Midterm Examination # 2

Mth 135 = Sta 104

Thursday, 2000 November 16, 2:15 – 3:30 pm

If you don't understand something in one of the questions, *please* ask me. You may use *your own* one-sided, $8\frac{1}{2}$" $\times$ 11" sheet of notes and calculator, but do not share materials or use any other notes or books on this **closed-book** exam. A blank worksheet, normal distribution table, and page of common pdf/pmf formulas are attached to the exam.

Each problem should take about 10–15 minutes. All problems count equally, even though they aren't all equally difficult. Point values for problem parts are indicated in parentheses.

You must **show** your **work** to get credit. Unsupported answers aren't acceptable, even if they're correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. Please acknowledge the Duke Honor Code:

> *I will not lie, cheat, or steal in my academic endeavors.*
> *I will give prompt written notification to an appropriate faculty member or dean when I observe academic dishonesty in any course.*
> *I join the undergraduate student body of Duke University in a commitment to this Code of Honor.*

Name: ______________ Signature: ____________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Three independent events A, B, and C have probabilities $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, respectively. Let X be the number of these events that occur $(0 \le X \le 3)$. Find:

a) (5) $\mathsf{P}[X = 0] =$ ______________

b) (5) $\mathsf{P}[X = 3] =$ ______________

c) (5) $\mathsf{E}[X] =$ ______________

d) (5) $\mathsf{P}[A \mid X = 1] =$ ______________

Problem 2: Let Y have a continuous probability distribution with pdf (density function) $y\,e^{-y}$, $0 < y < \infty$; then let X have a conditional distribution (given Y) that is uniform on the interval from 0 to Y.

a) (4) Give the joint density function for X and Y; be sure to give your answer correctly for all x and y:

$f(x, y) =$ ________________

b) (8) Find the marginal density functions for X and Y:

$f_X(x) =$ ________________ $f_Y(y) =$ ________________

c) (8) Find the indicated expectations. (Hint: Little or no integration is needed if you recognize both marginal distributions).

$\mathsf{E}[X] =$ ____________ $\mathsf{E}[Y] =$ ____________ $\mathsf{E}[\frac{1}{Y}] =$ ____________.

Problem 3: Microprocessors are damaged by radiation. The survival time T in days for a satellite's main microprocessor on Jupiter's moon Titan has (let us assume) an exponential distribution with $\mathsf{P}[T > t] = e^{-t/10}$ for $t > 0$. The battery onboard has a lifetime S in days that is uniform on the interval $[0, 20]$. You may assume S and T are independent.

a) (6) Find the probability density function for $X \equiv \sqrt{T}$.

$f_X(x) =$ ______________

b) (6) In order for the satellite to complete its mission, both the battery *and* the microprocessor must last at least two days. Find the probabilty that the mission can be completed, i.e., that both components last two days:

$\mathsf{P}[\min(S, T) > 2] =$ ______________

c) (6) Find the probability that the battery fails first:

$\mathsf{P}[S < T] =$ ___________________

d) (2) Find the probabilty density function for the time U (still in days) until at least one of these critical components fails (hard):

$f_U(u) =$ ______________

Problem 4: Let X and Y be independent, each with the exponential distribution with mean 1, and let $R \equiv Y/X$ be their ratio.

a) (5) Find the conditional density function for R, given X (Hint: Given X, $Y = XR$ is a *constant* multiple of R)

$f_{R|X}(r|x) =$ ____________________

b) (5) Find the joint density function for X and R:

$f_{X,R}(x, r) =$ ____________________

c) (5) The marginal density for R is $f_R(r) = (r+1)^{-2}$, $r > 0$; find the indicated conditional distribution and give its name and the value(s) of any parameter(s) (which may depend on r, of course):

$f_{X|R}(x|r) =$ ____________________
○ Exponential ○ Gamma ○ Geometric ○ Normal ○ Uniform
Parameter(s):

d) (5) Find the indicated probability. Show your work.

$\mathsf{P}[X > 2Y] =$ ____________________

Problem 5: Y is zero or one, with probability $\frac{1}{2}$ each, and Z has a normal distribution with mean zero and variance one. Let $X \equiv Y + Z$ be their sum. Find the expectation and probabilities below. (Hint: try conditioning on Y).

a) (5) $\mathsf{E}[X] =$ _______________

b) (5) $\mathsf{P}[X \geq 1] =$ _______________

c) (5) $\mathsf{P}[Y = 1 \mid X \geq 1] =$ _______________

d) (5) $\mathsf{P}[X \geq 2 \mid X \geq 1] =$ _______________

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt:$$

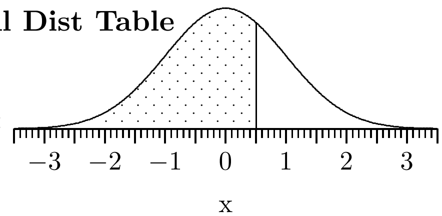

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		