triangle X_{t_{j+1}}-\mu_{t_j})^2}{2\int_{t_j}^{t_{j+1}}\sigma_s^2 ds}\right)$$

where $\mu_{t_j} \equiv \mu + \rho(Z_{\lambda t_{j+1}} - Z_{\lambda t_j})$, and let $s_{t_j} \equiv \int_{t_j}^{t_{j+1}}\sigma_s^2 ds$.

Recall that we choose a prior for μ as $N(0,\sigma_\mu^2)$, where $\sigma_\mu^2 = 1000$. Thus:

$$\pi(\mu|\triangle X_{\mathscr{T}}) \quad \propto \quad \pi(\mu)\exp\left(-\frac{1}{2}\sum_{j=0}^{n-1}\left[\frac{(\triangle X_{t_{j+1}}-\mu_{t_j})^2}{s_{t_j}}+\frac{\mu^2}{n\sigma_\mu^2}\right]\right)$$

and so $s = -\frac{1}{2}\sum_{j=0}^{n-1}\left[\frac{(\triangle X_{t_{j+1}}-\mu_{t_j})^2}{s_{t_j}}+\frac{\mu^2}{n\sigma_\mu^2}\right]$ is equal to:

$$s \quad = \quad -\frac{1}{2}\sum_{j=0}^{n-1}\left[\frac{(\triangle X_{t_{j+1}}-\mu_{t_j})^2}{s_{t_j}}+\frac{\mu^2}{n\sigma_\mu^2}\right]$$

implies that, after keeping the μ terms in s, we get:

$$-\frac{1}{2}\sum_{j=0}^{n-1}\left[\mu^2(\frac{1}{s_{t_j}}+\frac{1}{n\sigma_\mu^2})-2\frac{\mu}{s_{t_j}}(\triangle X_{t_{j+1}}-\rho(Z_{\lambda t_{j+1}}-Z_{\lambda t_j}))\right]$$

let $a \equiv \sum_{j=0}^{n-1}(\frac{1}{s_{t_j}}+\frac{1}{n\sigma_\mu^2})$, and so completing squares:

$$s^{'} \;\; = \;\; -\frac{a}{2}\left[\mu-\frac{\sum_{j=0}^{n-1}\frac{1}{\int_{t_j}^{t_{j+1}}\sigma_s^2 ds}(\triangle X_{t_{j+1}}-\rho(Z_{\lambda t_{j+1}}-Z_{\lambda t_j}))}{a}\right]^2$$

The posterior for μ given everything else is:

$$[\mu|\ldots] \;\; \sim \;\; N\left[\sum_{j=0}^{n-1}\frac{(\triangle X_{t_{j+1}}-\rho(Z_{\lambda t_{j+1}}-Z_{\lambda t_j}))}{a\int_{t_j}^{t_{j+1}}\sigma_s^2 ds},\frac{1}{\sum_{j=0}^{n-1}(\frac{1}{s_{t_j}}+\frac{1}{n\sigma_\mu^2})}\right]$$

- The posterior for ρ is:

$$\pi(\rho|\triangle X_{\mathscr{T}}) \;\; \propto \;\; \pi(\rho)\exp\left(-\frac{1}{2}\sum_{j=0}^{n-1}\left[\frac{(\triangle X_{t_{j+1}}-\mu_{t_j})^2}{s_{t_j}}+\frac{\mu^2}{n\sigma_\rho^2}\right]\right)$$

where $\pi(\rho)$ is the prior for ρ and is a $N(0,\sigma_\rho^2)$. Completing squares for ρ as we did for μ, we get:

$$[\rho|\ldots] \;\; \sim \;\; N\left[\sum_{j=0}^{n-1}\frac{(\triangle X_{t_{j+1}}-\mu)(Z_{\lambda t_{j+1}}-Z_{\lambda t_j})}{a'\int_{t_j}^{t_{j+1}}\sigma_s^2 ds},\frac{1}{\sum_{j=0}^{n-1}(\frac{(Z_{\lambda t_{j+1}}-Z_{\lambda t_j})^2}{s_{t_j}}+\frac{1}{n\sigma_\rho^2})}\right]$$

where $a^{'} \equiv \sum_{j=0}^{n-1}(\frac{(Z_{\lambda t_{j+1}}-Z_{\lambda t_j})^2}{s_{t_j}}+\frac{1}{n\sigma_\rho^2})$,

B.2 Solution of SDE for σ_t^2

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ^\star_{\lambda t}$$

Let $M_t \equiv \sigma_t^2 \exp(\lambda t)$, and $d\sigma_t^2 + \lambda\sigma_t^2 dt = dZ^\star_{\lambda t}$ so:

$$dM_t = \exp(\lambda t) d\sigma_t^2 + \lambda\sigma_t^2 \exp(\lambda t)$$

and so:

$\exp(-\lambda t) dM_t = d\sigma_t^2 + \lambda\sigma_t^2 dt = dZ^\star_{\lambda t}$ and so:

$$dM_t = \exp(\lambda t) dZ^\star_{\lambda t}$$

thus we have:

$$M_t - M_0 = \int_0^t \exp(\lambda s) dZ^\star_{\lambda s}$$

or:

$$\exp(\lambda t)\sigma_t^2 - \exp(0)\sigma_0^2 = \int_0^t \exp(\lambda s) dZ^\star_{\lambda s}$$

So finally:

$$\begin{aligned} \sigma_t^2 &= \exp(-\lambda t)\sigma_0^2 + \exp(-\lambda t) \int_0^t \exp(\lambda s) dZ^\star_{\lambda s} \\ &= \exp(-\lambda t)\sigma_0^2 + \int_0^t \exp\left[-\lambda(t-s)\right] dZ^\star_{\lambda s} \end{aligned}$$

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Biography

Enrique ter Horst was born in Caracas, Venezuela, on September the 17, 1975. He studied economics in Strasbourg, France, where he earned his B.Sc. in econometrics in 1998 and then his "Maitrise" in finance, in 1999. He decided to pursue his Ph.D in Statistics at ISDS in 1999, where he earned his M.Sc. in Statistics in 2001. He was awarded the fellowship from the Graduate School.