Midterm Examination # 1

Mth 135 = Sta 104

Thursday, 2000 October 12, 2:15 – 3:30 pm

If you don't understand something in one of the questions, please ask me. You may use *your own* calculator and a one-sided, $8\frac{1}{2}$" × 11" sheet of notes, but you may not share materials or use any other notes or books on this **closed-book** exam. A blank worksheet and a page of common pdf/pmf formulas are attached to the exam.

You should spend about 10–15 minutes on each problem. Problems all count equally, even though they are not equally difficult. Point values for problem parts are indicated in parentheses.

You must **show** your **work** to get credit. Unsupported answers are not acceptable, even if they are correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. After completing the exam please acknowledge the Duke Honor Code:

I have neither given nor received any unauthorized aid on this exam.

Signature: ____________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Lake Windemere is home to 1000 fish, all of which are equally likely to be caught by a fisherman. Just **one** of them has dangerously high levels of heavy metals (cadmium, lead, mercury) from swallowing an AAA battery carelessly dropped into the lake by a statistics professor speaking at a conference in nearby Ambleside, England. Tracy is sitting on a pier fishing; let X be the number of catches needed to find the toxic fish.

a) (8) If Tracy keeps each fish s/he catches, so that no fish may be caught more than once, what is the probability that the poison fish will be caught on the x^{th} try? Give your answer for **every** number $x \in \mathbb{R} \equiv (-\infty, \infty)$.

$\mathsf{P}[X = x] =$

b) (8) If Tracy replaces each fish s/he catches, so that each fish may be caught repeatedly, then what is the probability that the poison fish will be caught for the first time on the x^{th} try? Give your answer for every number $x \in \mathbb{R}$.

$\mathsf{P}[X = x] =$

c) (4) Write down an expression (sum or integral) for computing the expected number of fish Tracy must catch to find the toxic one, if fish **are replaced** (do not evaluate the expression).

$\mathsf{E}[X] =$

Do you expect this will be ◯ more or ◯ less than 500.5, the mean if fish are not replaced?

Problem 2: Alex wants to invest in the stock market, and plans to buy a 100-share block of ABC corporation stock, at \$1/share. S/he believes that ABC stock will either double in value to \$2/share, with probability p, or else will drop by half in value to \$0.50/share, with probability $1-p$, for some $0 < p < 1$. Let Y denote the total value of the stock investment *after* this event occurs. All shares have the same value.

a) (5) What are the possible values of Y?

$Y \in \mathcal{Y} = \{$ ____________________ $\}$

b) (5) What is the expected value of Y? (It will depend on p). **Simplify!**

$\mathsf{E}[Y] =$ ____________________

c) (5) For what values of p would Alex's investment be profitable in the sense that the expected value of Y would exceed (or at least equal) the initial investment?

________ $\le p \le$ ________

c) (5) Find the expectation of $g(y) = \sqrt{2y}$, for $p = 0.20$.

$\mathsf{E}\big[(2Y)^{1/2}\big] =$ ____________________

Problem 3: Two events, A and B, have probabilities $\mathsf{P}[A] = 0.5$ and $\mathsf{P}[B] = 0.3$, respectively. Find the probabilities asked, under the conditions given; each "If" condition applies *only* to that question.

a) (5) If A and B are independent, then $\mathsf{P}[A \cap B] =$ ____________

b) (5) If A and B are exclusive, then $\mathsf{P}[A \cup B] =$ ____________

c) (5) If $\mathsf{P}[A \cup B] = 0.74$, then $\mathsf{P}[A \mid B] =$ ____________

d) (5) If $\mathsf{P}[A \mid B] = 0.40$, then $\mathsf{P}[A \cup B] =$ ____________

Problem 4: The amount of time (in minutes) T that Darryl waits for the cross-campus bus has cumulative probability distribution function (CDF) given by

$$F_T(t) = \mathsf{P}[T \le t] = \begin{cases} \dfrac{t^2}{4+t^2} & t > 0 \\ 0 & t \le 0 \end{cases}$$

a) (4) Let S be the amount of time in *hours* that Darryl waits. Find the cumulative distribution function (CDF) for S. **Simplify!**

$F_S(s) =$

b) (4) What is the probability that Darryl waits between one and two minutes? Exactly ten minutes?

$\mathsf{P}[1 < T \le 2] =$ __________ $\mathsf{P}[T = 10] =$ __________

c) (4) In fifty days of bus-riding, two trips per day, how many times should Darryl expect to wait more than 14 minutes?

$\mathsf{E}\big[N_{[T>14]}\big] =$ ____________

d) (4) What is the probability density function (pdf) for Darryl's wait, in **hours**? **Simplify!**

$f_S(s) =$ ____________

e) (4) It turns out that $\mathsf{E}[T] = \pi$ (you don't have to show this). What does that mean? Mark any correct statements:

◯ $\mathsf{P}[T \le \pi] = 1/2$ ◯ On average, T will be about π

◯ $\int_{-\infty}^{\infty} t f_T(t)\,dt = \pi$ ◯ $T = \pi$ is the most probable value

Problem 5: A hat contains $n = 10$ coins, eight of which are fair (so that $\mathsf{P}[H] = 1/2$) and two of which are biased with $\mathsf{P}[H] = 2/3$. A single coin is drawn at random from the hat. All questions below are about this one coin; it is not replaced, and no other coin is drawn. Be sure to **simplify** your answers below. F and B denote the events that the coin is Fair and Biased, respectively, and H_i the event that it falls Heads on the i^{th} toss.

a) (5) On the first toss, it lands Heads. What is the (conditional) probability that it is a fair coin?

   $\mathsf{P}[F \mid H_1] =$ ______________

b) (5) It is tossed a second time. What is the (conditional) probability that it will land Heads this time, too, given Heads on the first toss? (note it could be fair or biased)

   $\mathsf{P}[H_2 \mid H_1] =$ ______________

c) (5) If it does land Heads on both the first and second tosses, what is the probability that it is a biased coin?

   $\mathsf{P}[B \mid H_1 \cap H_2] =$ ______________

d) (5) Denote by Z the number of Tails shown by this coin before the first Head. Give the pmf or pdf $f(z)$ for Z (first decide if Z has a discrete or continuous distribution). Do **not** condition on the outcomes of any tosses. Give your answer for **all** $z \in \mathbb{R}$.

   $f(z) =$ ___________

Extra worksheet, if needed:

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		